(1) A string having mass per unit length $\rho$ and length $L$ is whirled about one end, with angular velocity $\omega$, so that the motion is in a plane (neglect gravity). Using the property that the centripetal force exerted on a mass $m$ moving in a circle of radius $r$ with angular velocity $\omega$ is $F = mr\omega^2$, show that the tension in the string is

$$T(x) = \frac{1}{2} \rho \omega^2 (L^2 - x^2),$$

where $x$ is the distance from the stationary end. Show that the motion of the string is modeled by the differential equation

$$\rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x} \left( T \frac{\partial u}{\partial x} \right).$$

Specify appropriate boundary conditions.

(2) Here we are going to model the motion of a jump rope of length $L$, having mass per unit length $\rho$, that is rotating at constant angular velocity $\omega$. You can assume that it is fixed at both ends and that the tension $T$ is uniform throughout the rope. You can also neglect the effects of gravity. The out-of-plane displacement of the rope is denoted by $u(t, x)$. Finally, you can use the fact that the centripetal acceleration due to the angular velocity is $a = r\omega^2 \approx u\omega^2$.

(a) Show that the motion of the rope is modeled by the differential equation

$$\rho u\omega^2 + T \frac{d^2 u}{dx^2} = 0.$$  \(1\)

(b) Determine the solution to (1). Are there restrictions on $\omega$?

(3) Consider a rectangular membrane of length $L$ and width $a$ that is fixed along all four edges. You can assume that the tension $T$ is uniform at all points in the membrane and that the density (kg/m$^2$) is $\rho$. Determine the kinetic and potential energy of the membrane and use Lagrangian principles to develop a weak formulation of the model. Use integration by parts to verify that if weak solutions are sufficiently smooth, then the weak and stong formulations are equivalent.