MA131 Calculus for Life and Management Sciences A
Exam 1 Form 12 Solution

September 21, 2009

Instructions: Show all work relevant to the solution of each problem. i.e. no credit will be given for "just the answers." Please do *all* work in the Blue Books! There are **six** problems which carry a total 100 points. You will have until the end of class to complete this exam. Good luck!

(25 pts) **Problem 1.** Consider the difference equation $y_{n+1} = 2y_n + 10$, $y_0 = 3$.

a. Is the difference equation constant? How do you know?

(*) It is not constant. It would be if the quantities $\frac{b}{1-a}$ and y_0 were equal. However, $\frac{10}{1-2} \neq 3$.

b. Is the difference equation oscillating or monotonic? How do you know?

(*) It is monotonic. The quick answer is to note that $a > 0$.

c. Is the difference equation bounded or unbounded? How do you know?

(*) It is unbounded. The quick answer is to note $|a| > 1$.

d. Compute the first two terms in the sequence, y_1 and y_2.

(*) $y_1 = 2(3) + 10 = 16$.

$y_2 = 2(16) + 10 = 42$

e. Write down the general solution of the difference equation, $y_n =$.

(*) $y_n = \frac{10}{1-2} + (2)^n(3 - \frac{10}{1-2})$

f. Compute y_{20}.

(*) $y_{20} = \frac{10}{1-2} + (2)^{20}(3 - \frac{10}{1-2}) = 13631478$

(25 pts) **Problem 2.** Ira wishes to set up a retirement account. At age 35 she makes an initial deposit of $\$100,000$. The account's annual interest rate is 12%, compounded monthly. She wants to determine what her monthly deposit should be in order to obtain $\$1,000,000$ by the time she is age 65.

a. Write down the explicit difference equation $y_{n+1} = ay_n + b$ pertaining to this scenario.

> **(*)** Note that annual interest is 12%, so monthly interest is 1%, or .01. Then $a = 1 + .01 = 1.01$.
>
> $y_{n+1} = (1.01)y_n + b$

b. Write down the general solution of the difference equation, $y_n =$.

> **(*)** $y_n = \frac{b}{-.01} + (1.01)^n(10000 - \frac{b}{-.01})$.

c. Determine her monthly deposit so that she reaches her goal in time.

> **(*)** After 30 years, or 360 compounding periods, we have the relation
>
> $1000000 = \frac{b}{-.01} + (1.01)^{360}(10000 - \frac{b}{-.01})$.
>
> This simplifies to
>
> $1000000 = -100b + (1.01)^{360}(10000 + 100b)$.
>
> $1000000 = -100b + 35.9496(10000 + 100b)$.
>
> $1000000 = -100b + 359496 + 3594.96b$.
>
> $1000000 = 359496 + 3494.96b$.
>
> $640504 = 3494.96b$.
>
> $b = 183.317$.

d. Now suppose Ira is 65 and has reached her goal. She has stopped her monthly contributions. Consider the same interest rate and compounding. What is the maximum she can *withdraw* per month so that her savings last indefinitely?

> **(*)** To last indefinitely, we want the difference equation to remain constant. So we solve
>
> $y_0 = \frac{b}{1-a}$.
>
> y_0 is 1000000, the amount of money we start with once she has reached her goal.
>
> b is unknown.
>
> a is still 1.01.
>
> Solving $1000000 = \frac{b}{-.01}$, or $1000000 = -100b$, we get $b = -10000$.

(15 pts) **Problem** 3. Consider the function $f(x) = x^2 - 2$.

a. Compute the *difference quotient* of $f(x)$

(*) The difference quotient is $\frac{f(x+h)-f(x)}{h}$.

$$\begin{aligned}
\frac{f(x+h)-f(x)}{h} &= \frac{(x+h)^2-2-(x^2-2)}{h} \\
&= \frac{(x+h)^2-2-x^2+2}{h} \\
&= \frac{x^2+2hx+h^2-2-x^2+2}{h} \\
&= \frac{2hx+h^2}{h} \\
&= \frac{h(2x+h)}{h} \\
&= 2x+h
\end{aligned}$$

b. Compute the *instantaneous rate of change* of $f(x)$ at $x = 1$.

(*)
The instantaneous rate of change of $f(x)$ at $x = 1$ is otherwise defined as $f'(1)$.

$f'(x) = 2x$. Then $f'(1) = 2(1) = 2$.

c. Consider the line tangent to $f(x)$ at $x = 1$. Write the equation of a line parallel to the tangent line, with y intercept 0.

(*) The line tangent to $f(x)$ at $x = 1$ has the slope $f'(1)$, which is 2. A line parallel to this line must have the same slope. So it has the equation $y = 2x + b$.

To find b, note that the y intercept is 0. This is the definition of the b parameter. So the line has the equation $y = 2x$.

(15 pts) **Problem** 4. Consider the function $f(x) = 3x^3 - 3x + 9$

a. Compute $f'(x)$.

(*) Using the power rule, $f'(x) = 9x^2 - 3$.

b. Compute $f'(3)$.

(*) Plug in $x = 3$ in the derivative to get $f'(3) = 9(3)^2 - 3 = 78$.

c. Compute the equation of the line tangent to $f(x)$ at $x = 3$.

(*) The slope of this tangent line is $f'(3)$, or 78. So we have the equation $y = 78x + b$.

Note that when $x = 3$, then $y = f(3) = 81$. Then $81 = 78(3) + b$.

Solving for b, we get $b = -153$. Then $y = 78x - 153$.

(10 pts) **Problem 5.** Find the derivative of the function $f(x) = \frac{1}{x^2} + \sqrt{x} + 4$.

(*) Write $f(x)$ as $f(x) = x^{-2} + x^{\frac{1}{2}} + 4$.

Then use the power rule to get $f'(x) = -2x^{-3} + \frac{1}{2}x^{-\frac{1}{2}}$.

(10 pts) **Problem 6.** Milton the Cat's height as he leaps off a bookshelf 5 feet tall can be given as a function of time: $h(t) = -.2t^2 + .9t + 5$.

a. Compute the instantaneous rate of change of his height as a function of time.

(*) The instantaneous rate of change is the derivative. $h'(t) = -.4t + .9$.

b. Determine the time at which he is neither gaining, nor losing height. That is, the time at which his instantaneous rate of change is zero.

(*) Solve the equation $-.4t + .9 = 0$ to get $t = \frac{9}{4}$.