MA/ST 747, Spring 2009
Homework 5
Due: In-class, Thursday, April 23

1. Prove the following (stated in class): Let y be a recurrent state. Then $P_y\left(T_y^k < \infty\right) = 1$ for all $k \geq 1$. *Hint:* note $k = 1$ is true by definition. Use this with the Markov property to push out to all k.

2. Given a sequence of Bernoulli trials (i.e. a sequence of independent trials with $P(\text{success}) = p$ and $P(\text{failure}) = 1 - p = q$). Consider the Markov chain (X_n) where $X_n = k$, $k = 0, 1, 2, ...$, if the last failure was at $n - k$ (in other words the state is keeping track of the size of success runs and at a failure $X_n = 0$). Assume $X_0 = 0$. Show that the state 0 is recurrent. *Hint:* This is similar to a class example.

3. Consider the Markov chain (X_n) with transition probabilities $p(i,j)$ given by

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 1/4 & 1/4 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 1/5 & 2/5 & 2/5 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

   Find (if any)

   (a) The closed sets of states.
   (b) The irreducible sets of states.
   (c) The recurrent states (write as a union of closed and irreducible sets, if possible).
   (d) The transient states.
   (e) Absorbing states.

4. Durrett, Chapter 5, Exercise 3.1 *Hint:* Use the Strong Markov property repeatedly (i.e. various k), shifting using R_{k-1}.

5. Show that for a simple random walk on the integers $\boldsymbol{Z}$, the state 0 is recurrent. *Hint:* Find the formula for the n-step transition probability $p^n(0,0)$ and then apply Stirling's formula:

$$n! \sim n^n e^{-n} \sqrt{2\pi n},$$

   i.e. as $n \to \infty$ the ratio tends to 1.

6. Durrett, Chapter 5, Exercise 3.4 *Hint:* Some of the steps are similar to the first entrance decomposition example in class.

7. Durrett, Chapter 5, Exercise 3.5

8. Durrett, Chapter 5, Exercise 3.6 *Hint:* Use (3.7) for (i). *Added Hint*: Use (4.1) in Chapter 4 for (ii).