Acoustic Models

“It's easy to play any musical instrument: all you have to do is touch the right key at the right time and the instrument will play itself,” J.S. Bach
Motivating Applications

Jet Noise Reduction:

• Employ chevrons to improve mixing and decrease jet noise

• Boeing experiments demonstrate 4 dB noise reduction

• 3 dB reduction if 1 of 2 engines turned off

• Influences aeroacoustic coupling and requires 2-D and 3-D SMA models

Motivating Applications

Duct Noise: Use secondary sources or structure-borne actuators to reduce duct or fan noise

Noise Canceling Headphones: e.g., cockpit noise for pilots is a significant problem for both health and communication
Motivating Applications

**Improved Instrument and Amplifier Design:** What makes a Stradivarius unique?

**Remote Acoustic Sensing:** Can we listen to a conversation using a laser?
Motivating Applications

Seismology and Acoustic Medical Imaging: e.g., ultrasound
Motivating Applications

Medical Treatment Strategies:

• Ultrasound to break up kidney stones
• Acoustic barriers for EM treatment
• Ultrasonic dental tools
Motivating Applications

SAW (Surface Acoustic Wave) Devices:

- Used in filters, oscillators and transformers; e.g., bandpass filters in cell phones
- Relies on transduction capabilities of PZT

![Experimental image of SAW on a crystal of tellurium oxide.](image)

![Diagram illustrating RF Filter performance characteristics.](diagram)
Basic Definitions

**Acoustics:** The science of sound including its production, transmission, and effects.

**Sound:** A traveling wave created by a vibrating object and propagated through a medium (gas, liquid, or solid) due to particle interactions.

- Because it is due to particle interactions, it is a mechanical wave. Thus sound cannot travel through a vacuum --- “In space, no one can hear you scream,” *Alien* 1979
- Particle interactions yield oscillations in pressure resulting in local regions of compression and rarefaction.

**Noise:** Unwanted sound.
Sound Pressure and Acoustic Units

**Acoustic or Sound Pressure:** Difference between average local pressure of medium and pressure within sound wave at same point and time.

- Think about difference between changing pressure in an airplane (medium) and conversation (sound wave).

**Instantaneous Sound Pressure:** $p(t)$  
Units: Pascals ($\text{N/m}^2$)

**RMS Sound Pressure:**

$$p_{rms} = \sqrt{\frac{1}{t_f - t_0} \int_{t_0}^{t_f} [p(t)]^2 \, dt}$$  
Continuous Time

$$p_{rms} = \sqrt{\frac{1}{N_t} \sum_{i=1}^{N_t} [p(t_i)]^2}$$  
Discrete Time
Sound Pressure Levels

**Sound Pressure Level:** Employs logarithmic scale

\[ SPL = 10 \log_{10} \left( \frac{p_{rms}^2}{p_{ref}^2} \right) = 20 \log_{10} \left( \frac{p_{rms}}{p_{ref}} \right) \]

- **Air:** \( p_{ref} = 20 \mu Pa = 2 \times 10^{-5} \text{ N/m}^2 \)
- **Water:** \( p_{ref} = 1 \mu Pa \)

**Nondimensional Units:** Decibels (dB)
# Examples of Sound Pressure and Sound Pressure Levels

<table>
<thead>
<tr>
<th>Sound Source</th>
<th>RMS Sound Pressure</th>
<th>Sound Pressure Level</th>
</tr>
</thead>
<tbody>
<tr>
<td>Theoretical limit for undistorted sound</td>
<td>101,325 Pa</td>
<td>191 dB re 20 microPA</td>
</tr>
<tr>
<td>1883 Krakatoa eruption</td>
<td>Approx 180 at 100 miles</td>
<td></td>
</tr>
<tr>
<td>Stun grenades</td>
<td>170-180 dB re 20 microPA</td>
<td></td>
</tr>
<tr>
<td>Rocket launch</td>
<td>Approx 165 dB re 20 microPA</td>
<td></td>
</tr>
<tr>
<td>Hearing damage (short term exposure)</td>
<td>20 dB re 20 microPA</td>
<td>Approx 120 dB re 20 microPA</td>
</tr>
<tr>
<td>Jet engine at 100 m</td>
<td>6-200 dB re 20 microPA</td>
<td>110-140 dB re 20 microPA</td>
</tr>
<tr>
<td>Jackhammer/Disco</td>
<td>2 dB re 20 microPA</td>
<td>Approx 100 dB re 20 microPA</td>
</tr>
<tr>
<td>Hearing damage (long term exposure)</td>
<td>0.6 dB re 20 microPA</td>
<td>Approx 85 dB re 20 microPA</td>
</tr>
<tr>
<td>Normal Talking</td>
<td>0.002-0.02 dB re 20 microPA</td>
<td>40-60 dB re 20 microPA</td>
</tr>
<tr>
<td>Quiet rustling leaves</td>
<td>0.00006 dB re 20 microPA</td>
<td>10 dB re 20 microPA</td>
</tr>
</tbody>
</table>
Acoustic Sources

Direct Source:

Note: Nature of sources affects control strategies including choice of actuators and sensors.

Structural-Acoustic Coupling:

Aeroacoustic Coupling:
Physical Phenomena to be Modeled

Recall: Particle interactions yield oscillations in pressure resulting in local regions of compression and rarefaction.

Mechanisms:

- Particles move: Conservation of mass
- Particles transmit energy (momentum): Conservation of momentum
Conservation of Mass

Recall: (see Mass Conservation and Compartmental Analysis Notes)

1-D: Continuity Equation

\[ \frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0 \]

Fick’s law with transport

\[ q = -DA \frac{\partial \rho}{\partial x} + \rho A v \]

3-D: No diffusion

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0 \]
Conservation of Momentum: Lagrangian Description

Newton’s Second Law: \[ F = \frac{d}{dt}(mu) \]

Momentum of moving control volume at time \( t \):
\[
\int_{a(t)}^{b(t)} A\rho(t,x)u(t,x)\,dx
\]
\[
\Rightarrow \frac{d}{dt} \int_{a(t)}^{b(t)} A\rho(t,x)u(t,x)\,dx = A\rho(t,a(t)) - A\rho(t,b(t))
\]

(Leibniz Rule)
\[
\Rightarrow \rho(t,b)u(t,b)b'(t) - \rho(t,a)u(t,a)a'(t) + \int_{a(t)}^{b(t)} \frac{\partial(\rho u)}{\partial t} \,dx = p(t,a) - p(t,b)
\]
\[
\Rightarrow \rho(t,b)u^2(t,b) - \rho(t,a)u^2(t,a) + \int_{a(t)}^{b(t)} \frac{\partial(\rho u)}{\partial t} \,dx = p(t,a) - p(t,b)
\]

Note:
\[
\rho(t,b)u^2(t,b) = \rho(t,a)u^2(t,a) + \frac{\partial(\rho u^2)}{\partial x}(t,a)L + \mathcal{O}(L^2)
\]
\[
p(t,b) = p(t,a) + \frac{\partial p}{\partial x}(t,a)L + \mathcal{O}(L^2)
\]
\[
\int_{a(t)}^{b(t)} \frac{\partial(\rho u)}{\partial t} \,dx = \frac{\partial(\rho u)}{\partial t}(t,a)L + \mathcal{O}(L^2)
\]
Conservation of Momentum

Thus

$$\frac{\partial (\rho u^2)}{\partial x}(t, a)L + \frac{\partial (\rho u)}{\partial t}(t, a)L = -\frac{\partial p}{\partial x}(t, a)L + O(L^2)$$

$$\Rightarrow \frac{\partial (\rho u^2)}{\partial x} + \frac{\partial (\rho u)}{\partial t} = -\frac{\partial p}{\partial x} \quad (\ast)$$

Use continuity equation (conservation of mass)

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} = 0$$

and the product rule to simplify $(\ast)$:

$$u \frac{\partial (\rho u)}{\partial x} + \rho u \frac{\partial u}{\partial x} + u \frac{\partial \rho}{\partial t} + \rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x}$$

$$\Rightarrow \rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} \right) = -\frac{\partial p}{\partial x}$$

$$\Rightarrow \rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x}$$

Euler’s Equation
Conservation of Momentum: Eulerian Description

3-D Eulerian Description: Consider stationary volume

Principle:

\[
\begin{align*}
\{\text{Rate of momentum accumulation}\} \\
= \{\text{Rate of momentum in}\} \\
- \{\text{Rate of momentum out}\} \\
+ \{\text{Sum of forces acting on system}\}
\end{align*}
\]

Momentum Change on Face \(x\):

\[
(\text{Rate of mass in}) \cdot u \bigg|_x = \rho u^2 \bigg|_x \Delta y \Delta z
\]

Momentum Change on Face \(x + \Delta x\):

\[
(\text{Rate of mass out}) \cdot u \bigg|_{x+\Delta x} = \rho u^2 \bigg|_{x+\Delta x} \Delta y \Delta z
\]
Conservation of Momentum: Eulerian Description

Momentum Balance: x-direction

\[ \frac{\partial (\rho u)}{\partial t} \Delta x \Delta y \Delta z = \left[ \rho u^2 \bigg|_x - \rho u^2 \bigg|_{x+\Delta x} \right] \Delta y \Delta z \]
\[ + \left[ \rho vu \bigg|_y - \rho vu \bigg|_{y+\Delta y} \right] \Delta x \Delta z \]
\[ + \left[ \rho wu \bigg|_z - \rho wu \bigg|_{z+\Delta z} \right] \Delta x \Delta y \]
\[ + \Delta y \Delta z \left[ p \bigg|_x - p \bigg|_{x+\Delta x} \right] \]

Momentum: x-component

\[ \frac{\partial (\rho u)}{\partial t} = - \left[ \frac{\partial}{\partial x} (\rho u^2) + \frac{\partial}{\partial y} (\rho uv) + \frac{\partial}{\partial z} (\rho uw) \right] - \frac{\partial p}{\partial x} \]

Momentum: y-component

\[ \frac{\partial (\rho v)}{\partial t} = - \left[ \frac{\partial}{\partial x} (\rho uv) + \frac{\partial}{\partial y} (\rho v^2) + \frac{\partial}{\partial z} (\rho vw) \right] - \frac{\partial p}{\partial y} \]

Momentum: z-component

\[ \frac{\partial (\rho w)}{\partial t} = - \left[ \frac{\partial}{\partial x} (\rho uw) + \frac{\partial}{\partial y} (\rho vw) + \frac{\partial}{\partial z} (\rho w^2) \right] - \frac{\partial p}{\partial z} \]
Conservation of Momentum

Note: Combination with the continuity equation yields

$$\rho \left[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right] = -\frac{\partial p}{\partial x}$$

$$\Rightarrow \rho \left[ \frac{\partial u}{\partial t} + \vec{u} \cdot \nabla u \right] = -\frac{\partial p}{\partial x}$$

Euler’s Equations:

$$\rho \left[ \frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla) \vec{u} \right] = -\nabla p$$

$$\Rightarrow \rho \frac{D\vec{u}}{Dt} = -\nabla p$$
Equation of State

Conservation of Mass and Momentum:

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0
\]

\[
\rho \left[ \frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla) \vec{u} \right] = -\nabla p
\]

or

\[
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} = 0
\]

\[
\rho \left[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} \right] = -\frac{\partial p}{\partial x}
\]

Note: Need additional relation (constraint)

Barotropic Fluid: Pressure is a function only of density

\[ p = f(\rho) \]

Ideal Gas: \[ p = \rho RT \], \( R \) is ideal gas constant
Models for Sound Waves

Nonlinear System:

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0
\]

\[
\rho \left[ \frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla) \vec{u} \right] = -\nabla p
\]

\[p = f(\rho)\]

Assumption: Sound is perturbations of pressure and density about static or slowly time-varying state.

Static Quantities:

\[\rho_0(r) : \text{Static density}\]

\[p_0 : \text{Static pressure}\]

\[\vec{u}_0 : \text{Static velocity; } \vec{u} = 0 \Rightarrow p_0 \text{ constant in space}\]

\[\rho_0(r) \text{ constant in time}\]
Models for Sound Waves

Dynamic Perturbations:

\[ \rho(t, r) = \rho_0(r) + \hat{\rho}(t, r) \]
\[ p(t, r) = p_0 + \hat{p}(t, r) \]
\[ \vec{u}(t, r) = \hat{\vec{u}}(t, r) \]

**Strategy:** Linearize about steady, silent case

**Euler’s Equation:** Momentum

\[ (\rho_0 + \hat{\rho}) \frac{\partial \hat{\vec{u}}}{\partial t} + (\rho_0 + \hat{\rho})(\hat{\vec{u}} \cdot \nabla)\hat{\vec{u}} = -\nabla (p_0 + \hat{p}) \]

\[ \Rightarrow \rho_0 \frac{\partial \hat{\vec{u}}}{\partial t} + \hat{\rho} \frac{\partial \hat{\vec{u}}}{\partial t} + (\rho_0 + \hat{\rho})(\hat{\vec{u}} \cdot \nabla)\hat{\vec{u}} = -\nabla \hat{p} \]

\[ \Rightarrow \rho_0 \frac{\partial \hat{\vec{u}}}{\partial t} = -\nabla \hat{p} \]
Models for Sound Waves

Continuity Equation: Mass

\[
\frac{\partial}{\partial t}(\rho_0 + \hat{\rho}) + \nabla \cdot ((\rho_0 + \hat{\rho})\hat{u}) = 0
\]

\[
\Rightarrow \frac{\partial \hat{\rho}}{\partial t} + \nabla \cdot (\rho_0 \hat{u}) + \nabla \cdot (\hat{\rho} \hat{u}) = 0
\]

\[
\Rightarrow \frac{\partial \hat{\rho}}{\partial t} + \nabla \cdot (\rho_0 \hat{u}) = 0
\]

Equation of State:

\[
p = f(\rho)
\]

\[
= f(\rho_0 + \hat{\rho})
\]

\[
= f(\rho_0) + f'(\rho_0)\hat{\rho} + \text{H.O.T.}
\]

\[
= p_0 + f'(\rho_0)\hat{\rho} + \text{H.O.T.}
\]

First-Order Relation:

\[
\hat{p} = c^2 \hat{\rho}
\]

where \(c^2 \equiv f'(\rho_0) = \frac{\partial p}{\partial \rho} \bigg|_{\rho_0}\) is the speed of sound in the material.
Wave Equation in Pressure

First-Order Equations for Sound:

\[ \rho_0 \frac{\partial \hat{u}}{\partial t} = -\nabla \hat{p} \quad \text{(Euler)} \]
\[ \frac{\partial \hat{p}}{\partial t} + \nabla \cdot (\rho_0 \hat{u}) = 0 \quad \text{(Continuity)} \]
\[ \hat{p} = c^2 \hat{\rho} \quad \text{(State)} \]

Wave Equation in Pressure:

\[ \frac{\partial}{\partial t} \left( \frac{\hat{p}}{c^2} \right) = -\nabla \cdot (\rho_0 \hat{u}) \quad \text{(State into Continuity)} \]
\[ \Rightarrow \frac{1}{c^2} \frac{\partial^2 \hat{p}}{\partial t^2} = -\nabla \cdot \left( \rho_0 \frac{\partial \hat{u}}{\partial t} \right) \quad \text{(Differentiate wrt time)} \]
\[ \Rightarrow \frac{1}{c^2} \frac{\partial^2 \hat{p}}{\partial t^2} = -\nabla \cdot (-\nabla \hat{p}) \quad \text{(Substitute Euler)} \]

\[ \Rightarrow \frac{\partial^2 \hat{p}}{\partial t^2} = c^2 \Delta \hat{p} \]

Advantages: 1-D equation, no additional assumptions
Disadvantages: \( \hat{p} \) may or may not be easy to measure, may not facilitate multiphysics coupling
Wave Equation in Velocity

Wave Equation in Velocity: Note that

\[
\rho_0 \frac{\partial^2 \hat{u}}{\partial t^2} = -\nabla \hat{p}_t \quad \text{(Differentiate Euler)}
\]

and

\[
\hat{p}_t = c^2 \hat{p}_t \quad \text{(Differentiate state)}
\]

\[
= -c^2 \nabla \cdot (\rho_0 \hat{u}) \quad \text{(Continuity)}
\]

Thus

\[
\rho_0 \frac{\partial^2 \hat{u}}{\partial t^2} = \nabla \left[ c^2 \nabla \cdot (\rho_0 \hat{u}) \right] \quad \text{Note: Holds for variable } \rho_0(r)
\]

Additional Assumption: \( \rho_0 \text{ constant} \)

\[
\frac{\partial^2 \hat{u}}{\partial t^2} = c^2 \Delta \hat{u} + c^2 \nabla \times (\nabla \times \hat{u})
\]

\[
\nabla \times (\nabla \times \vec{w}) = \nabla [\nabla \cdot \vec{w}] - \Delta \vec{w}
\]

Irrotational Flow:

\[
\frac{\partial^2 \hat{u}}{\partial t^2} = c^2 \Delta \hat{u}
\]

Advantage: May be able to directly measure velocity

Disadvantage: Requires 3 variables, added assumption of constant \( \rho_0 \)
Wave Equation in Potential

Initial Assumption: $\rho_0$ constant

The curl of Euler’s equation yields

$$\nabla \times \left( \rho_0 \frac{\partial \hat{u}}{\partial t} \right) = -\nabla \times (\nabla \hat{p})$$

$$\Rightarrow \frac{\partial}{\partial t} (\nabla \times \hat{u}) = 0 \quad \text{since } \nabla \times (\nabla f) = 0 , \ \rho_0 \text{ constant}$$

Initial Vorticity: Suppose irrotational

$$\nabla \times \hat{u} \bigg|_{t=0} = 0$$

$$\Rightarrow \nabla \times \hat{u} = 0 \text{ for all time}$$

$$\Rightarrow \hat{u} \text{ is a conservative field}$$

$$\Rightarrow \text{There exists a nonunique scalar potential } \phi \text{ such that } \hat{u} = -\nabla \phi$$

Thus

$$\rho_0 \nabla \phi_t = -\nabla \hat{p} \quad \text{(Euler’s equation)}$$

$$\Rightarrow \nabla (\rho_0 \phi_t - \hat{p}) = 0$$

$$\Rightarrow \rho_0 \phi_t - \hat{p} = -k(t)$$
Wave Equation in Potential

WLOG: Take $k \equiv 0$. To see why, define new potential

$$\tilde{\phi} = \phi + \frac{1}{\rho_0} \int_0^t k(s)\,ds$$

$$\Rightarrow \nabla \tilde{\phi} = \nabla \phi = -\hat{u}$$

and

$$\rho_0 \tilde{\phi}_t = \rho_0 \phi_t + k(t) = \hat{p}$$

WLOG: Take $\hat{p} = \rho_0 \phi_t$. Thus

$$\frac{\partial \hat{p}}{\partial t} = -\nabla \cdot (\rho_0 \hat{u}) \quad \text{(Continuity equation)}$$

$$\Rightarrow \frac{\partial}{\partial t} \left( \frac{\hat{p}}{c^2} \right) = \nabla (\rho_0 \nabla \phi) \quad \text{(State equation)}$$

$$\Rightarrow \frac{\partial}{\partial t} \left( \frac{\rho_0 \phi_t}{c^2} \right) = \rho_0 \Delta \phi$$

$$\Rightarrow \phi_{tt} = c^2 \Delta \phi$$

Advantages: Relations $\hat{u} = -\nabla \phi$ and $\hat{p} = \rho_0 \phi_t$ can facilitate coupling with structural systems

Disadvantage: $\phi$ is difficult to measure directly
Wave Equation in Potential

Nonconstant Density: $\rho_0$ not constant in space


$$\Phi_{tt} = c^2 \Delta \Phi$$
Linear Wave Equations for Sound

**Pressure:**
\[ \frac{\partial^2 \hat{p}}{\partial t^2} = c^2 \Delta \hat{p} \]
Boundary Conditions
Initial Conditions

**Velocity:**
\[ \frac{\partial^2 \hat{u}}{\partial t^2} = c^2 \Delta \hat{u} \]
Boundary Conditions
Initial Conditions

**Potential:**
\[ \frac{\partial^2 \phi}{\partial t^2} = c^2 \Delta \phi \]
Boundary Conditions
Initial Conditions
Music

**Pitch:** This is simply the frequency of pure tones; e.g., Middle C is about 261.6 Hz as calculated in relation to A above Middle C which as a frequency of 440 Hz.

![Piano keyboard diagram](image)

**Note:** Going up an octave doubles the frequency. This is easily observed on a guitar where pressing at the twelfth fret cuts the string in half which doubles the frequency and raises the tone an octave.
Music

Note: Going up a fifth (e.g., C and G) causes frequencies to align every third interval for the G --- e.g., First two notes of *Star Wars* theme.

Reason: Equal tempering which spaces all 12 notes equally

- Each semitone is thus $\frac{12}{\sqrt{2}} \approx 1.059463$
- C sharp thus has a frequency of $261.6 \cdot 1.059463$
- G has a frequency of $261.6 \cdot (1.059463^7) = 261.6 \cdot 1.498307 \approx 261.6 \cdot (3/2)$
- A: $261.6 \cdot (1.059463^9) = 440$
Music

Note: Include a 3rd (E) and a fifth (G) -- C major chord

Note: Raising the key causes the confluence to occur more quickly.
Music

Note: The wavelengths of notes is related to their frequency by the expression

$$\lambda = \frac{c}{f}$$

where $c = 343 \text{ m/s}$ for air.

Example: The wavelength of Middle C is thus 1.31 m.