MA121 Elements of Calculus
Exam 1 Review Questions

September 7, 2008

1. Definitions and Concepts.

a. Give an example of an equation in which the value of y varies inversely with the value of x.

> **(*)** One example: $f(x) = \frac{k}{x}$

b. Give an example of a function whose domain is $\{x|x \neq 4\}$.

> **(*)** One example: $f(x) = \frac{k}{x-4}$

c. Give an example of a function whose domain is $\{x|x \geq 4\}$.

> **(*)** One example: $f(x) = \sqrt{x-4}$

d. Give an example of a rational function which has a vertical asymptote at $x = 3$.

> **(*)** One example: $f(x) = \frac{1}{x-3}$

e. Give an example of a graph of a function which is continuous on the interval $[0, 5]$.

> **(*)** One example: $f(x) = x$

f. What is the difference between continuity at a point and continuity over an interval?

> **(*)** If $f(x)$ is continuous over an interval $[a, b]$, then $f(x)$ is continuous at every point in the interval $[a, b]$.
>
> Another way to explain it:
> If $f(x)$ is continuous over an interval $[a, b]$, then I can trace the graph of the function from $x = a$ to $x = b$ without lifting my pen. If $f(x)$ is continuous at a point c, then it is possible for me to trace the graph of the function from the left side of c to the right side without lifting my pen.

g. If the limit as x approaches c from the left is the same as the limit as x approaches c from the right, what can I say about the general limit?

> **(*)** The limit exists and its value is the same as that of the two given limits.

h. True or False. If true, give a short statement as to why. If false, provide a counter-example. If a function is continuous at $x = 0$, then it must be continous on the interval $(-1, 1)$.

(*) False. See f for a discussion about the differences of continuity at a point and continuity over an interval.

For a counter-example, look at the graph of $f(x) = \frac{1}{x-\frac{1}{2}}$. The function is continuous at $x = 0$. But not at $x = \frac{1}{2}$. To see this, try to draw a segment of the graph of $f(x)$ from the left side of $x = \frac{1}{2}$ to the right side without lifting your pen. You should see this is impossible, because of the vertical asymptote.

This means, since there is at least one "break" in the graph in the interval $(-1, 1)$, the function is not continuous over that interval.

2. **a.** Simplify $\frac{(x^2y^3)^2}{(x^3y^2)^3}$

(*) $\frac{1}{x^5}$ or x^{-5}

b. Simplify $(x^{-2}y^{-2})^3(x^2y^3)^2$

(*) $\frac{1}{x^2}$ or x^{-2}

d. Express $(x-3)(x+2)$ as a quadratic of the form $Ax^2 + Bx + C$. What is the degree of the resulting polynomial? What are the x and y intercepts?

(*) $f(x) = x^2 - x - 6$
$x_1 = 3$
$x_2 = -2$
Degree = 2

e. Simplify $\frac{x^2+5x+6}{x^2+7x+12}$. What is the domain of the function? What are the locations of any asymptotes or holes?

(*) $\frac{x^2+5x+6}{x^2+7x+12} = \frac{(x+2)(x+3)}{(x+3)(x+4)} = \frac{x+2}{x+4}$
Domain = $\{x | x \neq -3, x \neq -4\}$
Because the term $(x + 3)$ is in the denominator, but I factored it out by simplifying, solving the term for 0 gives the location of a *hole*. $x + 3 = 0 \Rightarrow x = -3$.
Because the term $(x + 4)$ is in the denominator, but couldn't be factored out, solving the term for 0 gives the location of a *vertical asymptote*. $x + 4 = 0 \Rightarrow x = -4$.

Now notice the degree of the numerator and denominator are the same. So $\lim_{x\to 0} f(x) = \frac{a_n}{b_n}$ where a_n is the coefficient of the dominant term of the numerator, and b_n is the coefficient of the dominant term of the denominator. Then $\lim_{x\to 0} f(x) = \frac{1}{1} = 1$. So as we move further to the right, the graph of $f(x)$ moves closer to $y = 1$. This describes a horizontal asymptote at $y = 1$.

3. Consider the function $f(x) = x^2 - 1$.

a. Compute $f(-2)$.

(*) $f(-2) = (-2)^2 - 1 = 4 - 1 = 3$

b. Compute the difference quotient, $\frac{f(x+h)-f(x)}{h}$.

(*)

$$\begin{aligned} \frac{f(x+h)-f(x)}{h} &= \frac{(x+h)^2-1-(x^2-1)}{h} \\ &= \frac{x^2+2hx+h^2-1-x^2+1}{h} \\ &= \frac{2hx+h^2}{h} \\ &= \frac{h(2x+h)}{h} \\ &= 2x+h \end{aligned}$$

c. What is the average rate of change as x changes from 2 to 7?

(*)

$$\begin{aligned} \frac{f(b)-f(a)}{b-a} &= \frac{f(7)-f(2)}{7-2} \\ &= \frac{48-3}{5} \\ &= 9 \end{aligned}$$

d. Compute $\lim_{x\to 0} f(x)$. Is $f(x)$ continuous at $x = 0$? Why or why not?

(*) $\lim_{x\to 0} x^2 - 1 = -1$

Continuous at $x = 0$? Yes.

Why? Because $f(0) = -1$, then $\lim_{x\to 0} f(x) = f(0)$. Therefore, $f(x)$ is continuous at $x = 0$.

4. Find the equilibrium point for the given demand and supply functions. Consider only values which make sense with respect to the application.

$$D(p) = -p + 13$$
$$S(p) = p^2 - p - 3$$

(*)

$$\begin{aligned} D(p) &= S(p) \\ -p+13 &= p^2 - p - 3 \\ 16 &= p^2 \\ p &= \pm 4 \end{aligned}$$

Only the solution $p = 4$ makes sense with respect to the application. A negative price (-4) is not in the (restricted) domain.

5. Consider the piecewise function defined below.

$$f(x) = \begin{cases} x+2 & \text{if } x \neq 4; \\ 8 & \text{if } x = 4. \end{cases}$$

a. Evaluate $f(4)$.

(*) $f(4) = 8$

b. Evaluate $\lim_{x\to 4} f(x)$.

(*) $\lim_{x\to 4} f(x) = 6$

c. Is $f(x)$ continuous at $x = 3$? What about $x = 4$?

> **(*)** Continuous at $x = 3$? Yes. $\lim_{x\to 3} f(x) = f(3)$
>
> Continuous at $x = 4$? No. $\lim_{x\to 4} f(x) \neq f(4)$

d. Is $f(x)$ continuous on the interval $[0, 10]$? Why or why not?

> **(*)** No. $f(x)$ must be continuous at *all* points in the interval to be continuous on the interval, but $f(x)$ is not continuous at $x = 4$.

6. Evaluate.

a. $\lim_{x\to 1^-} \frac{1}{x-1}$

> **(*)** The rational is in reduced form, so there is no "removable discontinuity." This means there is a vertical asymptote at $x = 1$.
>
> We approach $x = 1$ starting on the left side. As the value of x moves to the right (increases) and closer to the asymptote, the value of $f(x)$ decreases. Because what we are moving toward is a vertical asymptote, the value of $f(x)$ will keep decreasing without bound. Hence, the limit is $-\infty$.

b. $\lim_{x\to 1} \frac{1}{x-1}$

> **(*)** As we approach $x = 1$ starting on the right side, and move to the left (x decreases), then $f(x)$ increases. So the limit is ∞. The left and right limits do not agree, so the general limit does not exist.

c. $\lim_{x\to \infty} \frac{1}{x-1}$

> **(*)** The degree of the numerator is less than the degree of the denominator, so the limit is 0.

d. $\lim_{x\to 4^+} \frac{x^2-16}{x-4}$

> **(*)**
>
> $$\begin{aligned}\lim_{x\to 4} \frac{x^2-16}{x-4} &= \lim_{x\to 4} \frac{(x-4)(x+4)}{x-4} \\ &= \lim_{x\to 4} x+4 \\ &= 8\end{aligned}$$
>
> After writing the rational in reduced form I end up with a polynomial. Polynomials are continuous, so the left and right side limits must match. Then $\lim_{x\to 4^+} = \lim_{x\to 4} = 8$.

e. $\lim_{h\to 0} hx^2 + h^2x + x + 3$

> **(*)** $x + 3$

f. $\lim_{x\to \infty} \frac{5x^2-19x+2}{x^2-3}$

> **(*)** The degrees of the numerator and denominator match. I divide the leading coefficients.
>
> $\lim_{x\to \infty} = \frac{5}{1} = 5$.

7. Watson Enterprises is considering producing and selling cat-proof keyboards. They plan to sell each keyboard for 40 dollars. Materials for each keyboard cost 20 dollars, and the machines to produce the keyboards cost $100,000$ dollars. The machines must only be purchased once.

a. Set up the linear function $C(x)$ which gives the total cost of producing x keyboards.

(*) $C(x) = 20x + 100000$.

The cost per unit is the cost of the materials plus an equal fraction of the cost of the machine. So, for instance, if five keyboards are produced, the cost per keyboard is 20 dollars plus one-fifth of the cost of the machine ($20,000$ dollars). So the cost per unit if x units are produced is given by

$$M(x) = \frac{20x+100000}{x}$$

b. What is the cost per unit if 10 keyboards are produced?

(*) $M(10) = \frac{20(10)+100000}{10} = \frac{100200}{10} = 10020$.

c. Compute $\lim_{x\to\infty} M(x)$.

(*) Notice the degree of the numerator and denominator are the same. So $\lim_{x\to 0} M(x) = \frac{a_n}{b_n}$ where a_n is the coefficient of the dominant term of the numerator (20), and b_n is the coefficient of the dominant term of the denominator (1). Then $\lim_{x\to 0} M(x) = \frac{20}{1} = 20$.

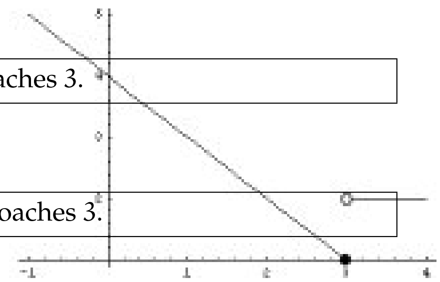
d. What does your answer to (c) tell us about the cost per unit?

(*) It tells us a few important things. The key results are:

1. The lowest cost per unit is 20 dollars.

2. The cost per unit decreases *to 20 dollars* as x increases. What do you suppose this suggests about new technologies which don't yet have strong markets?

8. The graph of $f(x)$ is given to the right.



a. Evaluate $\lim_{x \to 3^-} f(x)$.

(*) Coming in from the left, $f(x)$ approaches 0 as x approaches 3.

b. Evaluate $\lim_{x \to 3^+} f(x)$.

(*) Coming in from the right, $f(x)$ approaches 1 as x approaches 3.

c. Evaluate $\lim_{x \to 3} f(x)$.

(*) The limit does not exist because the limit approaching from the left does not match the limit approaching from the right.

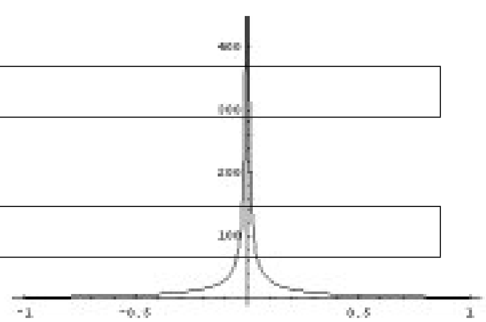
d. Is $f(x)$ continuous at $x = 3$?

(*) $f(x)$ is not continuous at $x = 3$ because the limit as x approaches 3 does not exist.

e. Is $f(x)$ continuous on the interval $[-1, 2]$?

(*) Yes. A simple and quick way to see this is because I can draw the portion of the graph from $x = -1$ to $x = 2$ without lifting my pen. Another explanation is that I cannot find a point in the interval such that $f(x)$ is not continuous at that point.

The graph of $g(x)$ is given to the right.



f. Evaluate $\lim_{x \to 0^-} g(x)$.

(*) $\lim_{x \to 0^-} g(x) = \infty$, or "DNE" (does not exist).

g. Evaluate $\lim_{x \to 0} g(x)$.

(*) $\lim_{x \to 0} g(x) = \infty$, or "DNE" (does not exist).

h. Is $g(x)$ continuous on the interval $[-1, 1]$?

(*) No, because $g(x)$ is not continuous at $x = 0$. I know $g(x)$ is not continuous at that point because the limit as x approaches 0 (from either side) does not exist.