MA 341
Introduction to Differential Equations
Chapter 4 Linear Second-Order Equations
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4.1 Introduction: The Mass-Spring Oscillator

Consider a mass attached to a spring. The spring is stretched and released, so the mass begins to move (1 dimension).
There are several forces that we need to consider:

(1) Restoring force of spring - proportional to displacement (Hooke's Law: $F_{spring} = -kx$)
(2) Friction/Damping - proportional to velocity ($F_{damping} = -bx'$)
(3) External "driving" force - may be non-existent ($F_{external}$)
(4) Total Force (Newton's Second Law: $F_{total} = mx''$)

From these we get $F_{total} = F_{spring} + F_{damping} + F_{external} \Rightarrow mx'' = -kx - bx' + F_{external}$ So we have:

$$mx'' + kx + bx' = F_{external}, \quad \text{A second-order, linear DE (and Homogeneous iff } F_{external} = 0)$$

Suppose no external force and negligible friction then $F_{external} = 0$ and $b = 0$ so we find $mx'' + kx = 0$
Solutions have form $x = c_1 \cos(\omega t) + c_2 \sin(\omega t)$ where $\omega = \sqrt{\frac{k}{m}}$ is the frequency.
These are sinusoidal oscillations that continue indefinitely.
Frequency increases for large k (stiff spring), small m (low mass).
Amplitude depends on initial conditions $x(0)$, $x'(0)$.

If friction is not negligible, but is small oscillations are damped out by friction.

If friction is large compared to mass and spring strength then the oscillations will not occur at all.

4.2 Homogeneous Linear Equations: The general solution

Suppose we have a second-order, linear, constant coefficient differential equation:

$$ay'' + by' + cy = f(t), \quad (a \neq 0)$$

We also have a special case when $f(t) = 0$, the homogeneous case. (i.e. A mass-spring oscillator vibrating freely, no external forces.)

By observation of the homogeneous case we see that a solution must have the property that its second derivative is expressible as a linear combination of its first and zeroth derivatives. This suggests that there is a solution of the form: $y = e^{rt}$. If we substitute this into the homogeneous equation we get:

$$ar^2e^{rt} + bre^{rt} + ce^{rt} = 0 \Rightarrow (ar^2 + br + c)e^{rt} = 0$$

Since e^{rt} is never zero we get that $ar^2 + br + c = 0$. This is called the auxiliary equation (also known as the characteristic equation). Hence we get that the roots are $r_1 = \frac{-b+\sqrt{b^2-4ac}}{2a}$ and $r_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$.

Example: Find two solutions to $y'' + 3y' + 2y = 0$.

This gives rise to three cases, when the discriminate is positive, negative, and zero. We explore the cases when the discriminate is positive and zero in this section. First a few definitions and theorems.

Note: Initial value problems for second-order linear constant coefficient homogeneous equations has the form:

$$ay'' + by' = cy = 0, \quad y(t_0) = Y_0, y'(t_0) = Y_1$$

Theorem: For any real numbers $a(\neq 0), b, c, t_0, Y_0, Y_1$ there exists a unique solution to the initial value problem

$$ay'' + by' + cy = 0; \quad y(t_0) = Y_0, y'(t_0) = Y_1$$

. The solution is valid for all t in $(-\infty, +\infty)$.

Definition: Two functions y_1 and y_2 are linear independent if they are **not** scalar multiples of each other.

For example $y_1 = 7x$ and $y_2 = 21x$ are linearly dependent since $y_2 = 3y_2$; however, if $y_2 = 21x^2$, y_1 and y_2 are linearly independent.

Theorem: If y_1 and y_2 are linearly independent solutions of the second-order linear homogeneous equation, then the general solution is given by

$$y(x) = c_1y_1(x) + c_2y_2(x)$$

where c_1 and c_2 are arbitrary constants.

In other words if we know two linearly independent solutions, then we know every solution.

Example: (# 14) Solve the IVP $y'' + y' = 0, \quad y(0) = 2, y'(0) = 1$

Theorem: Two equations y_1 and y_2 are linearly independent if their Wronskian is nonzero everywhere in the solution region; otherwise they are linearly dependent.

Definition: The Wronskian of y_1 and y_2 is defined to be

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = (y_1)(y_2') - (y_2)(y_1')$$

Recall there are three cases for these types of ODE's.
Let's first examine the case where the discriminate is positive. In this case we have two real roots to the characteristic equation, r_1 and r_2. This leads to 2 solutions $y_1 = e^{r_1 t}$ and $y_2 = e^{r_2 t}$ which are guaranteed to be linearly independent because $r_1 \neq r_2$. Check the Wronskian. So we have that $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ is the general solution in this case.

If the discriminate equals zero we get a repeated root. This leads to just one solution, $y_1 = e^{rt}$.
Let's try another solution $y_2 = te^{rt}$. Let's see if it is a solution and see if it is linearly independent from our first one.

First Linear independence:
$W = \begin{vmatrix} e^{rt} & te^{rt} \\ re^{rt} & rte^{rt} + e^{rt} \end{vmatrix} = e^{rt}(rte^{rt} + e^{rt}) - te^{rt}re^{rt} = e^{2rt} \neq 0$ Therefore y_2 and y_2 are linearly independent.
Now let's check that y_2 is a solution to the ODE.
$a(r^2te^{rt} + 2re^{rt}) + b(rte^{rt} + e^{rt}) + cte^{rt} = te^{rt}(ar^2 + br + c) + (2ar + b)e^rt = (2ar + b)e^{rt}$
Recall that $r = \frac{-b}{2a} \Rightarrow b = -2ar$
So we have $(2ar + b)e^{rt} = (2ar - 2ar)e^{rt} = 0$
Hence we have that y_2 is also a solution to the DE.

Example: Find a general solution to $y'' + 6y' + 9y = 0$

4.3 Auxiliary Equations with Complex Roots

Suppose we have $ax'' + bx' + cx = 0$ and the roots of the auxiliary equation are complex $\Rightarrow \quad b^2 + 4ac < 0$
Then the roots to the auxiliary equation will have the form $r = \alpha \pm i\beta$ where $\alpha = \frac{-b}{2a}$ and $\beta = \pm\frac{\sqrt{(4ac-b^2)}}{2a}$.
What will our solutions look like?

Recall Euler's equation: $e^{i\gamma} = \cos\gamma + i\sin\gamma$
Let's use the formula for the case with two distinct roots and see what happens.
$x(t) = k_1 e^{(\alpha+i\beta)t} + k_2 e^{(\alpha-i\beta)t} = e^{\alpha t}(k_1 e^{i\beta t} + k_2 e^{-i\beta t}) \quad = e^{\alpha t}(k_1(\cos(\beta t) + i\sin(\beta t)) + k_2((\cos(-\beta t) + i\sin(-\beta t)))) \quad = e^{\alpha t}(k_1(\cos(\beta t) + i\sin(\beta t)) + k_2((\cos(\beta t) - i\sin(\beta t)))) \quad = e^{\alpha t}((k_1 + k_2)\cos(\beta t) + i(k_1 - k_2)\sin(\beta t)) \quad = e^{\alpha t}(c_1\cos(\beta t) + c_2\sin(\beta t))$ where $c_1 = k_1 + k_2$ and $c_2 = i(k_1 - k_2)$
So the solution to a second order constant coefficient linear homogeneous ODE with discriminate of the auxiliary equation being negative is:
$x(t) = e^{\alpha t}(c_1\cos(\beta t) + c_2\sin(\beta t))$ where the roots to the auxiliary equation are $r = \alpha \pm i\beta$

Example: Find a general solution to $y'' - 2y' + 26y = 0$

4.4 Non-Homogeneous Equations: Undetermined Coefficients

We will now consider 2nd order linear constant coefficient non-homogeneous equations such as:

$$ay'' + by' + cy = f(x), \quad \text{where } f(x) \neq 0$$

Our experience from 4.3 is that there will be infinitely many solutions. Right now we are content in just finding one. We will use some strategic guessing to find one **particular solution**.

The Method of Undetermined Coefficients:

$f(x) =$	First try $y_p(x) =$
$C_n x^n + ... + C_1 x + C_0$	$A_n x^n + ... + A_1 x + A_0$
$C^e kx$	Ae^{kx}
$C\cos(kx) + D\sin(kx)$	$A\cos(kx) + B\sin(kx)$

Modification: If any term of y_p is a solution of the homogeneous equation, multiply y_p by x (or by x^2 if necessary)

Notice:

- Straightforward method
- Only works for a restricted class of functions $f(x)$
  - Exponential Functions
  - Polynomials
  - Sine or Cosine Functions
  - Any products and sums of the three

The process is most easily seen from doing examples:
Example: Find a particular solution to $y'' + 16y = 16x^2$

Example: Find a particular solution to $y'' + 2y' + y = e^{-x}$

Example: Find a particular solution to $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 8y = 5\sin(3x)$

Example: Find a particular solution to $y'' + 4y' - 5y = 36xe^x$

4.5 The Superposition Principle and Undetermined Coefficients Revisited

Theorem Superposition Principle:
Let y_1 be a solution to the differential equation:

$$ay'' + by' + cy = f_1(t)$$

and let y_2 be a solution to

$$ay'' + by' + cy = f_2(t)$$

The for any constants c_1 and c_2 the function $c_1y_1 + c_2y_2$ is a solution to the differential equation

$$ay'' + by' + cy = c_1f_1(t) + c_2f_2(t)$$

.

Proof:

Why do we care?
In 4.5 we found one particular solution to a non-homogeneous equation, but we want to find the general solution. We can use the superposition principle to achieve this.
Suppose we want to solve $ay'' + by' + cy = f(t)$
We let y_p be a particular solution to the above non-homogeneous equation and y_c be **complementary solution**, or the general solution to the associated homogeneous problem ($ay'' + by' + cy = 0$).
Then by the superposition principle $y = y_p + y_c$ is a solution to the original non-homogeneous problem.
In fact it is the general solution of the original equation.

Example: Find the general solution to $y'' + 16y = 16x^2$

Example: Find the general solution to $y'' + 2y' + y = e^{-x}$

Example: Find the general solution to $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 8y = 5\sin(3x)$

Example: Find the solution to the IVP $y'' + 2y' + y = t^2 + 1 - e^t, \quad y(0) = 0, y'(0) = 2.$

4.6 Variation of Parameters

Purpose: To find a particular solution to $ay'' + by' + cy = f(x)$ using two linearly independent solutions of the associated homogeneous differential equation when we cannot implore the method of undetermined coefficients.

Process:

- Solve the associated homogeneous DE (let $f(x) = 0$). You will obtain two solutions y_1 and y_2 and your solution will be $y_c + c_1y_1 + c_2y_2$.
- Replace c_1 and c_2 by u_1 and u_2, respectively.
- Look for the particular solution to the DE to be of the form $y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$.
- Write down the following system of equations satisfied by $u_1'(x)$ and $u_2'(x)$:
  $u_1'y_1 + u_2'y_2 = 0$
  $u_1'y_1' + u_2'y_2' = f(x)/a$
- Solve for u_1' and u_2' and integrate to obtain u_1 and u_2 (Set arbitrary constants of integration to 0)
- The final form of the particular solution is obtained by substituting the values of u_1 and u_2 into the function.

proof:

Example: Solve $y'' + y' = e^x$ using Variation of Parameters

Example: Solve $y'' + 3y' + 2y = \frac{1}{1+e^x}$

Example: Solve $y'' + 4y' + 4y = e^{-2t} \ln t$

Sometimes a combination method works best:
Example: $y'' + y = \tan t + e^3 t - 1$

4.8 A Closer Look at Free Mechanical Vibrations

Recall in 4.1 we introduced a mass-spring system modeled by:

$$mx'' + bx' + kx = F_{external}$$

where $x(t)$ represents the position at time t, $F_{external}$ represents the external force which is a function of time, m represents mass, b represents the damping constant, and k represents the spring strength.

We also discussed the case where $F_{external} = 0$ and $b = 0$ so we find $mx'' + kx = 0$.
We said solutions have form $x = c_1 \cos(\omega t) + c_2 \sin(\omega t)$ where $\omega = \sqrt{\frac{k}{m}}$ is the frequency.
$mx'' + kx = 0 \Rightarrow mr^2 + kr = 0 \Rightarrow r^2 = -\frac{k}{m} \Rightarrow r = \pm i\sqrt{\frac{k}{m}}$
If we let $\omega = \sqrt{\frac{k}{m}}$. Then $r = \pm i\omega \Rightarrow x(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t)$
These are sinusoidal oscillations and the solution can be written in the simpler form: $y(t) = A\sin(\omega t + \phi)$, where $A = \sqrt{c_1^2 + c_2^2} \geq 0$ is the amplitude, $\omega = \sqrt{\frac{k}{m}}$ is the frequency, $\frac{2\pi}{\omega}$ is the period, and ϕ is such that $\tan\phi = \left(\frac{c_1}{c_2}\right)$ is the phase shift. (To get this equation let $c_1 = A\sin\phi$, $c_2 = A\cos\phi$, and use the trigonometric identity $\sin(\alpha + \beta) = \cos\alpha \sin\beta + \sin\alpha\cos\beta$.)

Note that $\phi \neq \arctan\left(\frac{c_1}{c_2}\right)$ in some cases, as we must get ϕ in the appropriate quadrants for the signs of c_1 and c_2. This difficulty occurs because tangent has a period of π while the period of sine and cosine is 2π.

Example: (#2) A 2 kg mass is attached to a spring with stiffness $k = 50$ N/m. The mass is displaced 1/4 m left of the equilibrium, and given a velocity of 1 m/sec to the left. Neglecting damping, find the equation of motion, the amplitude, period, and frequency. How long after release does the mass pass through the equilibrium?

In most applications there will be some kind of frictional or damping force that will affect the vibrations. In this case $b > 0$ and we have

$$mx'' + bx + kx = 0 \quad \Rightarrow \quad r = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m} = -\frac{b}{2m} \pm \frac{1}{2m}\sqrt{b^2 - 4mk}$$

Now we have 3 cases for the solution depending on the discriminate ($\sqrt{b^2 - 4mk}$):

Case I: Under damping ($\sqrt{b^2 - 4mk} < 0$)
Here we have two complex roots $r = \alpha \pm i\beta$ with $\alpha = -\frac{b}{2m}$ and $\beta = \frac{1}{2m}\sqrt{4mk - b^2}$.
Solutions have the form $x(t) = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t)) = Ae^{\alpha t} \sin\beta t + \phi$.
Since $\alpha < 0$ the term $e^{\alpha t}$ is being "damped out" as time increases. The damping does not prevent the oscillations but it does bring their magnitude to zero exponentially.

Case II: Over damping ($\sqrt{b^2 - 4mk} > 0$)

Here we have two real roots $r_1 < 0$ and $r_2 < 0$ with $r_1 = \frac{-b+\sqrt{b^2-4mk}}{2m}$ and $r_2 = \frac{-b-\sqrt{b^2-4mk}}{2m}$
Solutions have the form $x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$.
Due to strong damping, at most one local max/min occurs; we do not have oscillations.
There are three possible graphs.

Case III: Critical damping ($\sqrt{b^2 - 4mk} = 0$)
damping ($\sqrt{b^2 - 4mk} = 0$)

Here we have one real root $r = \frac{-b}{2m}$ and solutions have the form $x(t) = c_1 e^{rt} + c_2 t e^{rt}$
Same behavior as over damped case.

Example: (#8) A 20-kg mass is attached to a spring with stiffness 200 N/m. The damping constant is 140 N·sec/m. If the mass is pulled 25cm right of equilibrium and given an initial leftward velocity of 1 m/sec, when will it first return to equilibrium?

4.9 A Closer Look at Forced Mechanical Vibrations

In 4.8 we explored mass-spring systems with free vibrations. In other words we had that $F_{external} \equiv 0$ in the model:

$$mx'' + bx' + kx = Fexternal$$

Now we want to explore the case when we do have an external force, in particular a sinusoidal force.

Example: (#13) A mass weighing 32lb is attached to a spring hanging from the ceiling and comes to rest at its equilibrium position. At time $t = 0$, an external force $F(t) = 3\cos(4t)$lb is applied to the system. If the spring constant is 5lb/ft and the damping constant is 2 lb·sec/ft, find the steady-state solution for the system.

Example: Determine the equation of motion for the following undamped system and sketch it.

$$\frac{d^2y}{dt^2} + y = 5\cos t, \quad y(0) = 0, y'(0) = 1$$

The above phenomenon (called resonance)stems form having the same frequency in the sinusoidal terms of the complementary solution (called the resonance frequency) as the frequency in the forcing term $5\cos t$. This can only occur if damping is small ($b \approx 0$); it can be useful or very destructive, depending on the physical situation.