MA/ST 747, Midterm
In-Class Part
You may use only your notes and the text.

1. Let (X_n) be a martingale. Show (e^{tX_n}) is a submartingale for $t \in \mathbb{R}$.

2. Let (Y_n), $n \geq 1$, be an arbitrary sequence of integrable random variables on $(\Omega, \mathcal{F}, P)$. Construct a martingale sequence (M_n), showing it is indeed a martingale sequence.

   Hint: use conditional expectation with filtration $\mathcal{F}_n = \sigma(Y_1, Y_2, ..., Y_n)$ and note $\mathcal{F}_0 \triangleq \{\emptyset, \Omega\}$.

3. Suppose (X_n) is $\mathcal{F}_n$-adapted and represents a population size. Furthermore, we have

$$E[(X_{n+1} - X_n)|\mathcal{F}_n] \leq 0$$

   for all $n \geq 0$, and $X_0 \equiv a$, $a \in \mathbb{R}^+$. Show

   (a) [corrected] $E[X_m|\mathcal{F}_n] \leq a$ for all $m > n = 0$.

   (b) $X_n \to X$ a.s. where $EX \leq a$.

4. Suppose (X_n) is a $\mathcal{F}_n$-martingale, $n \geq 0$. Let T be a stopping time with respect to $\mathcal{F}_n$, $n \geq 0$. Suppose $P(T < \infty) = 1$ and there exists a $0 < K < \infty$ such that for $n \geq 0$,

$$|X_{T \wedge n}| \leq K \quad \text{a.s.}$$

   Prove that $EX_T = EX_0$.

5. Consider a gambling system where $B > \beta_n > 0$ are the bets are made on i.i.d. outcomes X_n, $n \geq 1$, where $X_n = \pm 1$ and $X_1 = 1$ with probability $1/2$. Furthermore, (β_n) is a predictable sequence with respect to $\mathcal{F}_n = \sigma(X_1, X_2, ..., X_n)$, $n \geq 1$. The fortune at time n is given by

$$S_n = \sum_{i=1}^{n} \beta_i X_i + S_0,$$

   where S_0 is the initial fortune. Find a bound on the probability that the maximum magnitude of the fortune up to time m is greater than some large value $M < \infty$.