

Some Solutions to Review 4 (Rest to follow Sun 15 APR)

Instructions: Show all work for full credit. Refer to any work done on separate pages.

1. Definitions and Concepts.

- a. Give a function, f , which satisfies the relationship $\frac{df}{dx} = 1$.

(*) Another way to interpret the relationship is $f'(x) = 1$, then $f(x) = x + C$, the anti-derivative.

- b. Why does it not matter if I forget the “+C” when using *definite* integration?

(*) Here is an example of what will happen. Let's integrate $\int_0^2 f(x)dx$ with $f(x) = 3x^2$.

The definite integral is computed as $F(b) - F(a)$ where $F(x)$ is the anti-derivative of $f(x)$. The anti-derivative of $3x^2$ is $F(x) = x^3 + C$.

Now $F(2) - F(0) = (2^3 + C) - (0^3 + C) = 8 + C - C = 8$. See how the C 's canceled out? That should always happen. So it is the convention to just drop the '+C' altogether. Sometimes this can get you into trouble! (But not in anything we're doing in this class).

- c. Find the error in my proof that $0 = 1$.

Let $f(x) = 2x$ and $g(x) = 2x + 1$. Then $f'(x) = 2$ and $g'(x) = 2$. Anti-differentiating, $f(x) = 2x + C$ and $g(x) = 2x + C$. Then $f(x) = g(x)$. So it follows $2x = 2x + 1$. Subtracting $2x$ from both sides of the equation, I find $0 = 1$.

(*) C is not a variable. We don't know that the values of C in $f(x)$ and $g(x)$ are the same.

- d. What does it mean for an improper integral to converge? Diverge?

(*) An improper integral has one of the limits of integration at $\pm\infty$. So to compute $\int_1^\infty f(x)dx$, I write this instead as $\lim_{b \rightarrow \infty} \int_1^b f(x)dx$. If this limit exists, the integral converges. If not, it diverges.

- e. Give an example of a function, $f(x)$ such that the area bounded by the x -axis, $f(x)$, $x = 0$, and $x = 1$ can be determined exactly using one rectangle (as opposed to an infinite series of rectangles).

(*) Let $f(x) = 3$. Then the region itself is a rectangle!

- f. Give an example of a function, $f(x)$, such that $\int f(x) = 3x^2 + 2x$.

(*) I want a function such that its anti-derivative is $3x^2 + 2x$. So I take the derivative of $3x^2 + 2x$. This is $6x + 2$.

2. Find $\int f(x)dx$ for each of the following.

a. $f(x) = 3x^2 + 5x + 1$

(*) Using the power rule for anti-derivatives (add one to power, then divide by new power), I get $F(x) = x^3 + \frac{5}{2}x^2 + x + C$.

b. $f(x) = \frac{44}{x}$

(*) $\int \frac{44}{x} dx = 44 \int \frac{1}{x} dx = 44 \ln|x| + C$

c. $f(x) = \frac{1}{x}(\ln(3x))^5$

(*) Here is a substitution problem.

Let $u = \ln(3x)$, the "inside" function. Then $\frac{du}{dx} = \frac{1}{x}$. So we see the following:

$$\int \frac{1}{x}(\ln(3x))^5 dx = \int \frac{du}{dx} u^5 dx = \int u^5 du = \frac{1}{6}u^6 + C = \frac{1}{6}(\ln(3x))^6 + C$$

d. $f(x) = \frac{9x^2+3}{3x^3+3x}$

(*) Let $u = 3x^3 + 3x$. Then $\frac{du}{dx} = 9x^2 + 3$. So we see the following:

$$\int \frac{9x^2+3}{3x^3+3x} dx = \int \frac{du}{dx} \frac{1}{u} dx = \int \frac{1}{u} du = \ln|u| + C = \ln|(3x^3 + 3x)| + C$$

e. $f(x) = \sqrt{x}$

(*) Notice $f(x) = x^{\frac{1}{2}}$. Then, using the power rule for anti-derivatives, $F(x) = \frac{2}{3}x^{\frac{3}{2}} + C$.

3. Find $\int_0^2 f(x)dx$ for each of the following.

a. $f(x) = x^2 + x$

(*) $f(x) = x^2 + x$, so $F(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2$. Then $F(2) - F(0) = \frac{8}{3} + 2 = \frac{14}{3}$.

b. $f(x) = xe^{x^2}$

(*) To find the anti-derivative of $f(x)$, $\int xe^{x^2} dx$, I will need to use substitution. Let $u = x^2$, the inside function. Then $\frac{du}{dx} = 2x$.

Note that I can re-write the integral as:

$$\int \frac{1}{2}2xe^{x^2} dx = \frac{1}{2} \int 2xe^{x^2} dx = \frac{1}{2} \int \frac{du}{dx} e^u dx = \frac{1}{2} \int e^u du = \frac{1}{2}e^u = \frac{1}{2}e^{x^2}$$

So I get $F(x) = \frac{1}{2}e^{x^2}$. Then $F(2) - F(0) = \frac{1}{2}e^4 - \frac{1}{2}$.

4. Find the area of the region bounded by the x-axis and the curve $f(x) = -(x^2) + 4$.

(*) First I notice that the limits of integration have not been given to me, but I know that the region will be bounded at the points where the curve crosses the x-axis. So first I solve $f(x) = 0$. This gives me $x = \pm 2$.

If $f(x) = -(x^2) + 4$, then by the power rule for anti-derivatives, $F(x) = -\frac{1}{3}x^3 + 4x$.

$$F(2) - F(-2) = -\frac{8}{3} + 8 - \frac{8}{3} + 8 = 16 - \frac{16}{3} = \frac{32}{3}.$$

5. Find the volume of the solid constructed by rotating the above region about the x-axis.

(*) We derived a very handy formula in class for this.

The volume is given by $V = \int_a^b \pi(f(x))^2 dx$.

From the last problem I know the limits of integration are ± 2 . And π is a constant, so I can pull it out of the integral. So I have:

$$\begin{aligned} V &= \pi \int_{-2}^2 (-(x^2) + 4)^2 dx \\ &= \pi \int_{-2}^2 x^4 - 8x^2 + 16 dx \\ &= \pi \left(\frac{1}{5}x^5 - \frac{8}{3}x^3 + 16x \right) \Big|_{-2}^2 \\ &= \pi \left(\frac{1}{5}(32) - \frac{8}{3}(8) + 16(2) - \left(\frac{1}{5}(-32) - \frac{8}{3}(-8) + 16(-2) \right) \right) \\ &= \frac{512}{15}\pi \end{aligned}$$

6. Find the area of the region bounded by the curves $f(x) = \sqrt{x}$ and $g(x) = x^2$.

(*) First I need to know my limits of integration. I know the region is bounded by the points where the two curves meet, so I will find their intersections. If $\sqrt{x} = x^2$, then clearly $x = 0$ is a solution. Also, by squaring both sides I get $x = x^4$. Then $1 = x^3$. So $x = 1$ is a solution. So I know the limits of integration are 0 and 1.

Next I observe that on the interval $[0, 1]$, the curve $f(x) = \sqrt{x}$ is the higher curve, and $g(x) = x^2$ is the lower curve. So the area is given by $\int_0^1 f(x) - g(x) dx$.

$$\begin{aligned} A &= \int_0^1 f(x) - g(x) dx \\ A &= \int_0^1 \sqrt{x} - x^2 dx \\ A &= \left(\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{3}x^3 \right) \Big|_0^1 \\ A &= \left(\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{3}x^3 \right) \Big|_0^1 \\ A &= \left(\frac{2}{3}(1)^{\frac{3}{2}} - \frac{1}{3}(1)^3 \right) \\ A &= \frac{1}{3} \end{aligned}$$

7. Consider the following supply function, $S(x)$, and demand function, $D(x)$, where price is a function of quantity.

$$S(x) = x^2 - 2x + 2, D(x) = -2x + 11$$

- a. Find the equilibrium point.

(*) This is x such that $S(x) = D(x)$. So I solve $x^2 - 2x + 2 = -2x + 11$ to get $x^2 = 9$, or $x = \pm 3$. Now x can't be negative (by a restricted domain argument with respect to the application), so I only consider the case when $x = 3$. When $x = 3$, then $S(3) = D(3) = 5$. The equilibrium point is $(3, 5)$.

- b. Find the consumer's surplus at this point.

(*) The consumer's surplus is given by $\int_0^Q D(x)dx - QP$ where Q is the quantity and P is the price.

First, $Q = x = 3$ at the equilibrium point. So $P = D(3) = 5$. Then I have:

$$\begin{aligned} \text{surplus} &= \int_0^Q D(x)dx - QP \\ &= \int_0^3 -2x + 11 dx - (3)(5) \\ &= (-x^2 + 11x)|_0^3 - 15 \\ &= (-9 + 33) - 15 \\ &= 9 \end{aligned}$$

- c. Find the producer's surplus at this point.

(*) The producer's surplus is given by $QP - \int_0^Q S(x)dx$ where Q is the quantity and P is the price.

Recall $Q = x = 3$ at the equilibrium point. Also, $P = S(3) = 5$. Then I have:

$$\begin{aligned} \text{surplus} &= QP - \int_0^Q S(x)dx \\ &= (3)(5) - \int_0^3 x^2 - 2x + 2 dx \\ &= 15 - (\frac{1}{3}x^3 - x^2 + 2x)|_0^3 \\ &= 15 - 6 \\ &= 9 \end{aligned}$$

8. Find the area under the piece-wise defined function from $x = 0$ to $x = 99$.

$$f(x) = \begin{cases} 1 & \text{if } x \leq 50; \\ 2 & \text{if } x > 50. \end{cases}$$

(*) I start by attempting to compute $\int_0^{99} f(x)dx$. I have one complication though: $f(x)$ has two meanings, depending on the value of x . But I know I can “split” the integral. So I will do that in such a way that it is clear what I need to anti-differentiate.

First, $\int_0^{99} f(x)dx = \int_0^{50} f(x)dx + \int_{50}^{99} f(x)dx$. For the left integral it is clear I should use “1” for $f(x)$, as x is always less than or equal to 50. For the right integral it is clear I should use “2” for $f(x)$, as x is always greater than 50. So I have:

$$\begin{aligned} \int_0^{99} f(x)dx &= \int_0^{50} f(x)dx + \int_{50}^{99} f(x)dx \\ &= \int_0^{50} 1dx + \int_{50}^{99} 2dx \\ &= (x)|_0^{50} + (2x)|_{50}^{99} \\ &= (50) + (198 - 100) \\ &= 148 \end{aligned}$$

9. Find the amount of continuous money flow in which 5000 per year is being invested at 6%, compounded continuously, for 45 years.

(*) I can compute the amount of continuous money flow by calculating $\int_0^{45} 5000e^{-.06t}dt$. Do you remember why that is?

$$\begin{aligned} \text{funds} &= \int_0^{45} 5000e^{-.06t}dt \\ &= 5000 \int_0^{45} e^{-.06t}dt \\ &= 5000 \left(\frac{1}{-.06} e^{-.06(45)} - \frac{1}{-.06} e^{-.06(0)} \right) \\ &= 5000 \left(\frac{1}{.06} e^{-.06t} \right) \Big|_0^{45} \\ &\approx 1,156,644.31 \end{aligned}$$

I'm rich!

10. In 1897 the world's consumption of Watsonium-199 was 55,000 ft³. The amount used has steadily increased at an average rate of 10% per year. What is the consumption rate in 2007? What was the total amount of Watsonium-199 consumed?

(*) First I need to establish my model. Clearly we have an exponential growth situation, so I use $w(t) = Ce^{kt}$, where $w(t)$ is the amount of Watsonium-199 consumed at year t .

C , the initial amount, is 55000. k is .10. So I have $w(t) = 55000e^{.1t}$.

Then $w(110)$ is the amount consumed 110 years since 1897, or in 2007. $w(110) \approx 3,293,077,794$.

The total amount consumed in the 110 year time span is $\int_0^{110} w(t)dt$. This is given as:

$$\begin{aligned} \text{total} &= \int_0^{110} w(t)dt \\ &= \int_0^{110} 55000e^{.1t} dt \\ &= 55000 \int_0^{110} e^{.1t} dt \\ &= 55000 \left(\frac{1}{.1} e^{.1t} \right) \Big|_0^{110} \\ &= 55000 \left(\frac{1}{.1} e^{.1(110)} - \frac{1}{.1} e^{.1(0)} \right) \\ &\approx 32,930,227,943 \end{aligned}$$

11. Find $\int_1^\infty f(x) dx$ for each of the following.

a. $f(x) = \frac{1}{x}$

(*)

$$\begin{aligned}\int_1^\infty f(x) dx &= \lim_{b \rightarrow \infty} \int_1^b \frac{1}{x} dx \\ &= \lim_{b \rightarrow \infty} \ln(x) \Big|_1^b dx \\ &= \lim_{b \rightarrow \infty} \ln(b) - 0 dx \\ &= \infty\end{aligned}$$

The improper integral diverges.

b. $f(x) = \frac{1}{x^2}$

(*)

$$\begin{aligned}\int_1^\infty f(x) dx &= \lim_{b \rightarrow \infty} \int_1^b \frac{1}{x^2} dx \\ &= \lim_{b \rightarrow \infty} -\frac{1}{x} \Big|_1^b dx \\ &= \lim_{b \rightarrow \infty} -\frac{1}{b} + 1 dx \\ &= 1\end{aligned}$$

The improper integral converges.

c. $f(x) = e^{-x}$

(*)

$$\begin{aligned}\int_1^\infty f(x) dx &= \lim_{b \rightarrow \infty} \int_1^b e^{-x} dx \\ &= \lim_{b \rightarrow \infty} -e^{-x} \Big|_1^b dx \\ &= \lim_{b \rightarrow \infty} -e^{-b} + e^{-1} dx \\ &= \lim_{b \rightarrow \infty} -\frac{1}{e^b} + \frac{1}{e} dx \\ &= \frac{1}{e}\end{aligned}$$

The improper integral converges.

12. Find $\lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x_i$ where $f(x) = 2x$ on the closed interval $[3, 5]$.

(*) This is the limit definition of the definite integral, equivalent to $\int_3^5 2x dx$. The anti-derivative of $f(x)$, which we'll call $F(x)$, is x^2 . Then $F(5) - F(3) = 25 - 9 = 16$.