Math 573 — Final Exam

Due: 5 pm on Wednesday, December 12

(1) Suppose that fishing is regulated within a distance H from a shoreline, taken to be straight as depicted in Figure 1, but outside this zone, fishing is so heavy that the fish population is zero. Let x denote the distance from shore, t time, and $u(t,x)$ the population density (fish/distance from shore) of a species of fish. We assume that the population does not vary along the shoreline.

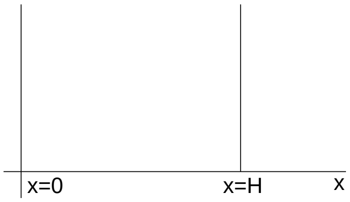

Figure 1: Orientation of the shore for Problem 1.

(a) Consider first the case where the population is constant in x. Assume that the fish population grows in a logistic fashion with birth rate b and death rate $d \cdot u$. Furthermore, suppose that within the regulated zone, fish are harvested at a rate Eu. Derive an ODE model quantifying the fish population. You should express the logistic term as $\alpha_0 u(1 - u/m)$. When $E = 0$, explain why α_0 and m are respectively termed the maximal growth rate and the carrying capacity.

(b) Determine the equilibria for the ODE in (a) and give a 1-D phase portrait. The stability of the equilibria depends on the relationship between E and α_0. Discuss the fish population behavior for differing values of E and α_0.

(c) We now want to model the situation in which fish exhibit a diffusion behavior in x with diffusivity κ. To do this, recall the relations

$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = k$$

$$q = f(u, u_x)$$

where ρ, q and k denote general density, flux and source or sink relations. Here f characterizes the constitutive behavior of the diffusion process. Determine an appropriate form of f and extend the ODE model of (a) to a PDE that characterizes this diffusion behavior.

(d) Determine appropriate boundary conditions at $x = 0$ and $x = H$.

(2) We now consider a predator-prey problem where $u_1(t,x)$ denotes the prey density and $u_2(t,x)$ denotes the predator density where $0 < x < 1$. The diffusivity, birth constants, and death constants are respectively denoted by κ_i, α_i and β_i where $i = 1, 2$. When deriving your model, you should make the following assumptions:

(i) Prey Birth Rate: α_1
(ii) Prey Death Rate: $\beta_1 u_2$
(iii) Predator Birth Rate: $\beta_2 u_1$
(iv) Predator Death Rate: α_2

(a) Using Problem 1, derive a coupled PDE model quantifying the predator-prey behavior under the assumption that both the predator and prey move by passive diffusion. You do not need to specify boundary conditions.

(b) Modify your model in Part (a) to account for the situation in which predators actively move toward higher prey concentrations and prey actively move away from higher predator concentrations. You should assume that no predators or prey leave the region. You do not need to specify boundary conditions.

(3) Consider two bars of length L that lie adjacent as depicted in Figure 2. We let $u_i(t,x)$ and κ_i, $i = 1,2$, denote the temperature and diffusivity of the bars. The boundary conditions for Bar 1 are $\frac{\partial u_1}{\partial x}(t,0) = 0$, $\frac{\partial u_1}{\partial x}(t,L) = 1$ whereas those for Bar 2 are $u_2(t,0) = u_2(t,L) = 0$. The initial conditions are $u_i(0,x) = 0$.

This is an example of a heat sink (Bar 2) bonded to an object that you want to cool (Bar 1).

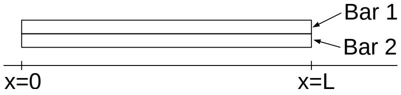

Figure 2: Orientation of the bars for Problem 3.

(a) Consider first the case in which the interface is insulated (no heat flow between the bars). Derive a model and discuss the behavior of the solutions u_1 and u_2 (use physical intuition and rather than finding an analytic solution).

(b) Now assume that heat is conduced between the bars. The rate at which heat flows from Bar 1 to Bar 2 is proportional to $u_1(t,x) - u_2(t,x)$ with proportionality constant α. A similar relations governs the heat flow between Bar 2 and Bar 1. Modify your model from Part (a) and discuss the physical behavior of the solutions (again, do not solve them explicitly).

(4) Consider the coupled system

$$\begin{aligned}\frac{\partial u}{\partial t} - \kappa_1 \frac{\partial^2 u}{\partial x^2} &= v \\ \frac{\partial v}{\partial t} - \kappa_2 \frac{\partial^2 v}{\partial x^2} &= u\end{aligned}$$

where $-L < x < L$. The boundary conditions are

$$u(t,-L) = u(t,L) = v(t,-L) = v(t,L) = 0.$$

(a) Use central differences in space and forward differences in time to construct a discrete numerical model. You can use the same number of gridpoints $m+1$ when discretizing u and v. Be sure to specify the spatial stepsize h in terms of m and designate the gridpoints x_i. Once discretized, you should formulate your numerical model in matrix form.

(b) Determine a weak formulation of the model. Be sure to specify the spaces of test functions.