MA/ST 747, Spring 2009
Homework 3
Due: In-class, Tuesday, March 10

1. Show that the map $X \to E(X|\mathcal{F})$ is a contraction in L^p, $1 \le p < \infty$: that is show
$$||E(X|\mathcal{F})||_p \le ||X||_p.$$

2. Show for (X_n) a martingale, (X_n^-) is submartingale, but for (X_n) a submartingale, (X_n^-) is not (in general) a submartingale.

3. Let (X_n) be a sequence of independent random variables representing the price of an asset, and suppose we can assign a mean price $\bar{X}$ to the asset. Set $Y_n \stackrel{\triangle}{=} X_n - \bar{X}$, and suppose someone decides to purchase quantities of the asset according to a strategy given by
$$F_1 = c \in I\!R, \quad \text{and } F_n \stackrel{\triangle}{=} F(X_1, ..., X_{n-1}).$$
   (a) Show that the amount of the asset held (relative to the mean) i.e.
   $$W_n = \sum_{i=1}^{n} F_n Y_n$$
   is a martingale.
   (b) Show that the increments in the asset holdings are uncorrelated; in other words
   $$E[W_{n+1} - W_n]\,[W_{k+1} - W_k] = 0, \quad k \ne n,$$
   i.e. the "martingale increments" are uncorrelated.

4. Durrett Chap. 4, Exercise 2.6

5. Durrett, Chapter 4, Exercise 2.7

6. Durrett, Chapter 4, Exercise 2.11. *Hint:* recall the scaling in the branching process example in class when trying to come up with the related supermartingale.

7. Durrett Chap 4, Exercise 3.1. *Hint:* Let $N = \inf\{n : X_n > M\}$, consider $X_{N \wedge n}$, and send $M \to \infty$.

8. Durrett Chap 4, Exercise 3.4 *Hint:* The supermartingale referred to in the Durrett's hint is $W_n = X_n - \sum_{m=1}^{n-1} Y_m$.

9. Suppose (X_n) is a submartingale and T the stopping time ($\lambda > 0$)
$$T = \inf\{k : X_k > \lambda \text{ or } k = n\}.$$
Let $A = \{\max_{0 \le k \le n} X_k > \lambda\}$. Prove
$$\int_A X_T \, dP \le \int_A X_n \, dP.$$

10. Durrett Chap. 4, Exercise 4.2 *Hint:* Use $K_n = 1_{M<n\leq N}$, a predictable process.

11. Durrett Chap. 4, Exercise 4.5

12. Durrett Chap. 4, Exercise 4.9

13. Durrett Chap. 4, Exercise 4.10 *Hint:* Use the previous exercise (but with a different martingale—here it is scaled by b_m) and the fact in lemma 8.5, Chapter 1 (Kronecker's lemma).

14. In class we saw the following: Let $(\Omega = [0,1), \mathcal{F} = \text{Borel}, P = \text{Lebesgue})$. We formed martingale (f_n) from
$$f_n \stackrel{\triangle}{=} E[f \mid \mathcal{F}_n], \quad f \in L_1[0,1) \text{ given.}$$
We then showed the collection $\{f_n \,; n\}$ is uniformly integrable and so by Theorem A in class, we have $f_n \to g$ a.s. and in L_1. Show $f = g$ a.s. *Hint:* apply Dynkin's π - λ Theorem.

15. Durrett, Chapter 4, Exercise 5.3.

16. Durrett, Chapter 4, Exercise 5.6. *Note:* You can just cite exercise 5.5 in your answer.

17. Durrett, Chapter 4, Exercise 5.8.