

# MA121 Elements of Calculus

## Exam 4 Form 42 Solution

24 November, 2008

*Instructions:* Show all work relevant to the solution of each problem. i.e. no credit will be given for “just the answers.” Please do *all* work in the Blue Books! There are **eight** problems which carry a total of 100 points. You will have until the end of class to complete this exam. Good luck!

(10 pts) **Problem 1.** Definitions and Concepts.

a. Briefly explain the concepts of convergent and divergent integrals. How do they differ?

(\*) When an improper integral evaluates to a finite number, it is convergent. Else it is divergent.

b. Write a function,  $f$ , which satisfies the relationship  $\frac{df}{dx} = x^2$ .

(\*)  $f(x) = \frac{1}{3}x^3 + C$

c. Write a function,  $f(x)$ , such that  $\int f(x)dx = \ln(x)$ .

(\*)  $f(x) = \frac{1}{x}$

d. In the expression  $\lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i)\Delta x_i$ , what does  $n$  represent?

(\*)  $n$  is the number of rectangles.

e. Suppose the function  $v(t)$  gives the speed of a car at time  $t$ . What is described by the value  $\int_0^5 v(t)dt$ ?

(\*) Distance travelled.

f. Identify the false statement amongst the three given.

1.  $\int kf(x)dx = k \int f(x)dx$
2.  $\int f(g(x))dx = \int f(x)dx \int g(x)dx$
3.  $\int f(x) + g(x)dx = \int f(x)dx + \int g(x)dx$

(\*) Statement 2 is false.

(30 pts) **Problem 2.** Compute each of the following:

a.  $\int 12x^3 + 4x dx$

(\*)  $3x^4 + 2x^2 + C$

b.  $\int \frac{4}{x} dx$

(\*)  $4\ln(x) + C$

c.  $\int 3x^2(x^3 - 5)^{10} dx$

(\*)  $\frac{1}{11}(x^3 - 5)^{11} + C$

d.  $\int_0^1 12x^2 + 2x dx$

(\*)  $F(x) = 4x^3 + x^2.$

$F(1) - F(0) = 5.$

(10 pts) **Problem 3.** Find the area of the region bounded by the x-axis, the curve  $f(x) = 4x + 1$ ,  $x = 1$  and  $x = 3$ .

(\*)  $\text{Area} = \int_1^3 4x + 1 dx.$

$F(x) = 2x^2 + x.$

$F(3) - F(1) = 18.$

(10 pts) **Problem 4.** Find the volume of the solid constructed by rotating the above region about the x-axis.

(\*)  $\text{Volume} = \int_1^3 \pi(4x + 1)^2 dx$

To simplify, write  $\text{Volume} = \pi \int_1^3 16x^2 + 8x + 2 dx.$

$F(x) = \frac{16}{3}x^3 + 4x^2 + 2x.$

$\pi(F(3) - F(1)) = \frac{524}{3}\pi.$

(10 pts) **Problem 5.**

a. Determine the anti-derivative of  $\frac{4x^3+9}{x^4+9x}$ .

(\*) To find  $\int \frac{4x^3+9}{x^4+9x} dx$ , use substitution.

Let  $u = x^4 + 9x$ . Then  $\frac{du}{dx} = 4x^3 + 9$ . Hence  $dx = \frac{du}{4x^3+9}$ .

$$\int \frac{4x^3+9}{x^4+9x} dx = \int \frac{u}{x^4+9x} \frac{du}{4x^3+9} = \int \frac{1}{u} du.$$

Antidifferentiate to find  $\int \frac{1}{u} du = \ln(u) + C = \ln(x^4 + 9x) + C.$

- b. Use your answer from (a) to compute  $\int_1^{\infty} \frac{4x^3+9}{x^4+9x} dx$ . If you could not determine the anti-derivative, explain how you would solve the rest of the problem.

$$(*) \int_1^{\infty} \frac{4x^3+9}{x^4+9x} dx = \lim_{b \rightarrow \infty} \int_1^b \frac{4x^3+9}{x^4+9x} dx$$

$F(x) = \ln(x^4 + 9x)$ , as determined above.

$$\lim_{b \rightarrow \infty} F(b) - F(1) = \lim_{b \rightarrow \infty} \ln(b^4 + 9b) - \ln(10) \rightarrow \infty.$$

Hence, the integral is divergent.

(10 pts) **Problem 6.**

- a. Find the point where the curves  $f(x) = x^3$  and  $g(x) = x^4$  intersect.

(\*) I want to find where  $f(x) = g(x)$ .

$$f(x) = g(x)$$

$$x^3 = x^4$$

$$0 = x^4 - x^3$$

$$0 = x^3(x - 1)$$

$$x = 0 \quad x = 1$$

- b. Find the area of the region bounded by the curves  $f(x)$  and  $g(x)$ .

(\*)

$$A = \int_0^1 f(x) - g(x) dx$$

$$= \int_0^1 x^3 - x^4$$

$$= \frac{1}{4}x^4 - \frac{1}{5}x^5 \Big|_0^1$$

$$= \frac{1}{4} - \frac{1}{5}$$

$$= \frac{1}{20}$$

(10 pts) **Problem 7.** In 1988 the Milton's consumption of cat food was 350 metric tons per year. The amount consumed has increased at an average rate of 2% per year. Find the *total amount* of cat food consumed from 1988 to 2008.

$$(*) \text{ Total food} = \int_0^{20} 350e^{.02t}$$

$$F(t) = \frac{350}{.02} e^{.02t}$$

$$F(20) - F(0) = 17500(e^{.4} - 1).$$

(10 pts) **Problem 8.** Find the area under the piece-wise defined function from  $x = 0$  to  $x = 30$ .

$$f(x) = \begin{cases} 2x & \text{if } x \leq 10; \\ -2x + 40 & \text{if } x > 10. \end{cases}$$

(\*)

$$\begin{aligned} A &= \int_0^{30} f(x) dx \\ &= \int_0^{10} f(x) dx + \int_{10}^{30} f(x) dx \\ &= \int_0^{10} 2x dx + \int_{10}^{30} (-x^2 + 40x) dx \\ &= x^2 \Big|_0^{10} + \left( -\frac{x^3}{3} + 20x^2 \right) \Big|_{10}^{30} \\ &= (100 - 0) + (300 - 300) \\ &= 100 \end{aligned}$$