MA121 Elements of Calculus
Exam 3 Review Solution

Edition 2(a)

(*) This is a review examination. The actual exam will not be as long. It is intended to help you review important concepts and give you an idea of *what kind* of questions may be asked (but *not* the questions themselves!).

Instructions: Show all work for full credit. Refer to any work done on separate pages.

1. Give an example of each of the following (You may present either a function of a graph of a function).

a. A function which satisfies the relationship $\frac{dP}{dt} = kP$.

(*) The relationship is true if and only if P is of the form $P = ce^{kt}$

b. A function which satisfies the relationship $\frac{dP}{dt} = -kP$.

(*) The relationship is true if and only if P is of the form $P = ce^{-kt}$

c. A function which is its own derivative ($f(x) = f'(x)$).

(*) The function is $f(x) = e^x$. Recall the general rule for derivatives of expoentials. If $f(x) = a^x$, then $f'(x) = \ln(a)a^x$. The special case, when $a = e$, yields $f'(x) = \ln(e)a^x$, and it is established that $\ln(e) = 1$.

d. A function on a closed interval, $[A, B]$, which has no relative mins or maxes, but has an absolute min and max

(*) Increasing and decreasing functions don't have relative mins and maxes, because there is no solution to $f'(x) = 0$. But when looking for an *absolute* min or max, we need to consider the end points of a *closed* interval. So the solution is to define an increasing or decreasing function on a closed interval.

e. A function which has an absolute maximum

(*) Any function will do as long as it is on a closed interval. Otherwise, I need to pick one which has a solution, c, to $f'(x) = 0$, and $f''(c) < 0$. For instance, $f(x) = -x^2$.

f. An increasing function with an absolute minimum

> **(*)** The only way to do this is to define the function on a closed interval. Then it will be guaranteed to have an absolute minimum and an absolute maximum.
>
> Note that this point was basically made three times. It must be important!

2. Differeniate.

a. $f(x) = \ln(3x)$

> **(*)** Recall the shortcut. I identify the "inside" function. Then I divide its derivative by it.
>
> The "inside" function is $3x$, and the derivative of $3x$ is 3. So the derivative of $\ln(3x)$ is $\frac{3}{3x} = \frac{1}{x}$.
>
> Do you remember how the shortcut was derived?

b. $f(x) = \log_4(3x^2)$

> **(*)** Here there is an added step because the log is not of base e. I divide the derivative of the "inside" function by the "inside" function as usual. Then I multiply by $\frac{1}{\ln(b)}$, where b is the base of the logarithm.
>
> The "inside" function is $3x^2$ and its derivative is $6x$. So I multiply $\frac{6x}{3x^2}$ by $\frac{1}{\ln(4)}$.

c. $f(x) = e^{3x^2+x}$

> **(*)** Because e^x is its own derivative, by the chain rule, I multiply e^{3x^2+x} by the derivative of the "inside" function (which is the exponent). The derivative of the exponent is $6x + 1$, so the derivative of $f(x)$ is $(6x + 1)e^{3x^2+x}$.

d. $f(x) = 5^{2x}$

> **(*)** Again, when the base is not e there is an added step. Here I multiply by $\ln(b)$, where b is 5 in this case. The "inside" function is $2x$, and its derivative is 2. So I multiply $(2)(5^{2x})$ by $\ln(5)$.
>
> *Warning*: $(2)(5^{2x})$ is *not* 10^{2x}.

3. Find the absolute maximum and minimum of $f(x) = x^2 + 3x$ on $[-4, 4]$.

> **(*)** Here is the process.
>
> First, I need to find the critical points. $f(x)$ is a polynomial, hence smooth and continuous, hence differentiable everywhere. So the only critical points are the solutions to $f'(x) = 0$. $f'(x) = 2x + 3$, so $x = -1.5$ is a critical point.
>
> Now I evaluate f at all critical points *and at the closed endpoints of an interval*, and make a list of their values.
>
> $f(-1.5) = -2.25$
> $f(-4) = 4$
> $f(4) = 28$
>
> Last, I identify the lowest and highest values. The absolute minimum is $(-1.5, -2.25)$ because -2.25 is the lowest value. The absolute maximum is $(4, 28)$, as 28 is the highest value.

4. Find the absolute maximum and minimum of $f(x) = \frac{x+3}{x}$ on $[-5, 5]$.

(*) Here I'll just compute everything I need. Refer to (3) for a step-by-step explanation.

$f'(x) = \frac{(x)(1)-(x+3)(1)}{x^2} = \frac{-3}{x^2}$. To find where $f'(x) = 0$ I just need to solve the numerator for 0. But there are no solutions to $-3 = 0$. However, there is a critical point at $x = 0$ because $f(x)$ is not continuos (or differentiable) here.

$f(-5) = .4$ and $f(5) = 1.6$. But are these the lowest/highest points in the domain?

No. Here we have to be very careful, because there is a vertical asymptote. As x approaches 0 from the left, $f(x)$ approaches $-\infty$. As x approaches 0 from the right, $f(x)$ approaches ∞. So $f(x)$ increases/decreases without bound. So there is no absolute minimum or maximum. To confirm, look at a graph and see if you can identify a highest and lowest point.

5. Find the absolute maximum and minimum of $f(x) = x^2 + \frac{250}{x}$

(*) Here we don't have to worry about the endpoints of a closed interval, so the only things to check for are critical points.

$f'(x) = 2x - \frac{250}{x^2}$. I solve $2x - \frac{250}{x^2} = 0$:

$$\begin{aligned} 2x - \tfrac{250}{x^2} &= 0 \\ 2x &= \tfrac{250}{x^2} \\ 2x^3 &= 250 \\ x^3 &= 125 \\ x &= 5 \end{aligned}$$

There's one critical point. The other is $x = 0$ because $f(x)$ is not differentiable there.

$f(5) = 75$, and $f''(5) = 6$, which is positive, so $(5, 75)$ is a local minimum.

Is it an absolute minimum? We need to investigate what is happening at the vertical asymptote. As x approaches 0 from the left, $f(x)$ approaches $-\infty$. As x approaches 0 from the right, $f(x)$ approaches ∞. So again there is no absolute minimum or maximum. Note this question is slightly different from the one on Webassign HW6. Do you see what is different and explain why it matters?

6. What is the function that satisfies $\frac{dP}{dt} = -.03P$?

(*) P must have the form ce^{-kt} and $k = .03$, so $P = ce^{-.03t}$.

7. What is the function that satisfies $P'(t) = .077P(t)$?

(*) P must have the form ce^{kt} and $k = .077$, so $P = ce^{.077t}$.

8. Watsonium-544 decays at a rate of 5.44% per year. Therefore, if $N(t)$ is the amount of Watsonium-544 at year t, then $\frac{dN}{dt} = -.0544N$.

a. Find the function $N(t)$ which satisfies the relationship and tells me how much Watsonium-544 I will have at time t if I initially had N_0 amount of Watsonium-544.

(*) $\frac{dN}{dt} = -.0544N$ is satisfied by $N = N_0 e^{-.0544t}$

b. Find the half-life of Watsonium-544.

(*) I need to find t such that $N = .5N_0$.

$$\begin{aligned} .5N_0 &= N_0 e^{-.0544t} \\ .5 &= e^{-.0544t} \\ \ln(.5) &= \ln(e^{-.0544t}) \\ \ln(.5) &= -.0544t \\ t &= \frac{\ln(.5)}{-.0544} \end{aligned}$$

c. Archaeologists investigating the great Watson Pyramids stumble upon the Idol of Cy. It is known that Watsonium-544 decays into Watsonium-322, and this is used to determine that 88.5% of the initial amount of Watsonium-544 in the Idol of Cy has decayed. How old is the Idol of Cy?

(*) If 88.5% has decayed, then 12.5% remains, or $.125N_0$. I need to find t so that $N = .125N_0$.

$$\begin{aligned} .125N_0 &= N_0 e^{-.0544t} \\ .125 &= e^{-.0544t} \\ \ln(.125) &= \ln(e^{-.0544t}) \\ \ln(.125) &= -.0544t \\ t &= \frac{\ln(.125)}{-.0544} \end{aligned}$$

9. Many savings accounts have notoriously low interest rates. I deposit 400 into a savings account which pays an interest of 0.13%, compounded continuously. To pay for a new car, I will leave the money in the account until it grows to $15,000$. How long will I wait?

(*) My balance at time t is $A(t) = Pe^{rt}$, with $P = 400$ and $r = .0013$. I want to find t such that $A(t) = 15000$.

$$\begin{aligned} 15000 &= 400e^{.0013t} \\ 37.5 &= e^{.0013t} \\ \ln(37.5) &= \ln(e^{.0013t}) \\ \ln(37.5) &= .0013t \\ t &= \frac{\ln(37.5)}{.0013} \\ t &\approx 2787.95 \end{aligned}$$

10. A box has a square base, and the sum of the length, width, and height must equal 60 inches. Depending upon how I set the length, width, and height within this constraint, the box will obtain varying volumes. Find the maximum volume I can obtain.

(*) Let's review the facts. First, because the box has a square base, its length equals its width. So the volume is given as $V = L^2h$. Also, $L + W + h = 2W + h = 60$. So $2L + h = 60$ is a *constraint equation*.

The function we want to optimize has two variables, L and h. A constraint equation can be used to write one variable in terms of another, hopefully with the result of reducing the number of independent variables. I can use the constraint equation to write h in terms of L. $2L+h = 60 \Rightarrow h = 60-2L$. I can then substitute $(60-2L)$ for h in the volume equation.

$V(L) = L^2(60 - 2L)$. Notice V is now a function of one variable, L, because I wrote h in terms of L.

Taking the derivative of $V(L) = L^2(60 - 2L) = 60L^2 - 2L^3$ I get $V'(L) = 120L - 6L^2$.

Factoring an L out, $V'(L) = L(120 - 6L)$. Then $L = 0$ and $L = 20$ are solutions to $V'(L) = 0$. $L = 0$ creates a box of volume 0, so we don't consider this solution.

$V''(L) = 120 - 12L$ and $V''(20) < 0$, so there must be a relative maximum at $L = 20$. $V(20) = 20^2(20) = 8000$. There are no other relative mins or maxes, or critical points. So $(20, 8000)$ is the absolute maximum. To get the maximum volume of 8000, I let $L = 20$.

11. Find t such that $f'(t) = 0$ if $f(t) = e^{t^2} - e$

(*) First I need to find $f'(t)$. To find the derivative of e^{t^2}, I note the inside function is t^2, and its derivative is $2t$. The derivative of e is 0 because e is a constant. So $f'(t) = 2te^{t^2}$.

$f'(t) = 0$ if $2t = 0$ or $e^{t^2} = 0$. $t = 0$ is a solution to $2t = 0$, and there are no solutions to $e^{t^2} = 0$. So $t = 0$.

12. Find the equation of the line tangent to $f(x)$ and through the point $(0, 5)$ if $f(x) = 5^{2x+1}$.

(*) We use the same routine. Let's review.

To find $f'(x)$, I multiply by the derivative of the exponent and the natural log of the base. So $f'(x) = (2)\ln(5)5^{2x+1}$.

Now a tangent line has the form $y = mx + b$, where m is $f'(0)$.

$m = f'(0) = 2\ln(5)5^{2(0)+1} = 2\ln(5)5 = 10\ln(5)$.

So $y = 10\ln(5)x + b$. Because the tangent line passes through $(0, 5)$, $5 = 10\ln(5)(0) + b \Rightarrow b = 5$.

So $y = 10\ln(5)x + 5$.

13. Find $f'(x)$ if $f(x) = \ln(\frac{3x^2+9}{x+3})$. Hint: This is easier if you review the properties of logarithms before diving in.

(*) In the interest of avoiding a migraine, instead of using the quotient rule to differentiate the "inside" function, I will write

$$f(x) = \ln(\tfrac{3x^2+9}{x+3}) = \ln(3x^2+9) - \ln(x+3).$$

Now $f'(x)$ is easy to find. Dividing the the derivative of the contents of each parenthesis by itself, I find

$$f'(x) = \tfrac{6x}{3x^2+9} - \tfrac{1}{x+3}.$$

14. When Watson Tech sells computers at a price of $1,000$, the company averages about $185,000$ sales. For each 50 increase in price, however, $5,000$ sales are lost. What is the domain of the profit function restricted to values which make sense in light of the application? What should Watson Tech sell their computers for to maximize revenue?

(*) I need to find a function which describes revenue as a function of price.

For each 50 increase in price, 5000 sales are lost. So for each 1 increase in price, 100 sales are lost. Using this, I can write sales as a function of price. Number of sales is 185000, minus 100 for each dollar in the price over 1000. $S(p) = 185000 - 100(p - 1000)$.

The revenue will be the number of sales multiplied by the price. $R(p) = p(185000 - 100(p - 1000)) = -100p^2 + 285000p$.

Now $R'(p) = -200p + 285000$ Solving $R(p) = 0$ I obtain $p = 1425$.

$R''(1425) = -200 < 0$, so $(1425, R(1425))$ is a relative maximum. Since there is only one critical point and there are no closed interval boundaries, it is the absolute maximum.

The maximum revenue will be obtained by selling the computers for 1425. The revenue is $R(1425) = 203,062,500$.

15. In 2006, Watsonville ISD had a total population of 34,000 students. The population grows at an average rate of 1.5% per year. The school district will need to consider constructing a new school building when the total population reaches 36,000. Using the uninhibited growth model, predict the year when Watsonville ISD will need to have the funds required to build a new school.

(*) First we need to model the population. Let the year 2006 be $t = 0$.

The population is growing exponentially at a rate of .015P per year. Then $\frac{dP}{dn} = .015P$. So $P(t) = P_0 e^{.015t}$. Now the initial population is 34000, so $P(t) = 34000e^{.015t}$.

I need to find t such that $P(t) = 36000$.

$$\begin{aligned} 36000 &= 34000e^{.015t} \\ \frac{18}{17} &= e^{.015t} \\ \ln(\frac{18}{17}) &= \ln(e^{.015t}) \\ \ln(\frac{18}{17}) &= .015t \\ t &= \frac{\ln(\frac{18}{17})}{.015} \\ t &\approx 3.81 \end{aligned}$$

WISD needs the funds by 2009 if it wants to have the new building finished before the population hits 36,000.

16. Find the derivatives.

a. $5^x x^5$

(*) Here is a product rule problem. I need to find the derivatives of 5^x and x^5. The derivative of 5^x is $\ln(5)5^x$ and the derivative of x^5 is $5x^4$. I now have all the pieces I need to assemble as the product rule dictates. So the derivative of $5^x x^5$ is $(5^x)(5x^4) + (\ln(5)5^x)(x^5)$.

b. $e^{x\ln(x)}$

(*) The derivative of e^x is itself, but the chain rule dictates that I multiply by the derivative of the "inside" function, which is the exponent. Now, the inside function is a product, so I must use the product rule to find its derivative. So, working carefully and step-by-step, let's first find the derivatives of x and $\ln(x)$.

The derivative of x is 1, and the derivative of $\ln(x)$ is $\frac{1}{x}$. Now I have all the components I need to use the product rule.

The derivative of $x\ln(x)$ is $x(\frac{1}{x}) + 1\ln(x) = 1 + \ln(x)$.

Now that I have the derivative of the exponent, I multiply $e^{x\ln(x)}$ by our result to finish the chain rule.

The derivative is $(1 + \ln(x))e^{x\ln(x)}$.

Now, do you remember the extra credit assignment where I asked you to find the derivative of x^x? Let us write $y = x^x$. Then I can "take the log of both sides" and write $\ln(y) = \ln(x^x)$. Then $\ln(y) = x\ln(x)$ thanks to the properties of logarithms (you should know them all!). Finally, I can treat the LHS and RHS of the equation as exponents of e, by writing $e^{\ln(y)} = e^{x\ln(x)}$. Now $e^{\ln(y)} = y$ because exponential functions and logarithm functions are inverses of each other. Then $y = x^x = e^{x\ln(x)}$. But wait, we just found the derivative of $e^{x\ln(x)}$. Since $x^x = e^{x\ln(x)}$, then the derivative of x^x is $(1 + \ln(x))e^{x\ln(x)}$.

c. $x^{55}\log_5(x)$

(*) The derivative of x^{55} is $55x^{54}$. The derivative of $\log_5(x)$ is $\frac{1}{\ln(5)}\frac{1}{x} = \frac{1}{x\ln(5)}$.

Now I have all the components necessary to use the product rule.

The derivative is $x^{55}\frac{1}{x\ln(5)} + 55x^{54}\log_5(x)$.

d. e^{x^2+4x+4}

(*) Again, the derivative of e^x is itself, but the chain rule dictates that I multiply by the derivative of the "inside" function, which is the exponent. The derivative of the exponent is $2x + 4$, so the derivative of e^{x^2+4x+4} is $(2x + 4)e^{x^2+4x+4}$.

17. Find $f(x)$ if $f'(x) = 3f(x)$.

(*) $f(x)$ must have the form $f(x) = ce^{kt}$, where $k = 3$.

18. Find the derivative of $f(x)$ if $f(x) = e^5$. What about if $f(x) = 2^5 3^5$?

> **(*)** All of these functions are simply constant functions. So their derivatives are 0.

19. I own a small house on the beach and I want to fence off a *rectangular* section of private beach along the oceanside. I have 180 feet of fencing I can use for the three edges, with the fourth edge of the lot being the water line (and hence, no need to be fenced). Find the maximum area I can enclose, and the lengths of all three edges which produce the maximum area.

> **(*)** Here is another constraint equation problem. I want to optimize the area of a rectangle ($A = lw$). Both l and w are variables. In order to reduce the area to a function of one variable, I write one of the variables in terms of the other using the constraint equation.
>
> Here the constraint is $2w + l = 180$. Then $l = 180 - 2w$. I can then make this substitution into the area formula.
>
> $A(w) = w(180 - 2w) = -2w^2 + 180w$.
>
> Now $A'(w) = -4w + 180$, so the solution to $A'(w) = 0$ is $w = 45$.
>
> $A''(45) < 0$ and it is the only critical point, so $(45, A(45)) = (45, 4050)$ is the absolute maximum.

20. Differentiate $f(x) = (e^{2x} + 50x)^5$

> **(*)** Here is an extended power rule problem. Do you still remember the extended power rule? I multiply the derivative of the inside by derivative of the outside.
>
> So let's find the derivative of the inside, $e^{2x} + 50x$. The derivative of e^{2x} is itself multiplied by the derivative of the exponent, and the derivative of $50x$ is 50. So the derivative of the inside is $(2e^{2x} + 50)$. Multiply that by the derivative of the outside, $5(e^{2x} + 50x)^4$, to get $(2e^{2x} + 50)5(e^{2x} + 50x)^4 = (10e^{2x} + 250)(e^{2x} + 50x)^4$

21. Integrate.

a. $\int x^2 + 2x + 4dx$

(*) Let's reverse the power rule!

Recall, for derivatives:
Step 1: Multiply coefficient by power
Step 2: Drop power by 1

So for integrals I will:
Step 1: Add 1 to the power
Step 2: Divide coefficient by power

And of course,
Step 3: Add C!
So after adding 1 to the power, I have $x^3 + 2x^2 + 4x$ (Note what happened to the constant 4. It is now 4x).
Then I divide by the power, to obtain $\frac{1}{3}x^3 + x^2 + 4x$.
Finally
$\int x^2 + 2x + 4dx = \frac{1}{3}x^3 + x^2 + 4x + C$

b. $\int \frac{15}{x}dx$

(*) Note $\frac{15}{x} = 15\frac{1}{x}$.

So I write $\int \frac{15}{x}dx = 15\int \frac{1}{x}dx$, and by defintion, $= 15\ln|x| + C$.

c. $\int_0^1 3x^2 - 5xdx$

(*) "Little f" is $3x^2 - 5x$.

So $F(x) = x^3 - \frac{5}{2}x^2$. (Note, I don't need "+C" for definite integrals).

Then $\int_0^1 3x^2 - 5xdx = F(1) - F(0) = x^3 - \frac{5}{2}x^2|_0^1 = (1)^3 - \frac{5}{2}(1)^2 - 0 + 0 = -\frac{3}{2}$.

d. $\int \sqrt{x} + \frac{1}{\sqrt{x}}dx$.

(*) Writing the integral as $\int x^{\frac{1}{2}} + x^{-\frac{1}{2}}dx$ helps me see that I should use the power rule for integration. Then it follows:

$\int x^{\frac{1}{2}} + x^{-\frac{1}{2}}dx = \frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + C$.

22. Find the area under the curve $f(x) = x^2 + 1$, bounded on the left by $x = 0$ and on the right by $x = 5$.

(*) The area described can be found by computing $\int_0^5 x^2+1\,dx = \frac{1}{3}x^3+x|_0^5 = \frac{1}{3}\times 125+5-0 = \frac{140}{3}$.

23. Find $f(x)$ such that $f'(x) = x - 6$ and $f(1) = 5$.

(*) I want to know the function who's derivative is $x - 6$. In other words, I want the *anti-derivative* of $x - 6$. So:

$f(x) = \int x - 6dx = \frac{1}{2}x^2 - 6x + C$.

Now $f(1) = 5$, so $5 = \frac{1}{2} - 6 + C$, or $C = 10.5$.

Then $f(x) = \frac{1}{2}x^2 - 6x + 10.5$.