Heat Conduction and the Heat Equation

“If you can’t take the heat, don’t tickle the dragon.”
Heat Transfer

Note: Energy is the conserved quantity

Conduction:

• Heat transfer due to molecular activity. Energy is transferred from more energetic to less energetic particles due to energy gradient
• Occurs in solids, fluids and gases
• Empirical relation: Fourier’s law

Convection:

• Energy transfer in fluid or gas due to bulk or macroscopic motion (advection)
• Convection: Advection + conduction
• Empirical relation: Newton’s law of cooling

Thermal Radiation:

• Energy emitted by matter due to changes in electron configurations that results in changes in energy via EM waves or photons
• Empirical relation: Stefan-Boltzmann law
Heat Conduction

1-D Assumptions:

- Temperature uniform over cross-sections
- Heat transfer is by conduction
- Heat transfer only along x-axis
- No heat escapes from sides (perfect insulation)

Relevant Quantities:

- $u(t, x)$: Temperature ($^\circ$C or K)
- $H(t, x)$: Amount of Heat (energy) – Units: Calories or Joules
  
  Note: 1 calorie = heat required to raise 1 g water 1 $^\circ$C
  
  $1 \text{ J} = 0.23885 \text{ cal}$
  $1 \text{ cal} = 4.19 \text{ J}$

  Note: $H = c_p m u$

- $c_p$: Specific heat – Units: $\frac{\text{J}}{\text{kg} \cdot \text{K}}$, e.g., Aluminum versus iron
- $m$: Mass, e.g., Consider 1 g Fe at 100 $^\circ$C in 10 g water versus 10 g Fe at 100 $^\circ$C in 10 g water
Heat Conduction

Thermal Energy Density: Units: J/m$^3$ or cal/m$^3$

$$\rho_{th}(t, x) = H(t, x)/m^3 = c_p \rho(t, x) u(t, x)$$

Rate of Heat Transfer:

$$q(t, x): \text{Units: Watts (power)} \text{ where } 1 \text{ W} = 1 \text{ J/s}$$

Flux:

$$\mu(t, x) = q(t, x)/A$$

Conservation of Energy: See mass conservation with constant $A$

$$\frac{\partial \rho_{th}}{\partial t} + \frac{\partial \mu}{\partial x} = 0$$

$$\Rightarrow c_p \rho \frac{\partial u}{\partial t} = -\frac{\partial \mu}{\partial x}$$

if $c_p$ and $\rho$ are constant

Note: $c_p \rho$: Volumetric heat capacity
(ability of material to store heat)
1-D Heat Equation

Constitutive Relation:

\[ q = k A \frac{u(t, x) - u(t, x + \Delta x)}{\Delta x} \]

\[ \Rightarrow q = -k A \frac{\partial u}{\partial x} \]

Fourier’s law of heat conduction

or

\[ \mu = -k \frac{\partial u}{\partial x} \]

Note: \( k \) provides measure of material’s ability to conduct heat

1-D Unforced Heat Equation:

\[ \frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} \]

Note: Thermal diffusivity \( \alpha = \frac{k}{\rho c_p} \) has units \( \frac{m^2}{s} \)
Boundary and Initial Conditions

Initial Condition:
\[ u(0, x) = \psi(x), \quad 0 < x < L \]

Dirichlet Boundary Condition: Specify temperature
\[ u(t, 0) = u_1(t), \quad u(t, L) = u_2(t) \]

Neumann Boundary Condition: Specify \( q(t, x) \) or \( \mu(t, x) \) at boundaries
  e.g., Perfectly insulated at \( x = 0 \) implies \( k \frac{\partial u}{\partial x}(t, 0) = 0 \)

Robin Boundary Condition:
  e.g., \( k \frac{\partial u}{\partial x}(t, L) + h u(t, L) = g(t) \)
Boundary Conditions

Robin Boundary Condition: Motivation

**Newton’s Law of Cooling:**

\[ q = h A (u_s - u_f) \]

- \( u_s \) – Temperature of solid
- \( u_f \) – Temperature of fluid or convective medium
- \( h \) – Convective heat transfer coefficient

<table>
<thead>
<tr>
<th>Mechanism</th>
<th>( h \ (W/m^2K) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Still air</td>
<td>2.8-23</td>
</tr>
<tr>
<td>Moving air</td>
<td>11.3-55</td>
</tr>
<tr>
<td>Moving water</td>
<td>280-17,000</td>
</tr>
<tr>
<td>Condensing steam</td>
<td>5,700-28,000</td>
</tr>
</tbody>
</table>

Note:

\[-\mu(t, L) = h[u_f - u(t, L)]\]

\[\Rightarrow k \frac{\partial u}{\partial x}(t, L) = h[u_f - u(t, L)]\]

\[\Rightarrow k \frac{\partial u}{\partial x}(t, L) + hu(t, L) = hu_f\]
3-D Heat Equation

Fourier’s Law:
\[ \mu = -k \nabla u \cdot \hat{n} \]

Conservation of Energy: Let \( f \) denote heat source or sink
\[
\frac{d}{dt} \int_V \rho_{th} dV = - \int_S \mu dS + \int_V f dV
\]
\[ \Rightarrow \int_V c_p \rho \frac{\partial u}{\partial t} = \int_S k \nabla u \cdot \hat{n} dS + \int_V f dV \]
\[ \Rightarrow \int_V \left( c_p \rho \frac{\partial u}{\partial t} - \nabla \cdot (k \nabla u) - f \right) dV = 0 \]
\[ \Rightarrow c_p \rho \frac{\partial u}{\partial t} = k \nabla^2 u + f \quad \text{if} \ k \text{ is constant} \]

Note: \[ \nabla^2 u = \Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \] in cartesian coordinates
Thermal-Based Damage Detection in Porous Materials

Current Research: H.T. Banks, Amanda Criner (NCSU), William Winfree (NASA LaRC)

Detail: CRSC Technical Report CRSC-TR08-11

Goal: Use active thermography to detect subsurface anomalies in porous materials; e.g., for aeronautic and aerospace structures

Homogenized Model:

\[
\frac{\partial u}{\partial t} - k \Delta u = 0 \quad \text{in} \quad \Omega \\
\frac{\partial u}{\partial \eta} \bigg|_{\bigcup_i \partial \Omega_i} = \frac{\partial u}{\partial \eta} \bigg|_{\bigcup_{i=1}^3 \omega_i} = u\big|_{\omega_5} = 0 \\
\frac{\partial u}{\partial \eta} \bigg|_{\omega_4} = S_0 \chi_{[t_0, t_s]}(t) \\
u(0, x) = u_0
\]