MA121 Elements of Calculus
Exam 4 Form 41

15 April, 2009

Instructions: Show all work relevant to the solution of each problem. i.e. no credit will be given for "just the answers." Please do *all* work in the Blue Books! There are **eight** problems which carry a total of 106 points. You will have until the end of class to complete this exam. Good luck!

(10 pts) **Problem 1.** Definitions and Concepts.

a. Write a function, $f(x)$, such that $\int f(x)dx = \ln(x)$.

(*) $f(x) = \frac{1}{x}$

b. Write a function, f, which satisfies the relationship $\frac{df}{dx} = -2x$.

(*) $f(x) = -x^2 + C$

c. Briefly explain the concepts of convergent and divergent integrals. How do they differ?

(*) If an improper integral evaluates to a finite value, it is said to converge to that value. If it evaluates to infinity, it is divergent.

d. In the expression $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x_i$, what does $f(x_i)$ represent?

(*) The expression computes the sum of the areas of each of the rectangles used to divide the region. x_i is the x coordinate of the ith rectangle. $f(x_i)$ is the height of that rectangle.

e. Suppose the function $A(t)$ gives the rate of oil consumption at year t. What is described by the value $\int_0^5 A(t)dt$?

(*) The total oil consumed from now to 5 years into the future.

(24 pts) **Problem 2.** Compute each of the following:

a. $\int \frac{1}{x} - \sqrt{x}dx$

(*) $\ln(x) - \frac{3}{2}x^{\frac{2}{3}} + C$

b. $\int x^2 + \frac{3}{x}dx$

(*) $\frac{1}{3}x^3 + 3\ln(x) + C$

c. $\int 20x^3(5x^4-100)^5\,dx$

(*) Let $u = 5x^4 - 100$.

Then $\frac{du}{dx} = 20x^3$. Solve for dx to get $dx = \frac{du}{20x^3}$

Hence, I can rewrite the integral as $\int 20x^3u^5\frac{du}{20x^3}$.

Simplify to get $\int u^5\,du = \frac{1}{6}u^6 + C = \frac{1}{6}(5x^4-100)^6 + C$

(12 pts) **Problem** 3. Find the area of the region bounded by the x-axis, the curve $f(x) = e^x$, $x = 0$ and $x = 1$.

(*) $Area = \int_0^1 e^x\,dx$.

$f(x) = e^x$, so $F(x) = e^x$.

$F(1) - F(0) = e^1 - e^0 = e - 1$.

(12 pts) **Problem** 4. Find the volume of the solid constructed by rotating the above region about the x-axis.

(*) $Vol = \int_0^1 \pi(e^x)^2\,dx$

$= \pi\int_0^1 e^{2x}\,dx$

$f(x) = e^{2x}$, so $F(x) = \frac{1}{2}e^{2x}$.

$\pi(F(1) - F(0)) = \pi(\frac{1}{2}e^2 - \frac{1}{2})$

(12 pts) **Problem** 5. In the year 1900, Milton City switched from a coal to a petroleum-based economy. As such, the amount of coal used by Milton City in year t is modelled using the decreasing function $A(t) = 15000e^{-.05t}$.

a. Find the anti-derivative of $A(t)$.

(*) $\int A(t) = -300000e^{-.05t} + C$

b. Compute the total amount of coal used by Milton City from 1900 to indefinitely into the future.

(*) Let t be years since 1900.

$\int_0^\infty 15000e^{-.05t}dt = \lim_{b\to\infty}\int_0^\infty 15000e^{-.05t}dt$

Now we already found the anti-derivative in part a. So we have
$\lim_{b\to\infty}(-300000e^{-.05b} - -300000e^0)$

Which simplifies to $\lim_{b\to\infty} 300000 - 300000e^{-.05b}$.

As b approaches infinity, $e^{-.05b}$ approaches zero. Hence, the total amount of oil consumed is
$300000 - 300000(0) = 300000$.

(12 pts) **Problem 6.** Find the area of the region bounded by the curves $f(x) = x$ and $g(x) = x^2$.

(*) Set $f(x) = g(x)$ and solve for x to get the limits.

$x = x^2 \Rightarrow x^2 - x = 0 \Rightarrow x(x-1) = 0 \Rightarrow x = 0, x = 1.$

Now to evaluate $\int_0^1 x - x^2 dx$, let

$f(x) = x - x^2$. Then $F(x) = \frac{1}{2}x^2 - \frac{1}{3}x^3$.

$F(1) - F(0) = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$.

(12 pts) **Problem 7.** Suppose I am saving for retirement by investing into an account which pays 1% interest, compounded continuously. There is a continuous flow of money into the account, at the rate of 4500 dollars per year. Find the total amount of money that will have accumulated in 40 years.

(*) $\mathtt{Total} = \int_0^{40} 4500e^{.01t} dt$.

$f(t) = 4500e^{.01t}$. So $F(t) = 450000e^{.01t}$.

$\mathtt{Total} = F(40) - F(0) = 450000e^{.4} - 450000e^0 = 221321$

(12 pts) **Problem 8.** Find the area under the piece-wise defined function from $x = -10$ to $x = 10$.

$$f(x) = \begin{cases} -x & \text{if } x \leq 0; \\ x & \text{if } x > 0. \end{cases}$$

(*) $\int_{-10}^{10} = \int_{-10}^{0} -x dx + \int_0^{10} x dx$

$= -\frac{1}{2}x^2|_{-10}^0 + \frac{1}{2}x^2|_0^{10}$

$= -0 - (-50) + (50) - 0 = 100$