MA 573 — PROJECT 3

Due: Wednesday, March 19

1. Consider a thin curved beam having width b, thickness h, and constant radius of curvature R as depicted in Figure 1. The circumferential and transverse displacements are denoted by v and w. The density, Young's modulus, and Kelvin–Voigt damping coefficients are respectively denoted by ρ, Y and c. You can consider the beam to be fully clamped at both ends.

   Consider first the case with no damping so that $c = 0$. Derive appropriate expressions for the kinetic and potential energies and use these relations, in combination with Hamilton's principle to derive a weak formulation of the curved beam model. Be sure to specify your space of test functions. Show that if the solutions have sufficient smoothness, the weak formulation is equivalent to the strong formulation detailed in Section 7.7.3 of the book by Smith.

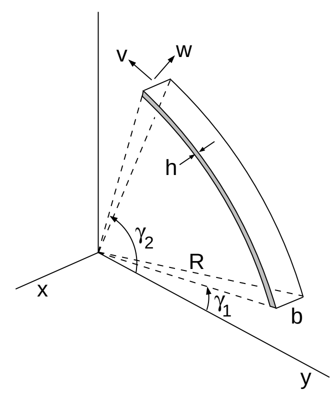
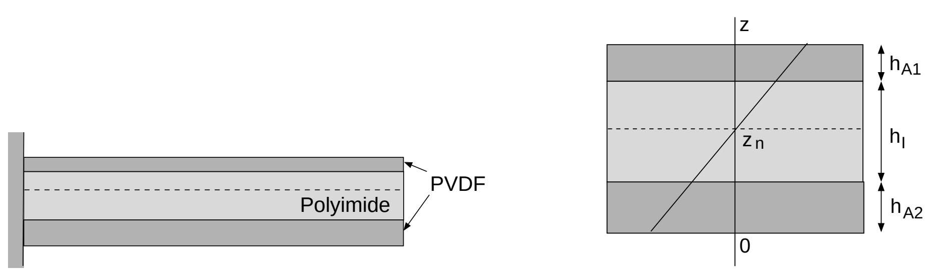


Figure 1: Curved beam in which circumferential and transverse motion are coupled due to curvature.

2. Consider the structure depicted in Figure 2 that is comprised of two active PVDF layers and an inactive polyimide layer. Bimorphs of this type are presently being investigated for applications ranging from flow control to artificial lens cleaners. We are going to model this structure using thin beam theory.

   Let w and f respectively denote the transverse displacement and distributed out-of-plane force. The effective linear density (units of Kg/m), Young's modulus, and Kelvin–Voigt damping coefficients for the composite structure are denoted by ρ, Y and c whereas material properties for constituent components are delineated by subscripts. The geometric and material



(a) (b)

Figure 2: (a) Asymmetric polymer bimorph comprised of two active PVDF layers and an inactive polyimide layer. (b) Geometry used to compute the neutral line z_n.

properties for the active PVDF layers and inactive polyimide layer are respectively delineated by the subscripts A and I. Both layers are assumed to have width b and the bimorph is assumed to have length ℓ. Finally, we assume fixed-end conditions at $x = 0$ and free-end conditions at $x = \ell$.

For linear inputs, we employ the constitutive relation

$$\sigma = \begin{cases} Y_A\varepsilon + c_A\dot{\varepsilon} - Y_A\dfrac{d_{31}}{h_{A1}}V & \text{, Active layer 1} \\ Y_A\varepsilon + c_A\dot{\varepsilon} - Y_A\dfrac{d_{31}}{h_{A2}}V & \text{, Active layer 2} \\ Y_I\varepsilon + c_I\dot{\varepsilon} & \text{, Inactive layer} \end{cases} \tag{1}$$

where d_{31} is a piezoelectric coupling coefficient, σ is a stress, and ε is a strain. As illustrated for the stress profile depicted in Figure 2(b), the moment arm at height z in the bimorph has length $z - z_n$. If we denote the total moment by

$$M = M_e + M_d + M_{ext},$$

where the subscripts e, d and ext indicate elastic, damping and external components, then

$$M = \int_0^{h_{A1}+h_I+h_{A2}} b(z - z_n)\sigma\, dz.$$

The following questions illustrate aspects of the model construction. You can use Sections 7.4.1 and 7.9 of Smith as references.

(a) To specify the neutral line z_n, assume a linear strain profile $\varepsilon(z) = \kappa(z - z_n)$ where

$$\kappa = -\frac{\partial^2 w}{\partial x^2}$$

quantifies the change in curvature due to bending. It is easiest if you take the origin at the base of the structure as depicted in Figure 2(a). Use force balancing to determine a relation for z_n in terms of h_{A1}, h_{A2}, h_I, Y_I and Y_A. Does your relation make sense when $h_{A1} = h_{A2}$?

(b) Compute the Young's modulus Y in terms of $z_n, h_I, h_{A1}, h_{A2}, Y_I$ and Y_A. Use the fact that

$$M_e = -\int_0^{h_{A1}+h_I+h_{A2}} bY\frac{\partial^2 w}{\partial x^2}(z - z_n)^2\, dz$$

to specify the moment of inertia I.

(c) We will take

$$M_d = -cI\frac{\partial^3 w}{\partial x^2 \partial t}$$

so you do not need to do anything here.

(d) Use the constitutive relation (1) to determine the constant k_p in the external moment relation

$$M_{ext} = k_p V(t).$$

(e) Determine an expression for the composite density ρ in terms of $\rho_A, \rho_I, h_{A1}, h_{A2}, h_I$ and b.

(f) Determine strong and weak forms of an Euler–Bernoulli beam model to characterize transverse displacements w of the structure. Be sure to specify appropriate boundary conditions.