MA/ST 747, Spring 2009
Homework 2
Due: In-class, February 12

1. (a) Given $(\Omega, \mathcal{F}, P)$ and $\mathcal{F}$-measurable r.v. X. The conditional probability of $A \in \mathcal{F}$ given X (i.e. $P(A|\sigma(X))$) can be defined as any $\sigma(X)$-measurable r.v. satisfying

$$P(A \cap \{X \in B\}) = \int_{\{X \in B\}} P(A|X) dP, \quad \forall\, B \in \mathcal{B}(I\!R). \tag{1}$$

Using the definition of conditional expectation, show

$$P(A|X) = E(1_A(\omega)|X) \quad \text{a.s. wrt. } P.$$

(b) We can also write equation (1) as

$$P(A \cap \{X \in B\}) = \int_B P(A|X = x)\mu(dx), \quad \forall\, B \in \mathcal{B}(I\!R).$$

Suppose X can only be integers with $P(X = j) > 0$. Show that any version of the conditional probability satisfy (integers j)

$$P(A|X = j) = \frac{P(A \cap \{X \in B\})}{P(X = j)}.$$

(c) Durrett, Chapter 4, Exercise 1.2

2. Suppose that random variables X and Y have a joint density $f(x, y)$, $f(x) = \int f(x, y)\, dy$, and assume $f(x) > 0$. Then show

$$P(Y \in B|X = x) = \int_B \frac{f(y, x)}{f(x)}\, dy, \quad \text{a.s. wrt. } \mu.$$

3. Durrett, Chapter 4, Exercise 1.4

4. Durrett, Chapter 4, Exercise 1.6. Also, write a comment about what the inequality says from the geometric point of view (ie. under proper assumptions, projections in $L_2(P)$).

5. Durrett, Chapter 4, Exercise 1.7

6. Durrett, Chapter 4, Exercise 1.10 and use the result of Exercise 1.9. (FYI: An example could be Y_i models data packets and N models the number of arrivals by fixed time T, modeled by a Poisson distribution with rate λ, so $EN^2 < \infty$).

7. Suppose in $(\Omega, \mathcal{F}, P)$ we have that sets in $\mathcal{F}$ only have measure 0 or 1. Show for integrable random variable Y,

$$E[Y|\mathcal{F}] = EY \quad \text{a.s.}$$

8. Let X be a $\mathcal{F}$-measurable (e.g. $\mathcal{F} = \sigma(X)$) random vector on $(\Omega, \mathcal{F}, P)$ taking values in $(I\!R^d, \mathcal{B}(I\!R^d))$ and measurable $f : (I\!R^d, \mathcal{B}(I\!R^d)) \to (I\!R, \mathcal{B}(I\!R))$. Let Y be a random variable on $(\Omega, \mathcal{F}, P)$, $E|Y| < \infty$. Also suppose $E|f(X)| < \infty$ and $E|Y f(X)| < \infty$. Prove

$$E(f(X)Y|\mathcal{F}) = f(X)E(Y|\mathcal{F}), \quad \text{a.s. wrt. } P.$$

9. In class, we stated: A r.v. Y is $\sigma(Z)$-measurable $\Leftrightarrow$ $Y = f(Z)$, some f Borel measurable.

   Prove the direction $\Rightarrow$.

   Hint: Y can be written as the limit of $Y_n = \sum_i c_i 1_{A_i}$, where $A_i = \{\omega : Z(\omega) \in B_i\}$, $B_i \in \mathcal{B}(\mathbb{R})$.

10. Suppose X and Y are independent r.v., X is countable, and $\psi : \mathbb{R}^2 \to \mathbb{R}$. Show (supplying any needed conditions along the way) for

$$f(x) \triangleq E[\psi(x, Y)],$$

   we have $f(X)$ is a version of $E[\psi(X, Y)|X]$.