MA/ST 747, Midterm
Take-Home Part
Due: In-class, Tuesday, March 31

Rules: work independently, using only your notes and the text.

1. Let $(Y_n)_{n\geq 1}$ be a sequence of i.i.d. random variables and let $\{\mathcal{F}_n\}_{n\geq 1}$ be a filtration such that:

   (a) Y_n is $\mathcal{F}_n$ -measurable and

   (b) $\mathcal{F}_n$ and $\sigma(Y_k, k \geq n+1)$ are independent for $n \geq 1$ (i.e. random variables that are $\mathcal{F}_n$ -measurable are independent of random variables that are $\sigma(Y_k, k \geq n+1)$ -measurable).

   Let $T \geq 1$ be a finite stopping time with respect to $\{\mathcal{F}_n\}_{n\geq 1}$ and $ET < \infty$. Finally, let $S_n = Y_1 + ... + Y_n$, $n \geq 1$.

   Prove that $E|Y_1| < \infty$ implies $E|S_T| < \infty$.

2. Let (Z_n) be a martingale. Show that the following are equivalent:

   (a) $\sup_n E|Z_n| < \infty$;

   (b) there exists a martingale (X_n) such that $|Z_n| \leq X_n$ for each n;

   (c) there exists a martingale (X_n) with $X_n \geq 0$ and $Z_n \leq X_n$ for each n;

   (d) there exists martingales (X_n) and (Y_n), with $X_n \geq 0$ and $Y_n \geq 0$ for each n, such that $Z_n = X_n - Y_n$.

   Hint: To show (a) implies (b), consider the Doob decomposition: $|Z_n| = M_n + A_n$, and take $X_n = M_n + E(A_\infty|\mathcal{F}_n)$.

   If using the hint or not, justify all your steps.

3. (a) Let $\{X_n; n = 1, 2, ...\}$ be a collection of integrable random variables on $(\Omega, \mathcal{F}, P)$. If

   $$K \overset{\triangle}{=} \sup\left\{\int |X_n|^{1+\epsilon}\, dP : n = 1, 2, ...\right\} < \infty$$

   for some $\epsilon > 0$, then show that $\{X_n; n = 1, 2, ...\}$ is uniformly integrable.

   (b) Let (X_n) in (a) be a submartingale. Show that X_n converges to some limit X in L_1 and a.s.