Stochastic Models, Estimators and Emulators

"The best model of a cat is another cat or, better yet, the cat itself," Norbert Wiener

Stochastic Models

Motivation:

- No mathematical system model is perfect.

  -- Even physical laws are often approximations (e.g., Newton versus Einstein)

  -- Physical laws provide framework but parameters are unknown

- Dynamic systems often driven by disturbances that we can neither control nor model deterministically

- Sensors do not provide perfect measurements

Emulators

Definition: A mathematical model and the computer program used to implement it are often referred to as a *simulator*.

Definition: An *emulator* is a statistical approximation of a simulator.

- It is ideally designed to be simpler and more efficient than the original model. Hence it is much quicker to run.
- If the approximation is sufficiently accurate, it can be used to provide uncertainty and sensitivity measures.
- Because the emulator is a statistical approximation, it can be used to construct a probability function for the original model.
- Bayesian approaches often considered to incorporate prior knowledge.

References:

- A. O'Hagan, ``Bayesian analysis of computer code outputs: A tutorial''
- M.C. Kennedy and A. O'Hagan, ``Bayesian calibration of computer models,''

Frequentist Versus Bayesian Analysis

Frequentist Analysis:

• Based on the interpretation of the probability of an event as the long term limiting frequency with which the event occurs if the experiment is repeated an infinite number of times.

• This is limited if event can be repeated only a few times (e.g., once!).

-- e.g., the permeability of a certain oil region may only be tested once.

• Aleatory Uncertainty: uncertainty in repeated events due to intrinsic randomness and unpredictability.

• Epistemic Uncertainty: Uncertainty in non-repeatable events due simply to lack of knowledge.

Bayesian Inference:

• Statistical inference in which evidence or observations are used to update or infer the probability that a hypothesis is true.

-- As evidence is gained, the degree of belief in the hypothesis changes and is updated.

Bayesian Inference

Bayesian Inference:

• Example: For billions of years, the sun has risen after it has set. The sun has set tonight. With very high probability, the sun will rise tomorrow. With very low probability, the sun will not rise tomorrow.

Bayes Theorem:

$$P(H_0|E) = \frac{P(E|H_0)P(H_0)}{P(E)}$$

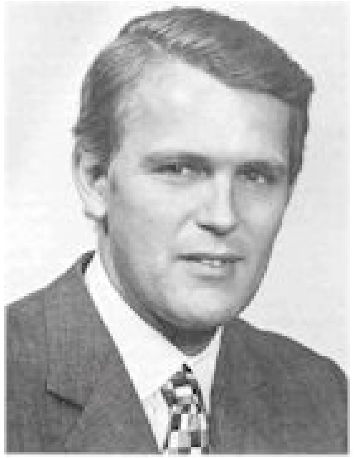
Here

- H_0: Null hypothesis inferred before new evidence E became available
- $P(H_0)$: prior probability of H_0
- $P(E|H_0)$: Conditional probability of observing evidence E given that H_0 is true. Often termed the likelihood function when expressed as a function of E given H_0.
- $P(E)$: Marginal probability of E. Probability of observing new evidence E under all mutually exclusive hypotheses. Computed as $\Sigma P(E|H_i)P(H_i)$.
- $P(H_0|E)$: posterior probability of H_0 given E.

Note: $P(E|H_0)/P(E)$ is impact that evidence has on belief in hypothesis.

Kalman Filter

Rudolf Kalman:



References:

- P.S. Maybeck, Stochastic Models, Estimation, and Control, Academic Press, New York, 1979
- Material on the website http://www.cs.unc.edu/~welch/kalman/

High Level Definition: An optimal recursive data processing algorithm

- Optimal with regard to most sensible criteria
- Recursive implies that previous data does not have to be stored and reprocessed every time a new measurement is taken.

Kalman Filter

Basic Assumptions:

• System can be described by a linear model

-- For many applications, linear models are adequate and, if not we can often linearize about a nominal point or trajectory

-- For certain nonlinear applications, the theory can be extended to obtain nonlinear filters

• System and measurement noise are white and Gaussian

-- ``White'' noise implies that the noise value is not correlated in time.

-- This also implies noise has equal power at all frequencies which cannot be true since it would result in infinite noise.

-- However, physical systems have a natural ``bandpass'' above which effects are basically negligible.

-- Assumption of Gaussian influences amplitude and shape at certain time.

-- Motivated by Central Limit Theorem.

-- Requires knowledge only of first and second-order statistics (mean and variance). Hence highly tractable.

Kalman Filter

Motivating Example: