14. Applying Dynkin's π-λ theorem:

(i) Let π-system be $\bigcup_n \mathcal{F}_n$.

(a) Note that g is measurable wrt $\sigma\left(\bigcup_{n=1}^{\infty} \mathcal{F}_n\right)$

(b) Also note that $\int_A f\, dP = \int_A f_n\, dP$ if $A \in \mathcal{F}_m$, $n \geq m$, using the defn. cond. expectation.

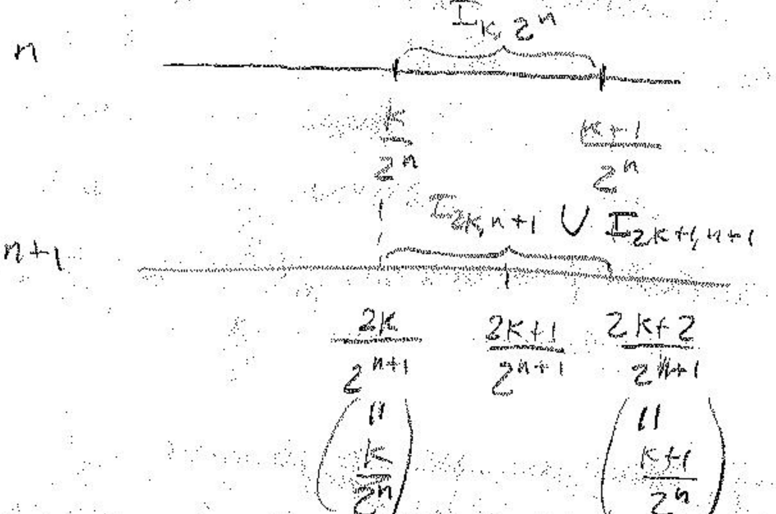
By the convergence result ($f_n \to g$ a.s. & in L_1) & (a) we have $\int_A f\, dP = \int_A g\, dP$ (1), $A \in \bigcup_{m=1}^{\infty} \mathcal{F}_m$,

ie. the π-system

(ii) Let the λ-system be sets $\{A : \int_A f\, dP = \int_A g\, dP\}$

Then Dynkin's π-λ system says (1) is true for $A \in \sigma\left(\bigcup_{n=1}^{\infty} \mathcal{F}_n\right)$, so $f = g$ a.s.

15. Here, given n, $0 \le k \le 2^n - 1$.



Perhaps easiest way to show martingale is to use calculations like in Hw 2, #4 which use Durrett formula on p. 220:

Note $\mathcal{F}_n = \sigma\left([\tfrac{0}{2^n}, \tfrac{1}{2^n}), [\tfrac{1}{2^n}, \tfrac{2}{2^n}), \dots, [\tfrac{2^n-1}{2^n}, 1)\right)$

$$E[X_{n+1} \mid \mathcal{F}_n] = \frac{E[X_{n+1}; [\frac{2k}{2^n}, \frac{2k+1}{2^n})]}{\frac{1}{2^n}} \text{ on } \left[\tfrac{2k}{2^n}, \tfrac{2k+1}{2^n}\right)$$

$$= \frac{\frac{2^{n+1}}{2^{n+1}}\left[\left(f\left(\frac{2k+2}{2^{n+1}}\right) - f\left(\frac{2k+1}{2^{n+1}}\right)\right) + \left(f\left(\frac{2k+1}{2^{n+1}}\right) - f\left(\frac{2k}{2^{n+1}}\right)\right)\right]}{\frac{1}{2^n}}$$

$$= \left[\frac{f\left(\frac{k+1}{2^n}\right) - f\left(\frac{k}{2^n}\right)}{2^{-n}}\right] = X_n \text{ so}$$

a martingale. By the Lipschitz continuity

$0 \le |X_n| \le K$, so $\{X_n, n\}$ is u.i. By Thm. A in class (or (5.6) Durrett) $X_n \to X_\infty$ a.s. and in L_1. With the corollary to Thm. A in class (or ~~st~~ (5.5) Durrett) $X_n = E[X_\infty | \mathcal{F}_n]$. In other words

$$f\left(\tfrac{k+1}{2^n}\right) - f\left(\tfrac{k}{2^n}\right) = \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} X_\infty(\omega)\,d\omega = \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} X_n(\omega)\,d\omega \text{ , each } K.$$

By linearity of the integral & taking limits, noting f is continuous (Lipschitz continuity is "stronger" condition than continuity) one can get

$$f(a) - f(b) = \int_a^b X_\infty(\omega)\,d\omega,$$

$a, b \in \mathbb{R} \cap [0, 1]$.

16. p_K is prob. of having K-male children (each male, each generation since ξ_i^n; $i, n \geq 0$ iid) ie $p_K = P(\xi_i^n = K)$.

If $p_0 > 0$, then $P(Z_{n+1} = 0 \mid Z_1, \ldots, Z_n) \geq p_0^K$ on $\{Z_n \leq K\}$ by the iid assumption on $\{\xi_i^n\}$ (& noting $p_0 < 1$, of course). Let $D \triangleq \{Z_n = 0$ for some $n \geq 1\}$ & note $f(K)$ in Exercise 5.5 corresponds to p_0^K here. Also note D is also $\{\lim_n Z_n = 0\}$ (once hit 0, stay at zero). Thus

$P(\lim_n Z_n = 0 \text{ or } \infty) = 1$ results from Exercise 5.5.

<u>FYI</u>: In Exercise 5.5, when applying Lévy's 0-1 law, $\mathcal{F}_n = \sigma(X_1, \ldots X_n)$,
$P(D \mid X_1, \ldots, X_n) \to 1_D$ a.s. by Lévy 0-1 law and from this one can show
[Replace ∞ by M below & let $x = M+1$, eg. $\delta(x) = M+1$. Then let $M \to \infty$] $D \supset \{\liminf X_n < \infty\}$ a.s. Thus conclusion holds.

17. (Note this result is an analog to DCT for L_1 case.)

By triangle inequality,

$$E\left|E[Y_n|\mathcal{F}_n]-E[Y|\mathcal{F}_\infty]\right| \le E\left|E[Y_n|\mathcal{F}_n]-E[Y|\mathcal{F}_n]\right|$$

$$+E\left|E[Y|\mathcal{F}_n]-E[Y|\mathcal{F}_\infty]\right|$$

By Jensen

$$E\left|E(Y_n|\mathcal{F}_n)-E[Y|\mathcal{F}_n]\right|$$

$$\le E[E(|Y_n-Y||\mathcal{F}_n)]$$

$$=E[|Y_n-Y|]\to 0 \text{ (in } L_1)$$

since $Y_n\to Y$ in L_1.

~~Since $Z_n \triangleq E[Y|\mathcal{F}_n]$ is a mtgle, Thm A~~ 5.7 + Corollary

By (Cor (5.7) Durrett) ~~says $Z_n\to Z=E[Y|\mathcal{F}_\infty]$~~

Ym L_1, i.e.

$$E\left|E[Y|\mathcal{F}_n]-E[Y|\mathcal{F}_\infty]\right|\to 0 \text{ (in } L_1).$$