MA131 Calculus for Life and Management Sciences A
Exam 1 Review Questions - Solution (Revision 1)

September 13, 2009

(*) *Typos / Errors corrected*

12 **Sept** 2009 Problem 7 Solution

1. Given the difference equation $y_{n+1} = -2y_n + 5, y_0 = 1$, compute y_1, y_2, and y_3.

   (*) $y_1 = -2y_0 + 5$
   $y_1 = -2(1) + 5 = 3$

   $y_2 = -2y_1 + 5$
   $y_2 = -2(3) + 5 = -1$

   $y_3 = -2y_2 + 5$
   $y_3 = -2(-1) + 5 = 7$

2. Solve the above difference equation. (Find y_n)

   (*) $y_n = \frac{b}{1-a} + a^n(y_0 - \frac{b}{1-a})$.

   Note $a = -2$ and $b = 5$. So

   $y_n = \frac{5}{3} + (-2)^n(1 - \frac{5}{3})$.

3. Find y_{20}

   (*) Using the above equation, $y_{20} = \frac{5}{3} + (-2)^{20}(1 - \frac{5}{3}) = -669049$

4. Is the difference equation oscillating or monotonic?

   (*) $a < 0$, so it is oscillating.

5. Is the difference equation bounded or unbounded?

   (*) $|a| > 1$, so it is unbounded.

6. 1000 dollars is deposited into an account paying 5% interest, compounded annually. A deposit of 200 dollars is made at the end of each year. What is the difference equation that describes y_{n+1} in terms of y_n?

   (*) Note that a is $1 + \frac{\text{interest rate}}{\text{number of compounds per year}} = 1 + \frac{.05}{1} = 1.05$.
   $y_{n+1} = (1.05)y_n + 200, y_0 = 1000$.

7. Solve the difference equation. (Find y_n)

   (*) $y_n = \frac{200}{-.05} + (1.05)^n(1000 - \frac{200}{-.05})$.

   Which gives

   $y_n = -4000 + (1.05)^n(5000)$.

8. Compute the balance after 20 years

   (*) $y_{20} = -4000 + (1.05)^{20}(5000) = 9266.49$.

9. Is the difference equation oscillating or monotonic?

   (*) $a > 0$, so it is monotonic. This is the behavior we'd expect with a growing bank account.

10. Is the difference equation bounded or unbounded?

    (*) $|a| > 1$, so it is unbounded. Again, this is the behavior we'd expect with a growing bank account.

11. Is the difference equation constant?

    (*) A way to check if it is constant is to compute $\frac{b}{1-a}$. If this value equals y_0, then it is constant. However, this value is -4000. But y_0 is 1000. So it is not constant.

12. Write the appropriate equation and find the expected value after 2 years, assuming you deposit 10,000 dollars intially and 10% interest, compounded monthly, with no regular deposits.

    (*) First note that a is $1 + \frac{\text{interest rate}}{\text{number of compounds per year}} = 1 + \frac{.1}{12} \approx 1.00833$.

    Then we have the equation $y_{n+1} = 1.00833y_n + 0, y_0 = 10000$.

    After 2 years, 24 compounding periods have passed.

    Hence, after 24 periods, $y_{24} = \frac{0}{-.00833} + (1.00833)^{24}(10000 - \frac{0}{-.00833}) = 12202.90$.

13. Write the appropriate equation and find the expected value after 2 years, assuming you deposit 10,000 dollars intially and 10.5% interest, compounded quarterly, with a regular deposit of 1000 per quarter.

(*) First note that a is $1 + \frac{\text{interest rate}}{\text{number of compounds per year}} = 1 + \frac{.105}{4} = 1.02625$.

Then we have the equation $y_{n+1} = 1.02625y_n + 1000, y_0 = 10000$.

After 2 years, 8 compounding periods have passed.

Hence, after 8 periods, $y_8 = \frac{1000}{-.02625} + (1.02625)^8(10000 - \frac{1000}{-.02625}) = 21078.30$.

14. For the difference equation $y_{n+1} = -3y_n + 4, y_0 = 1$, determine if it is

Constant
Monotonic
Oscillating
Bounded
Unbounded

(*) $\frac{b}{1-a} = \frac{4}{1-(-3)} = 1$, which is precisely y_0. This means the equation is constant.

A constant equation is not oscillating. It is also bounded.

15. For the difference equation $y_{n+1} = -\frac{1}{2}y_n - 8, y_0 = 3$, determine if it is

Constant
Monotonic
Oscillating
Bounded
Unbounded

(*) $\frac{b}{1-a} = \frac{-8}{1-(-\frac{1}{2})} = -\frac{16}{3}$, which is not y_0. This means the equation is not constant.

$a < 0$, so the equation is Oscillating.

$|a| < 1$, so the equation is Bounded.

16. For the difference equation $y_{n+1} = 5y_n + 2, y_0 = 0$, determine if it is

Constant
Monotonic
Oscillating
Bounded
Unbounded

(*) $\frac{b}{1-a} = \frac{2}{1-5} = -2$, which is not y_0. This means the equation is not constant.

$a > 0$, so the equation is Monotonic.

$|a| > 1$, so the equation is Unbounded.

17. For the difference equation $y_{n+1} = \frac{3}{4}y_n + 5$, $y_0 = 10$, determine if it is

    Constant

    Monotonic

    Oscillating

    Bounded

    Unbounded

    (*) $\frac{b}{1-a} = \frac{5}{1-(\frac{3}{4})} = 20$, which is not y_0. This means the equation is not constant.

    $a > 0$, so the equation is Monotonic.

    $|a| < 1$, so the equation is Bounded.

18. How much money can you borrow at 12% interest, compounded monthly, if the loan is to be paid off in exactly 10 years with a payment of 660 per month?

    (*) First, set up the parameters.
    a is $1 + \frac{\text{interest rate}}{\text{number of compounds per year}} = 1 + \frac{.12}{12} = 1.01$.
    b is -660. (Remember, it is a payment. This means we are subtracting from the balance. So it is negative)
    y_0 is the amount we borrow. We're looking for this value.
    y_{120} is our "goal." This value is 0 because we want the loan paid off after 10 years, or 120 compounding periods.

    Next, set up the equation.

    $y_n = \frac{b}{1-a} + a^n(y_0 - \frac{b}{1-a})$.

    $0 = \frac{-660}{1-1.01} + (1.01)^{120}(y_0 - \frac{-660}{1-1.01})$.

    Finally, solve for the unknown.

    $0 = \frac{-660}{-.01} + 3.30039(y_0 - \frac{-660}{-.01})$.

    $0 = 66000 + 3.30039(y_0 - 66000)$.

    $0 = 66000 + 3.30039y_0 - 217826)$.

    $0 = 3.30039y_0 - 151826$

    $151826 = 3.30039y_0$

    $y_0 = 46002.40$

19. If you borrow 10,000 dollars at 10% interest, compounded monthly, and want to pay the loan off in 20 years, how much will you have to pay every month?

(*) First, set up the parameters.
a is $1 + \frac{\text{interest rate}}{\text{number of compounds per year}} = 1 + \frac{.12}{12} = 1.01$.
b is unknown.
y_0 is the amount we borrow. This is 10000.
y_{240} is our "goal." This value is 0 because we want the loan paid off after 20 years, or 240 compounding periods.

Next, set up the equation.

$y_n = \frac{b}{1-a} + a^n(y_0 - \frac{b}{1-a})$.

$0 = \frac{b}{1-1.01} + (1.01)^{240}(10000 - \frac{b}{1-1.01})$.

Finally, solve for the unknown.

$0 = \frac{b}{1-1.01} + (1.01)^{240}(10000 - \frac{b}{1-1.01})$.

$0 = \frac{b}{-.01} + (1.01)^{240}(10000 - \frac{b}{-.01})$.

$0 = -100b + 10.8926(10000 + 100b)$.

$0 = -100b + 108926 + 1089.26b$.

$0 = 108926 + 989.26b$.

$-108926 = 989.26b$.

$b = -110.109$

Note that since b is a payment, we should expect a negative number.

20. If you were to borrow 10,000 dollars at 10% interest, compounded monthly, and want to pay off the loan eventually, how much should you have to pay every month?

(*) A difference equation is constant if $y_0 = \frac{b}{1-a}$. We know y_0 is 10000 and a is $1 + \frac{.1}{12} = 1.00833$. We don't know b. We want to find b so that the difference equation is constant. Then we'll know that if we don't pay at least this much, the loan will never be paid off. (If the equation is constant, the balance will never hit 0).

$10000 = \frac{b}{-.00833}$

$b = -83.3$.

We should pay at least 83.30 per month.

21. Given $f(x) = x^2 + 1$, Find the average rate of change of $f(x)$ as x moves from 1 to 3.

(*) The average rate of change is given by the slope of the secant line: $\frac{f(b)-f(a)}{b-a}$.

$\frac{f(b)-f(a)}{b-a} = \frac{f(3)-f(1)}{3-1} = \frac{10-2}{2} = 4$

22. Given $f(x) = 2x^2 + x$, Find the average rate of change of $f(x)$ as x moves from -1 to 1.

(*) The average rate of change is given by the slope of the secant line: $\frac{f(b)-f(a)}{b-a}$.

$\frac{f(b)-f(a)}{b-a} = \frac{f(1)-f(-1)}{1-(-1)} = \frac{3-1}{2} = 1$

23. Given $f(x)$ as defined above, find the slope of the secant line as x moves from -1 to 1.

(*) We want to find an equation of the form $y = mx + b$.

m is the slope of the secant line as x moves from -1 to 1. We already computed that above. It is 1.

So we have the equation $y = x + b$. To find b, we can plug in a point that is on the line and solve for it.

I know that when $x = 1$, then $y = f(1) = 3$. So I have $3 = 1 + b$. Solve for b to get $b = 2$.

$y = x + 2$.

24. Given $f(x) = x^2 + 2x + 1$, find $f'(x)$.

(*) Use the power rule to get $f'(x) = 2x^{2-1} + 2x^{1-1} + 0 = 2x + 2$.

25. Given $f(x) = \sqrt{x} + 1$, find $f'(x)$.

(*) Note that $\sqrt{x} = x^{\frac{1}{2}}$. Then I can use the power rule to get

$f'(x) = \frac{1}{2}x^{\frac{1}{2}-1} + 0$.
$f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$

26. Given $f(x) = \sqrt{5} + \pi$, find $f'(x)$.

(*) Both $\sqrt{5}$ and π are constants. The derivative of a constant function is 0.

$f'(x) = 0$

27. Compute $\frac{d}{dx}(x + \frac{1}{x})$.

(*) Write $\frac{1}{x}$ as x^{-1}.

$\frac{d}{dx}(x + x^{-1}) = 1 - (-1)x^{-1-1} = 1 - x^{-2}$

28. Compute $\frac{d}{dx}(\frac{1}{\sqrt{x}})$.

(*) Write $\frac{1}{\sqrt{x}}$ as $x^{-\frac{1}{2}}$.

$\frac{d}{dx}(x^{-\frac{1}{2}}) = -\frac{1}{2}x^{-\frac{1}{2}-1} = -\frac{1}{2}x^{-\frac{3}{2}}$

29. For $f(x) = x^2 + 9$, find $f'(2)$.

(*) First take the derivative. $f'(x) = 2x + 0 = 2x$.

Then $f'(2) = 2(2) = 4$.

30. Given $f(x)$ as defined above, compute the equation of the tangent line at $x = 2$.

(*) We're looking for the equation of a line, $y = mx + b$.

The slope, m, is the slope of the tangent line at $x = 2$, or $f'(2)$. We already computed this. $m = 4$.

So we have $y = 4x + b$. To find b, plug in a point and solve for it.

I know when $x = 2$, then $y = f(2) = 13$. Note that I used the *original* function, **not** its derivative!

Solve $13 = 4(2) + b$ for b to get $13 = 8 + b$, or $b = 5$.

$y = 4x + 5$

31. Given $f(x) = Ax^2 + Bx + C$, find $f'(x)$. For what value(s) of x is $f'(x) = 0$?

(*) Use the power rule to find the derivative. $f'(x) = 2Ax + B$.

Now let $f'(x) = 0$ and solve for x.

$0 = 2Ax + B$

$-B = 2Ax$

$x = -\frac{B}{2A}$.

Do you recognize this formula?