MA 574 — Final Project

Due: Wednesday, May 5

1. Consider the structure depicted in Figure 1 that is comprised of two active PVDF layers and an inactive polyimide layer. Bimorphs of this type are presently being investigated for applications ranging from flow control to artificial lens cleaners. We are going to model this structure using thin beam theory.

Let \( w \) and \( f \) respectively denote the transverse displacement and distributed out-of-plane force. The effective linear density (units of Kg/m), Young’s modulus, and Kelvin–Voigt damping coefficients for the composite structure are denoted by \( \rho, Y \) and \( c \) whereas material properties for constituent components are delineated by subscripts. The geometric and material properties for the active PVDF layers and inactive polyimide layer are respectively delineated by the subscripts \( A \) and \( I \). Both layers are assumed to have width \( b \) and the bimorph is assumed to have length \( \ell \). Finally, we assume fixed-end conditions at \( x = 0 \) and free-end conditions at \( x = \ell \).

For linear inputs, we employ the constitutive relation

\[
\sigma = \begin{cases} 
Y_A \varepsilon + c_A \dot{\varepsilon} - Y_A \frac{d_{31}}{h_{A1}} V, & \text{Active layer 1} \\
Y_A \varepsilon + c_A \dot{\varepsilon} - Y_A \frac{d_{31}}{h_{A2}} V, & \text{Active layer 2} \\
Y_I \varepsilon + c_I \dot{\varepsilon} & \text{Inactive layer}
\end{cases}
\]

where \( d_{31} \) is a piezoelectric coupling coefficient, \( \sigma \) is a stress, and \( \varepsilon \) is a strain. As illustrated for the stress profile depicted in Figure 1(b), the moment arm at height \( z \) in the bimorph has length \( z - z_n \). If we denote the total moment by

\[
M = M_e + M_d + M_{ext},
\]

where the subscripts \( e, d \) and \( ext \) indicate elastic, damping and external components, then

\[
M = \int_0^{h_{A1} + h_I + h_{A2}} b(z - z_n) \sigma \, dz.
\]

The following questions illustrate aspects of the model construction. You can use Sections 7.4.1 and 7.9 of Smith as references.

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Figure 1: (a) Asymmetric polymer bimorph comprised of two active PVDF layers and an inactive polyimide layer. (b) Geometry used to compute the neutral line \( z_n \).
(a) To specify the neutral line \( z_n \), assume a linear strain profile \( \varepsilon(z) = \kappa(z - z_n) \) where

\[
\kappa = -\frac{\partial^2 w}{\partial x^2}
\]

quantifies the change in curvature due to bending. It is easiest if you take the origin at the base of the structure as depicted in Figure 1(a). Use force balancing to determine a relation for \( z_n \) in terms of \( h_{A1}, h_{A2}, h_I, Y_I \) and \( Y_A \). Does your relation make sense when \( h_{A1} = h_{A2} \)?

(b) Compute the Young’s modulus \( Y \) in terms of \( z_n, h_I, h_{A1}, h_{A2}, Y_I \) and \( Y_A \). Use the fact that

\[
M_e = -\int_0^{h_{A1}+h_I+h_{A2}} bY \frac{\partial^2 w}{\partial x^2} (z - z_n)^2 \, dz
\]

to specify the moment of inertia \( I \).

(c) We will take

\[
M_d = -cI \frac{\partial^3 w}{\partial x^2 \partial t}
\]

so you do not need to do anything here.

(d) Use the constitutive relation (1) to determine the constant \( k_p \) in the external moment relation

\[
M_{ext} = k_p V(t).
\]

(e) Determine an expression for the composite density \( \rho \) in terms of \( \rho_A, \rho_I, h_{A1}, h_{A2}, h_I \) and \( b \).

(f) Determine strong and weak forms of an Euler–Bernoulli beam model to characterize transverse displacements \( w \) of the structure. Be sure to specify appropriate boundary conditions.