Lyapunov Analysis

Time Invariant LPDF and PDF:

Definition: (a) A continuous function $W : \mathbb{R}^n \rightarrow \mathbb{R}$ is a locally positive definite function (lpdf) if and only if

(i) $W(0) = 0$

(ii) $W(x) > 0$ for all $x \neq 0$ which belongs to the ball

$$B_r = \{x \in \mathbb{R}^n \mid \|x\| \leq r\}$$

e.g., $W(x) = x_1^2 + \sin^2 x_2$

(b) W is a positive definite function (pdf) if and only if

(i) $W(0) = 0$

(ii) $W(x) > 0$ for all $x \neq 0$

(iii) $W(x) \rightarrow \infty$ as $\|x\| \rightarrow \infty$

e.g., $W(x) = x_1^2 + x_2^2$

Class-K Functions

Definition: A continuous function $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be of class-k if

$$\alpha(0) = 0$$

$$\alpha(\rho) > 0 \quad \forall \rho > 0$$

$$\alpha \text{ is nondecreasing}$$

$$\text{e.g.,} \quad \alpha(\rho) = \min_{\rho \leq \|x\| \leq r} V(x)$$

$$\beta(\rho) = \max_{0 \leq \|x\| \leq \rho} V(x)$$

Time Varying PDF and LPDF

Result: (a) A continuous function $V(t,x) : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$ is a lpdf if and only if there exists an lpdf $W : \mathbb{R}^n \to \mathbb{R}$ such that $V(t,x) \geq W(x)$ for all $t \geq 0$ and for all $x \in B_r(x)$.

(b) A continuous function $V(t,x) : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$ is a pdf if and only if there exists an pdf $W : \mathbb{R}^n \to \mathbb{R}$ such that $V(t,x) \geq W(x)$ for all $t \geq 0$ and for all $x \in \mathbb{R}^n$.

Result: A function $V(t,x)$ is lpdf (pdf) if and only if there exists a class-k function α such that $V(t,0) = 0$ and

$$V(t,x) \geq \alpha(\|x\|)$$

for all $t \geq 0$ and all $x \in B_r(x)$ (or whole state space).

e.g., $V(t,x) = e^t(x_1^2 + x_2^2)$ is pdf

e.g., $V(t,x) = e^{-t}(x_1^2 + x_2^2)$ is not pdf or lpdf

Decrescent Functions

Definition: The function $V(t,x)$ is decrescent if $V(t,0)=0$ and there exists a pdf $W(x)$ such that

$$V(t,x) \leq W(x)$$

for all $t \geq 0$.

Result: A function $V(t,x)$ is locally (or globally) decrescent if and only if there exists a class-k function β such that $V(t,0)=0$ and

$$V(t,x) \leq \beta(\|x\|)$$

for all $t \geq 0$ and all $x \in B_r(x)$ (or whole state space).

e.g., $V(t,x) = (1+\sin^2(t))(x_1^2 + x_2^2)$ is decrescent

Stability Theorems

Nonlinear System:

$$\frac{dx}{dt} = f(t, x(t)) \qquad (1)$$
$$x(t_0) = x_0$$

Theorem: (a) The equilibrium point $x_e = 0$ at time t_0 of (1) is stable if there exists a continuously differentiable lpdf V such that

$$\dot{V}(t, x) \leq 0$$

for all $t \geq t_0$ and all $x \in B_r(x)$.

(b) If additionally V is decrescent, then $x_e = 0$ is uniformly stable over $[t_0, \infty)$.

e.g., $\ddot{y} + \dot{y} + (2 + \sin t)y = 0$ Here $x_0 = 0$ is uniformly stable.

Stability Theorems

Theorem: (a) The equilibrium state $x_0 = 0$ of (1) is uniformly asymptotically stable over $[t_0, \infty)$ if there exists a decrescent lpdf V such that $-\dot{V}$ is an lpdf.

(b) The stability is global if there exists a decrescent pdf V such that

$$\dot{V}(t,x) \le -W(x) \quad \forall t \ge t_0 \,, \forall x \in \mathbb{R}^n$$

where $W(0) = 0, W(x) > 0$ for $x \neq 0$.

e.g.,
$$\dot{x}_1 = x_1(x_1^2 + x_2^2 - 1) - x_2$$
$$\dot{x}_2 = x_1 + x_2(x_1^2 + x_2^2 - 1)$$

Uniformly asymptotically stable but not global (numerical check)

Stability Theorems

Theorem: Let $V(t,x)$ with $V(t,0)=0$ be decrescent. Let $\dot{V}$ be lpdf and $V(t_0,x)>0$ where x is arbitrarily close to 0. Then $x_e=0$ at time t_0 is unstable.

e.g.,
$$\dot{x}_1 = x_1 - x_2 + x_1 x_2$$
$$\dot{x}_2 = -x_2 - x_2^2$$

Choose $V(x) = (2x_1 - x_2)^2 - x_2^2$

Lyapunov's Indirect Method

Autonomous System: Consider

$$\dot{x} = f(x)$$
$$x(0) = x_0$$

and the linearized system $\dot{x} = Ax$ where A is the Jacobian at $x_e = 0$.

Theorem: If f is continuously differentiable and if A is asymptotically stable, then 0 is a locally stable equilibrium point of $\dot{x} = f(x)$.

Lyapunov's Indirect Method

Nonautonomous System: Consider

$$\dot{x} = f(t,x)$$
$$x(0) = x_0$$

and linearized system $\dot{x} = A(t)x$ where $A(t)$ is the Jacobian at $x_e = 0$.

Theorem: Assume that f is continuously differentiable and

$$\lim_{||x||\to 0} \sup_{t\geq 0} \frac{||f_1(t,x)||}{||x||} = 0$$

$A(\cdot)$ is bounded

Then if the equilibrium point $x_e = 0$ of $\dot{x} = A(t)x$ is uniformly asymptotically over $[0,\infty)$, then the equilibrium $x_e = 0$ of $\dot{x} = f(t,x)$ is locally uniformly stable over $[0,\infty)$.