MA 573
Mathematical and Experimental Modeling of Physical Processes I

THERMAL EXPERIMENT PROJECT DUE DATE: NOVEMBER 26, 2007

The goal of this project is to validate the mathematical model for heat transfer in a rod using data collected from experiments that you performed in the CRSC/Math Instructional and Research Laboratory. To this end, recall that under the assumptions

(i) Heat is transferred along the x-axis only;

(ii) Temperature is uniform over a cross-section and;

(iii) The rod is *perfectly insulated*,

a one-dimensional heat equation describing the heat conduction in a rod was developed. This has the form

$$\rho c_p \frac{\partial u(t,x)}{\partial t} = k \frac{\partial^2 u(t,x)}{\partial x^2},$$

where c_p is the specific heat, ρ is the mass density, and k is the thermal conductivity.

1. The appropriate boundary conditions for this experiment are

$$k u_x(t,0) = Q, \qquad u(t,L) = u_{\text{ambient}},$$

   where Q is an unknown constant determining the constant heat flux at the source, u_{ambient} is the known ambient (room) temperature, and L is the length of the rod . The specified boundary condition at $x = L$ implies that we have a constant temperature end. Is this an accurate assumption? If not, what would be a reasonable alternative?

   Formulate the least squares problem to estimate the parameters k and Q using the *steady-state* temperature values along the rod. The optimization problem associated with the inverse least squares technique can be solved numerically by using the MATLAB routines *fminsearch*, *fminunc* or *fmincon*. Note that since steady-state temperature values are used as data, we only have to solve a steady-state one dimensional heat equation, which is a linear two-point boundary value problem for which an exact solution can be derived. Derive this exact solution. Also, discuss all data or measurements that would be required in order to carry out the parameter estimation problem.

2. Collect steady-state values of the temperature distribution in a copper rod. Using this data set, estimate the unknown parameters k and Q. Recall that the MATLAB routines *fminunc*, *fmincon* and *fminsearch* require an initial guess for the parameters set. Are the estimated parameters k and Q unique with respect to different initial guesses? Explain the results. Plot the solution of the mathematical model against the data. How well does the model fit the data?

3. To compare the rates of heat flow in different materials under the same conditions, set-up a second experiment involving an aluminum bar. Repeat Part 2. Are the estimated values of k and Q changed (or unchanged) for the two rods? How do thermal conductivities of copper and aluminum bar affect their heat conduction?

4. Instead of assuming a nonhomogeneous Dirichlet boundary condition at $x = L$ as in Part 1, consider a convective cooling boundary condition of the form

$$ku_x(t, L) = h[u_{\text{ambient}} - u(t, L)],$$

   where h is the convective heat transfer coefficient. Repeat Part 2 (where you now estimate k, h, and Q) and compare the results.

5. In practice, it is very difficult to ensure perfect insulation. In fact, in the design of the experiment, we have an uninsulated metal bar which is heated at one end and allows heat to escape along its entire length.

   Under this new assumption of no insulation and assumptions (i) and (ii) above, derive the following equation for the conduction of heat in the rod:

$$\rho c_p \frac{\partial u(t, x)}{\partial t} = k \frac{\partial^2 u(t, x)}{\partial x^2} - \frac{2(a + b)}{ab} h(u(t, x) - u_{\text{ambient}}) \ , \ 0 \leq x \leq L$$

   where a and b are the dimensions of the cross-section area of the rod. The minus term on the right hand side of the above equation comes from the heat loss term along the length of the rod, which should be modeled by the Newton's law of cooling.

   Repeat Steps 1–4 using this model.

6. You should obtain improved fits to your data. Give a comparison and discussion of your findings for the two different models.