Analytic Techniques for Advection-Diffusion Equations

“Furious activity is no substitute for understanding,” H.H. Williams
Properties of Fourier Series

Fourier Series: Consider the representation

\[ f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos \left( \frac{n\pi x}{L} \right) + b_n \sin \left( \frac{n\pi x}{L} \right) \right] \]

for a \(2L\) periodic, continuous function \(f(x)\). If the series converges uniformly, it is the Fourier series for \(f(x)\).

Note:

\[ \int_{-L}^{L} f(x) dx = a_0 L \]

\[ \int_{-L}^{L} \sin \left( \frac{m\pi x}{L} \right) \sin \left( \frac{n\pi x}{L} \right) dx = \int_{-L}^{L} \cos \left( \frac{m\pi x}{L} \right) \cos \left( \frac{n\pi x}{L} \right) dx = 0 \quad \text{for} \ m \neq n \]

\[ \int_{-L}^{L} \sin \left( \frac{m\pi x}{L} \right) \cos \left( \frac{n\pi x}{L} \right) dx = 0 \quad \text{for all} \ m, n \]

\[ \int_{-L}^{L} \sin^2 \left( \frac{m\pi x}{L} \right) dx = \cos^2 \left( \frac{n\pi x}{L} \right) dx = L \quad \text{for} \ n \geq 1 \]
Properties of Fourier Series

Fourier Coefficients:

\[ a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \left( \frac{n\pi x}{L} \right) \, dx \quad \text{for } n = 0, 1, 2, \ldots \]

\[ b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \left( \frac{n\pi x}{L} \right) \, dx \quad \text{for } n = 1, 2, \ldots \]

Theorem: Suppose that \( f \) and \( f' \) are piecewise continuous on the interval \(-L \leq x \leq L\). Further, suppose that \( f \) is defined outside the interval \(-L \leq x \leq L\) so that it is periodic with period \( 2L \). Then \( f \) has a Fourier series

\[ f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos \left( \frac{n\pi x}{L} \right) + b_n \sin \left( \frac{n\pi x}{L} \right) \right] \]

with the above coefficients. The Fourier series converges to \( f(x) \) at all points where \( f \) is continuous, and to \([f(x+) + f(x-)]/2\) at all points where \( f \) is discontinuous (see Boyce and DiPrima).
Analytic Solution Techniques: Diffusion Equation

Recall: General advection-diffusion equation

\[
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho \bar{u})}{\partial x} = \frac{\partial}{\partial x} \left( D \frac{\partial \rho}{\partial x} \right) + b - d
\]

Diffusion Equation: Consider \( \bar{u} = b = d = 0 \)

\[
\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x} \left( D \frac{\partial \rho}{\partial x} \right)
\]

\[\rho(t,0) = \rho_L(t), \quad \rho(t,L) = \rho_R(t)\]

\[\rho(0,x) = \rho_0(x)\]

Simplifying Assumption: \( D \) constant, \( \rho_L(t) = \rho_R(t) = 0 \)
Analytic Solution Techniques: Diffusion Equation

Separation of Variables: Consider solutions of the form
\[ \rho(t, x) = X(x)T(t) \]

\[ \Rightarrow X(x)\dot{T}(t) = DX''(x)T(t) \]

\[ \Rightarrow \frac{X''(x)}{X(x)} = \frac{\dot{T}(t)}{DT(t)} = c \]

Hence
\[ X''(x) - cX(x) = 0 \quad \text{and} \quad \dot{T}(t) = cDT(t) \]
\[ X(0) = X(L) = 0 \quad \Rightarrow T(t) = \alpha e^{cDt} \]

Note:
\[ \int_0^L [XX'' - cX^2] \, dx = - \int_0^L \left[ -(X')^2 - cX^2 \right] \, dx = 0 \]

If \( c \geq 0 \), this implies that \( X(x) = k = 0 \). Thus \( c < 0 \) so take \( c = -\lambda^2, \lambda > 0 \).
Analytic Solution Techniques: Diffusion Equation

Boundary Value Problem:

\[ X''(x) + \lambda^2 X(x) = 0 \]
\[ X(0) = X(L) = 0 \]

Solution: \[ X(x) = A \cos(\lambda x) + B \sin(\lambda x) \]
\[ X(0) = 0 \Rightarrow A = 0 \]
\[ X(L) = 0 \Rightarrow \lambda L = n\pi \]

Thus
\[ X_n(x) = B_n \sin(\lambda_n x) \quad , \quad \lambda_n = \frac{n\pi}{L} \quad , \quad B_n \neq 0 \]

PDE Solution:

\[ \rho(t, x) = \sum_{n=1}^{\infty} \alpha_n e^{-D\lambda_n^2 t} \sin(\lambda_n x) \]
Analytic Solution Techniques: Diffusion Equation

Initial Condition:

\[ \rho(0, x) = \sum_{n=1}^{\infty} \alpha_n \sin(\lambda_n x) = \rho_0(x) \]

\[ \Rightarrow \alpha_n = \frac{2}{L} \int_{0}^{L} \rho_0(x) \sin(\lambda_n x) dx \]

Example: \( \rho_0(x) = \sin(\pi x/L) \)

\[ \alpha_n = \frac{2}{L} \int_{0}^{L} \sin \left( \frac{\pi x}{L} \right) \sin \left( \frac{n\pi x}{L} \right) dx \]

\[ = \begin{cases} 
\frac{2}{L} \cdot \frac{L}{2} & , \quad n = 1 \\
0 & , \quad n \neq 1 
\end{cases} \]

Solution

\[ \rho(t, x) = e^{-D(\pi/L)^2 t} \sin(\pi x/L) \]
Analytic Solution Techniques: Diffusion Equation

Forced Response:

\[ \rho_t = D \rho_{xx} + f(t, x) \]
\[ \rho(t, 0) = \rho(t, L) = 0 \]
\[ \rho(0, x) = \rho_0(x) \]

Consider solution of the form (variation of parameters)

\[ \rho(t, x) = \sum_{n=1}^{\infty} \alpha_n(t) e^{-D \lambda_n^2 t} \sin(\lambda_n x) \]

Expand force:

\[ f(t, x) = \sum_{n=1}^{\infty} f_n(t) e^{-D \lambda_n^2 t} \sin(\lambda_n x) \]

where

\[ f_n(t) e^{-D \lambda_n^2 t} = \frac{2}{L} \int_0^L f(t, x) \sin(\lambda_n x) \]
Analytic Solution Techniques: Diffusion Equation

Assumption: Piecewise differentiation yields

\[
\sum_{n=1}^{\infty} \left[ \alpha_n'(t) - D\lambda_n^2 \alpha_n(t) + D\lambda_n^2 \alpha_n(t) - f_n(t) \right] e^{-D\lambda_n^2 t} \sin(\lambda_n x) = 0
\]

\[\Rightarrow \alpha_n'(t) = f_n(t)\]

\[\Rightarrow \alpha_n(t) = \alpha_n + \int_{0}^{t} f_n(\tau) d\tau\]

since \[\alpha_n(0) = \frac{2}{L} \int_{0}^{L} \rho_0(x) \sin(\lambda_n x) dx = \alpha_n\]

Formal Solution:

\[\rho(t, x) = \sum_{n=1}^{\infty} \alpha_n e^{-D\lambda_n^2 t} \sin(\lambda_n x) + \sum_{n=1}^{\infty} \left[ \int_{0}^{t} f_n(\tau) d\tau \right] e^{-D\lambda_n^2 t} \sin(\lambda_n x)\]
Analytic Solution Techniques: Diffusion Equation

**Special Case:** Periodic source \( f(t, x) = f(x)e^{i\omega t} \)

Thus

\[
  f_n(t)e^{-D\lambda_n^2 t} = \frac{2}{L} \int_0^L f(x)e^{i\omega t} \sin(\lambda_n x)\,dx = \mathcal{F}_n e^{i\omega t}
\]

where

\[
  \mathcal{F}_n = \frac{2}{L} \int_0^L f(x) \sin(\lambda_n x)\,dx
\]

So...

\[
  f_n(t) = \mathcal{F}_n e^{(D\lambda_n^2 + i\omega) t} \Rightarrow \int_0^t f_n(\tau)\,d\tau = \frac{\mathcal{F}_n}{D\lambda_n^2 + i\omega} \left[ e^{(D\lambda_n^2 + i\omega) t} - 1 \right]
\]

**Formal Solution:**

\[
  \rho(t, x) = \sum_{n=1}^{\infty} \alpha_n e^{-D\lambda_n^2 t} \sin(\lambda_n x) - \sum_{n=1}^{\infty} \frac{\mathcal{F}_n}{D\lambda_n^2 + i\omega} e^{-D\lambda_n^2 t} \sin(\lambda_n x)
\]

\[
  + \sum_{n=1}^{\infty} \frac{\mathcal{F}_n}{D\lambda_n^2 + i\omega} e^{i\omega t} \sin(\lambda_n x) \quad \text{Steady State Solution}
\]
Green’s Function

Consider:
\[ \rho_t = D\rho_{xx} \]
\[ \rho(t, 0) = \rho(t, L) = 0 \]
\[ \rho(0, x) = \rho_0(x) \]

Solution:
\[ \rho(t, x) = \sum_{n=1}^{\infty} \alpha_n e^{-D\lambda_n^2 t} \sin(\lambda_n x) \quad , \quad \alpha_n = \frac{2}{L} \int_0^L \rho_0(y) \sin(\lambda_n y) dy \]

With uniform convergence, we can write
\[ \rho(t, x) = \int_0^L \left[ \frac{2}{L} \sum_{n=1}^{\infty} e^{-D\lambda_n^2 t} \sin(\lambda_n x) \sin(\lambda_n y) \right] \rho_0(y) dy \]
\[ = \int_0^L G(x, y, t) \rho_0(y) dy \]

Green’s Function:
\[ G(x, y, t) = \frac{2}{L} \sum_{n=1}^{\infty} e^{-D\lambda_n^2 t} \sin(\lambda_n x) \sin(\lambda_n y) \]

Analogy:
\[
\dot{y} = ay \\
y(0) = y_0 \Rightarrow y(t) = e^{At}y_0
\]
Analytic Solution Techniques: Wave Equation

Wave Equation: Assume that $\bar{u}$ is constant and $D = b = d = 0$

\[
\frac{\partial \rho}{\partial t} + \bar{u} \frac{\partial \rho}{\partial x} = 0
\]

\[
\rho(0, x) = \rho_0(x)
\]

Solution: $\rho(t, x) = \rho_0(x - \bar{u}t)$

Verification: Let $z = x - \bar{u}t$

Thus

\[
\rho_t = \frac{d \rho}{dz} \frac{\partial z}{\partial t} = \rho'_0(x - \bar{u}t)(-\bar{u})
\]

\[
\rho_x = \frac{d \rho}{dz} \frac{\partial z}{\partial x} = \rho'_0(x - \bar{u}t)
\]

Hence

\[
\rho_t + \bar{u}\rho_x = -\bar{u}\rho'_0(x - \bar{u}t) + \bar{u}\rho'_0(x - \bar{u}t) = 0
\]

\[
\rho(0, x) = \rho_0(x)
\]