MA131 Calculus for Life and Management Sciences A
Exam 1 Review Questions

October 4, 2009

1. Find the first and second derivatives for each of the following functions.

a. corrected $f(x) = x^2 + 2x + 1$

> **(*)** $f'(x) = 2x + 2$.
>
> $f''(x) = 2$

b. $f(x) = \sqrt{x}$

> **(*)** $f(x) = x^{\frac{1}{2}}$.
>
> $f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$.
>
> $f''(x) = -\frac{1}{4}x^{-\frac{3}{2}}$.

c. $f(x) = \frac{1}{\sqrt{x}}$

> **(*)** $f(x) = x^{-\frac{1}{2}}$.
>
> $f'(x) = -\frac{1}{2}x^{-\frac{3}{2}}$.
>
> $f'(x) = \frac{3}{4}x^{-\frac{5}{2}}$.

d. $f(x) = (x^2 + 2x + 5)^8$

> **(*)** $f'(x) = 8(x^2 + 2x + 5)^7(2x + 2)$.
>
> We haven't discussed what we need to compute $f''(x)$.

e. $f(x) = (2x + 7)^{-3}$

> **(*)** $f'(x) = -6(2x + 7)^{-4}$
>
> $f''(x) = 48(2x + 7)^{-5}$

f. $f(x) = \frac{1}{\sqrt{2x+1}}$

> **(*)** $f(x) = (2x+1)^{-\frac{1}{2}}$.
>
> $f'(x) = -(2x+1)^{-\frac{3}{2}}$.
>
> $f''(x) = 3(2x+1)^{-\frac{5}{2}}$.

2. Let $f(3) = 4$ and $f'(3) = 6$. Let $g(3) = 7$ and $g'(3) = 2$. Determine $h(3)$ where $h(x)$ is defined as follows:

a. $h(x) = f(x) + g(x)$.

> **(*)** $h(3) = f(3) + g(3) = 4 + 7 = 11$
>
> $h'(x) = f'(x) + g'(x)$
>
> $h'(3) = f'(3) + g'(3) = 6 + 2 = 8$

b. $h(x) = (f(x))^3$.

> **(*)** $h(3) = (f(3))^3 = (4)^3 = 64$
>
> $h'(x) = 3(f(x))^2 f'(x)$
>
> $h'(3) = 3(f(3))^2 f'(3) = 3(4^2)6 = 288$

c. $h(x) = \frac{1}{g(x)}$.

> **(*)** $h(3) = \frac{1}{g(3)} = \frac{1}{7}$
>
> $h(x) = (g(x))^{-1}$.
>
> $h'(x) = -1(g(x))^{-2} g'(x)$
>
> $h'(3) = -1(g(3))^{-2} g'(3) = -1(7)^{-2} 2 \approx -0.0408163265$

d. $h(x) = (f(x) + g(x))^3$.

> **(*)** $h(3) = (f(3) + g(3))^3 = (4+7)^3 = 11^3 = 1331$
>
> $h'(x) = 3(f(x) + g(x))^2 (f'(x) + g'(x))$.
>
> $h'(3) = 3(f(3) + g(3))^2 (f'(3) + g'(3)) = 3(121)(8) = 2904$.

3. Evaluate the limits that exist. If the limit does not exist, write "DNE". You do not need to distinguish between "Infinity" and "DNE".

a. $\lim_{x \to 1} \frac{1}{x-1}$

(*) As we approach $x = 1$ starting on the right side, and move to the left (x decreases), then $f(x)$ increases. So the limit is ∞. The left and right limits do not agree, so the general limit does not exist.

b. $\lim_{x\to\infty} \frac{1}{x-1}$

(*) The degree of the numerator is less than the degree of the denominator, so the limit is 0.

c. $\lim_{x\to 4^+} \frac{x^2-16}{x-4}$

(*)

$$\begin{aligned}\lim_{x\to 4} \frac{x^2-16}{x-4} &= \lim_{x\to 4} \frac{(x-4)(x+4)}{x-4} \\ &= \lim_{x\to 4} x+4 \\ &= 8\end{aligned}$$

After writing the rational in reduced form I end up with a polynomial. Polynomials are continuous, so the left and right side limits must match. Then $\lim_{x\to 4^+} = \lim_{x\to 4} = 8$.

d. $\lim_{h\to 0} hx^2 + h^2x + x + 3$

(*) $x + 3$

e. $\lim_{x\to\infty} \frac{5x^2-19x+2}{x^2-3}$

(*) The degrees of the numerator and denominator match. I divide the leading coefficients.

$\lim_{x\to\infty} = \frac{5}{1} = 5$.

f. $\lim_{x\to\infty} \frac{x^3+9x^2+5}{765942x^2+34245343x+10000000000}$

(*) The degree of the numerator is greater than that of the denominator. So the limit is ∞.

g. $\lim_{x\to 0} |x|$.

(*) Let x approach 0 from the left.

x	$\|x\|$
$-.1$	$.1$
$-.01$	$.01$
$-.001$	$.001$

Hence, $\lim_{x \to 0^-} |x| = 0$.

Now let x approach 0 from the right.

x	$\|x\|$
$.1$	$.1$
$.01$	$.01$
$.001$	$.001$

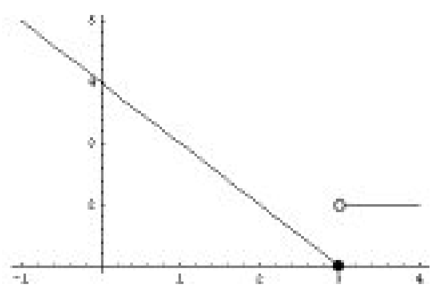
Hence, $\lim_{x \to 0^+} |x| = 0$.

The two sides match. The general limit exists. $\lim_{x \to 0} |x| = 0$.

h. $\lim_{x \to 1} \sqrt{x^2 + x - 6}$

(*) I can't simplify this expression, so I plug in 1 to get $\sqrt{1 + 1 - 6} = \sqrt{-4}$.

This is not a real number, so the limit does not exist.



4. The graph of $f(x)$ is given to the right.

a. Evaluate $\lim_{x \to 3^-} f(x)$.

(*) Coming in from the left, $f(x)$ approaches 0 as x approaches 3.

b. Evaluate $\lim_{x \to 3^+} f(x)$.

(*) Coming in from the right, $f(x)$ approaches 1 as x approaches 3.

c. Evaluate $\lim_{x\to 3} f(x)$.

(*) The limit does not exist because the limit approaching from the left does not match the limit approaching from the right.

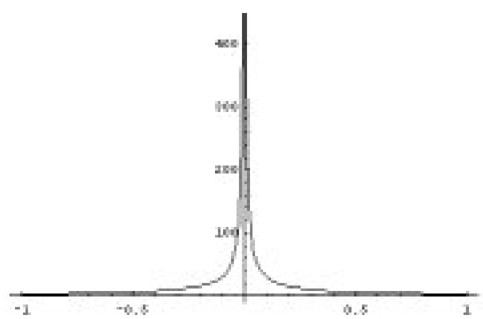
d. Is $f(x)$ continuous at $x = 3$?

(*) $f(x)$ is not continuous at $x = 3$ because the limit as x approaches 3 does not exist.

e. Is $f(x)$ continuous on the interval $[-1, 2]$?

(*) Yes. A simple and quick way to see this is because I can draw the portion of the graph from $x = -1$ to $x = 2$ without lifting my pen. Another explanation is that I cannot find a point in the interval such that $f(x)$ is not continuous at that point.

The graph of $g(x)$ is given to the right.



f. Evaluate $\lim_{x\to 0^-} g(x)$.

(*) $\lim_{x\to 0^-} g(x) = \infty$, or "DNE" (does not exist).

g. Evaluate $\lim_{x\to 0} g(x)$.

(*) $\lim_{x\to 0} g(x) = \infty$, or "DNE" (does not exist).

h. Is $g(x)$ continuous on the interval $[-1, 1]$?

(*) No, because $g(x)$ is not continuous at $x = 0$. I know $g(x)$ is not continuous at that point because the limit as x approaches 0 (from either side) does not exist.

5. Watson Enterprises is considering producing and selling cat-proof keyboards. They plan to sell each keyboard for 40 dollars. Materials for each keyboard cost 20 dollars, and the machines to produce the keyboards cost 100,000 dollars. The machines must only be purchased once.

a. Set up the linear function $C(x)$ which gives the total cost of producing x keyboards.

(*) $C(x) = 20x + 100000$.

The cost per unit is the cost of the materials plus an equal fraction of the cost of the machine. So, for instance, if five keyboards are produced, the cost per keyboard is 20 dollars plus one-fifth of the cost of the machine (20,000 dollars). So the cost per unit if x units are produced is given by

$$M(x) = \frac{20x+100000}{x}$$

b. What is the cost per unit if 10 keyboards are produced?

(*) $M(10) = \frac{20(10)+100000}{10} = \frac{100200}{10} = 10020.$

c. Compute $\lim_{x\to\infty} M(x)$.

(*) Notice the degree of the numerator and denominator are the same. So $\lim_{x\to 0} M(x) = \frac{a_n}{b_n}$ where a_n is the coefficient of the dominant term of the numerator (20), and b_n is the coefficient of the dominant term of the denominator (1). Then $\lim_{x\to 0} M(x) = \frac{20}{1} = 20.$

d. What does your answer to (c) tell us about the cost per unit?

(*) It tells us a few important things. The key results are:

1. The lowest cost per unit is 20 dollars.

2. The cost per unit decreases *to 20 dollars* as x increases. What do you suppose this suggests about new technologies which don't yet have strong markets?

e. Give a model for the marginal cost as a function of x.

(*) $C'(x) = 20$

6. Suppose I am producing cats. When I charge 20 dollars for a bag, I sell 30 units. When I charge 40 dollars for a bag, I sell 15 units.

a. Write down the *slope* and *y-intercept* of the linear *demand equation*, $D(p)$, which gives the number of sales as a function of the price, p.

(*) Units sold y is a function of price p. I have the points $(20, 30)$ and $(40, 15)$.

Slope is $\frac{y_2-y_1}{p_2-p_1} = \frac{15-30}{40-20} = -\frac{3}{4}.$

$D(p) = mp + b$. $m = -\frac{3}{4}$.

$D(p) = -\frac{3}{4}p + b$. When $p = 20$ then $D(p) = 30$.

$30 = -\frac{3}{4}(20) + b$

$b = 45.$

$D(p) = -\frac{3}{4}p + 45.$

b. Write down a model giving the *revenue* (product of sales and price) as a function of price.

(*) As revenue is the product of price and sales, I have $R = pD(p) = p(-\frac{3}{4}p + 45) = -\frac{3}{4}p^2 + 45p$.

c. Write down a model for the marginal revenue.

(*) $R'(p) = -\frac{3}{2}p + 45$

d. Suppose I charge 19 dollars per bag. Use your model from (c) to determine if I would increase or decrease revenue by raising the price.

(*) $R'(19) = \frac{33}{2} > 0$, so at $p = 19$, the revenue function is *increasing* as price increases. So I should expect revenues to decrease.

7. Consider the piecewise function defined below.

$$f(x) = \begin{cases} x + 2 & \text{if } x \neq 4; \\ 8 & \text{if } x = 4. \end{cases}$$

a. Evaluate $f(4)$.

(*) $f(4) = 8$

b. Evaluate $\lim_{x \to 4} f(x)$.

(*) $\lim_{x \to 4} f(x) = 6$

c. Is $f(x)$ continuous at $x = 3$? What about $x = 4$?

(*) Continuous at $x = 3$? Yes. $\lim_{x \to 3} f(x) = f(3)$ Continuous at $x = 4$? No. $\lim_{x \to 4} f(x) \neq f(4)$

d. Is $f(x)$ continuous on the interval $[0, 10]$? Why or why not?

(*) No. $f(x)$ must be continuous at *all* points in the interval to be continuous on the interval, but $f(x)$ is not continuous at $x = 4$.

8. Let $f(x) = x^3 - 12x + 16$.

a. Locate the relative extrema on $f(x)$.

(*) $f'(x) = 3x^2 - 12$

Setting $f'(x) = 0$, I get:

$$\begin{aligned} 3x^2 - 12 &= 0 \\ 3x^2 &= 12 \\ x^2 &= 4 \\ x &= \pm 2 \end{aligned}$$

b. On what interval(s) is $f(x)$ increasing?

(*) I divide the real number line into three intervals, with the divisions being the critical points.

For $(-\infty, -2)$, $f'(-5) > 0$. Hence, $f(x)$ is increasing on this interval.

For $(-2, 2)$, $f'(0) < 0$. Hence, $f(x)$ is decreasing on this interval.

For $(2, \infty)$, $f'(5) > 0$. Hence, $f(x)$ is increasing on this interval.

c. Locate the inflection points of $f(x)$.

(*) $f''(x) = 6x$.

Solve $f''(x) = 0$ to get $x = 0$.

d. On what interval(s) is $f(x)$ concave up?

(*) To the right of $x = 0$: $f''(-5) < 0$. Hence, $f(x)$ is concave down on the interval $(-\infty, 0)$.

To the left of $x = 0$: $f''(5) > 0$. Hence, $f(x)$ is concave up on the interval $(0, \infty)$.

9. The graph of $g(x)$ is given below.

a. At which points is $g(x)$ *not* continuous?

(*) B

b. At which points is $g(x)$ *not* differentiable?

(*) B, D

c. At which points is the slope of the line tangent to $g(x)$ horizontal?

(*) A, C, E, F, G

d. Which points, if any, are the critical points?

(*) A, B, C, D, E, F, G

e. Which points, if any, are relative extrema?

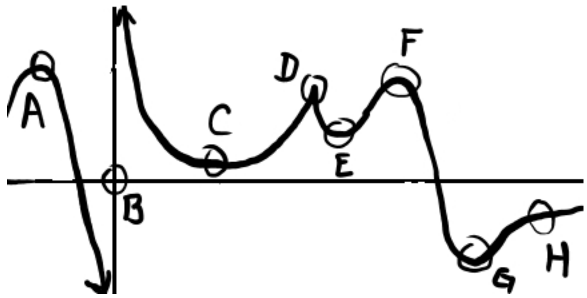
(*) A, C, E, F, G

f. Which points, if any, are relative maxima?

(*) A, F

g. Which point, if any, is the relative minima?

(*) C, E, G


A
B
C
D
E
F
G
H