Math 573 — Exam 1

(1) In this problem, we are going to model 1-D foot traffic between classes. Pertinent variables and their units are defined as follows:

\( \rho(x, t) \) : Density or number of students per unit length of sidewalk \((\text{students/m})\)

\( q(x, t) \) : Rate at which students flow past \( x \) \((\text{students/s})\)

\( c \) : Speed at which students are moving \((\text{m/s})\)

![Figure 1: Orientation of foot traffic flow.](image)

(a) Consider the control ‘volume’ from \( x \) to \( x + dx \) depicted in Figure 1. Use the following steps to determine a PDE relating \( \rho \) and \( q \).

(i) Specify the number of students in the interval \( dx \) in terms of \( \rho \) as well as the rate at which this number is changing.

(ii) For flow from left to right, determine the rate at which students are entering the interval at \( x \), in terms of \( q \). At what rate are they leaving at \( x + dx \)?

(iii) Specify a modeling PDE in terms of \( \rho \) and \( q \).

(b) The PDE from (a) provides one equation in terms of the two unknowns \( \rho \) and \( q \). Determine a second, algebraic, expression specifying \( q \) in terms of \( c \) and \( \rho \). Note that you can determine this relation through either physical arguments or simply by checking the units.

(c) Combine your results from (a) and (b) to specify a PDE model for foot traffic flow in terms of \( \rho \) and \( c \).

(d) For heavy foot traffic flow, the speed \( c \) will probably not be constant and will more likely decrease as the density of students \( \rho \) increases. Determine an expression for \( c \) that has a maximum value of 1 when \( \rho = 0 \) and a minimum of \( c = 0 \) when \( \rho = 1 \). Rewrite your modeling PDE solely in terms of \( \rho \).

(e) Specify an initial condition \( \rho(x, 0) \) that is 0 for \( x < 0 \), 1/2 for \( x > 1 \), and linearly varying between the two for \( 0 \leq x \leq 1 \).

(2) In this problem, we will model the flow of fluid in a vessel that splits (e.g., a blood vessel) as depicted in Figure 2. You can assume that the flow is steady (no temporal variation) and has a density \( \rho \) that is constant in both space and time. The cross-sectional area and velocity at the inlet are \( A_{in}, v_{in} \) whereas the areas and velocities at the outlets are \( A_{out_1}, v_{out_1} \) and \( A_{out_2}, v_{out_2} \).

(a) Formulate the continuity equation that quantifies the conservation of mass.

(b) If \( v_{in} = 0.5 \text{ m/s} \) and \( A_{out_1} = A_{out_2} = \frac{1}{3} A_{in} \), what is \( v_{out} = v_{out_1} = v_{out_2} \)?

![Figure 2: Fluid flow in a vessel that splits.](image)

(3) State the divergence theorem and fundamental theorem of calculus. How are the two related?