# Math 731 Project 1

Due Thursday, February 5

(1) Consider the spring

$$\ddot{x} + g(x) = 0$$

with nonlinear stiffness. Use Lyapunov theory to determine conditions on the nonlinearity $g$ so that the equilibrium $x_e = (0,0)$ is uniformly stable.

(2) The following criterion can be used to determine the instability of a system. Let $V(t,x)$ with $V(t,0) = 0$ have continuous first partials and be decrescent. Let $\dot{V}$ be l.p.d.f. and $V(t_0, x) > 0$ where $x$ is arbitrarily close to 0. Then $x_e = 0$ at time $t_0$ is unstable.

(a) Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 + ax_1(x_1^2 + x_2^2) \\ \dot{x}_2 &= -x_1 + ax_2(x_1^2 + x_2^2)\,. \end{aligned} \tag{1}$$

Show that for $a = 1$ the solution $x_e = (0,0)$ of the linearization of (1) is stable but that it is an unstable equilibrium for (1). When choosing your Lyapunov candidate, think simple!

(b) Now determine conditions on the parameter $a$ so that the equilibrium state of (1) is (i) Stable and (ii) Globally Asymptotically Stable.

(3) Determine sufficient conditions for the stability of the system

$$\dot{x} = Ax + bf(x_1)$$

where

$$A = \begin{bmatrix} -2 & -1 \\ 0 & -2 \end{bmatrix} \quad , \quad B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

(4) Consider the second-order equation

$$\ddot{y} + \dot{y} + e^{-t}y = 0.$$

Using the Lyapunov candidate

$$V(t,x) = x_1^2 + e^t x_2^2\,,$$

where $x_1 = y, x_2 = \dot{y}$, show that the equilibrium $x_e = 0$ is stable at $t = 0$.

(5) Consider the system

$$\begin{aligned} \dot{x}_1 &= 3x_1 + x_2^2 - \text{sat}(2x_2 + u) \\ \dot{x}_2 &= \sin(x_1) - x_2 + u \end{aligned} \tag{2}$$

where $u$ is the control input and

$$\text{sat}(f) = \begin{cases} f \ , & |f| \le 1 \\ \text{sign} f \ , & |f| > 1 \end{cases}.$$

(a) Use Lyapunov's Indirect Method to determine the stability of $x_e = 0$ when $u(t) = 0$.

(b) For $u(t) = 1$, use Matlab to solve both the nonlinear system (2) and the linearized system for a couple of choices of initial conditions. Discuss the stability of the two systems.