MA 341
Introduction to Differential Equations
Chapter 2
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2.2 Separable Equations

Def: A differential equation, $\frac{dy}{dx} = f(x,y)$, is called **separable** if $f(x,y) = g(x)h(y)$ for some functions g and h.

Method to solve separable equations:
Suppose we have a separable differential equation: $\frac{dy}{dx} = g(x)p(y)$. Further suppose $h(y) = \frac{1}{p(y)}$
Multiply both sides of the differential equation by $h(y)$ and dx $h(y)dy = g(x)dx$
Integrate both sides: $\int h(y)dy = \int g(x)dx$
Now we have an implicit solution to the differential equation: $H(y) = G(x) + C$

Once you have identified a differential equation as separable the difficulty often becomes integrating.
Example: In falling body example what are $g(x)$ and $p(y)$?

Example: Is this separable? $\dfrac{dy}{dx} = x + y$

Example: Is this separable? $\dfrac{ds}{dt} = t\ln(x^{2t}) + 8t^2$

Example: $x\dfrac{dv}{dx} = \dfrac{1-4v^2}{3v}$

Example: $\dfrac{dy}{dx} = (1+y^2)\tan x, \quad y(0) = \sqrt{3}$

2.3 Linear Equations

Def: A **linear first order differential equation** is one that can be expressed in the form:

(1) $a(x)\dfrac{dy}{dx} + b(x)y = c(x)$, where a(x),b(x), and c(x) depend only on the independent variable x, not y.

Clearly $a(x)$ cannot be zero otherwise we do not have a differential equation.
To put a linear first order differential equation in **standard form** divide the original equation (1) by $a(x)$ and we get:

$$\frac{dy}{dx} + P(x)y = Q(x), \text{ where } P(x) = \frac{b(x)}{a(x)}, \text{ and } Q(x) = \frac{c(x)}{a(x)}$$

Notice if we can find a function $\mu(x)$ such that $\mu' = \mu P$ we can multiply both sides of the standard equation to get::

$$\mu(x)\frac{dy}{dx} + \mu(x)P(x)y = \mu(x)Q(x) \quad \Rightarrow \quad \mu(x)\frac{dy}{dx} + \mu'(x)y = \mu(x)Q(x)$$

$$\Rightarrow \quad \frac{d}{dx}[\mu(x)y] = \mu(x)Q(x) \quad \Rightarrow \quad y = \frac{1}{\mu(x)}\left[\int \mu(x)Q(x)dx + C\right]$$

Method for solving linear equations:
(1) Write the equation in standard form: $\frac{dy}{dx} + P(x)y = Q(x)$
(2) Calculate the integrating factor $\mu(x) = exp\left[\int P(x)dx\right]$
(3) Multiply (1) by $\mu(x)$ so that the left hand side becomes: $\frac{d}{dx}[\mu(x)y]$. So now you have $\frac{d}{dx}[\mu(x)y] = \mu(x)Q(x)$
(4) Finally integrate the (3) and solve for y by dividing by $\mu(x)$

Example: (#8) $\dfrac{dy}{dx} = \dfrac{y}{x} + 2x + 1$

Example: (#16) $(x^2+1)\dfrac{dy}{dx} = x^2 + 2x - 1 - 4xy$

Example: $\frac{dy}{dx} + ye^{x^2} = 2x, \quad y(0) = 2$

The integrating factor is: $\mu(x) = exp\left(\int e^{x^2} dx\right)$, but e^{x^2} does not have an elementary antiderivative. Therefore we are forced to use an approximation method such as Euler's method.

Example: (#20) $\dfrac{dy}{dx} + \dfrac{3y}{x} + 2 = 3x, \quad y(1) = 1$

Theorem Existence and Uniqueness of Solution: Suppose $P(x)$ and $Q(x)$ are continuous on an open interval (a, b) that contains the point x_0. Then for any choice of initial value y_0, there exists a unique solution $y(x)$ on (a, b) to the initial value problem

$$\frac{dy}{dx} + P(x)y = Q(x), \quad y(x_0) = y_0$$

Difference from other existence and uniqueness theorem is that the solution must exist on the entire interval (a, b) rather than some smaller interval around x_0.

2.4 Exact Equation

For a function $F(x,y)$, if we move along level curves of F (which have the form $F(x,y)=c$ fro some number c), then F remains constant so its total differential is zero. This means:

$$dF(x,y) = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = 0$$

So the slope of the tangent line to the level curve at any point can be found by manipulating the preceding equation

$$\frac{dy}{dx} = -\frac{\partial F/\partial x}{\partial F/\partial y}$$

(Think of the Implicit Function Theorem from Calc III)

For some differential equations, we can use this idea in reverse to solve the DE. First let's observe the following.

- Any 1st order DE $\frac{dy}{dx}$ can be rewritten as $P(x,y)dx + Q(x,y)dy = 0$
- We will call the differential form $P(x,y)dx + Q(x,y)dy$ **exact** if there exists a function $F(x,y)$ such that $\frac{\partial F}{\partial x} = P$ and $\frac{\partial F}{\partial y} = Q$.
- This is completely the same idea as you applied in Calc III to find a potential function for a conservative vector field, so that you could use the Fundamental Theorem of Line Integrals.
- If P and Q have continuous 1st partials and F exists, then

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right) = \frac{\partial^2 F}{\partial y \partial x}$$

$$\frac{\partial Q}{\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right) = \frac{\partial^2 F}{\partial x \partial y}$$

These must be equal, so $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

- So our test for exact differential form is whether $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.
  - If so, it is exact, and we will find and use F to solve the DE.
  - If not, it is not exact, so we must abandon this method and try another.

Now let's assume we have an exact DE $P(x,y)dx + Q(x,y)dy = 0 \quad \Rightarrow P_y = Q_x$. Since we have an exact form we know:

$$\frac{\partial F}{\partial x} = P \quad \Rightarrow F(x,y) = \int P(x,y)dx + g(y)$$

and

$$\frac{\partial F}{\partial y} = Q \quad \Rightarrow F(x,y) = \int Q(x,y)dy + h(x)$$

Now we reconcile these two forms to get $F(x,y)$ and the solution to the DE is given implicitly by $F(x,y)=c$ (level curves of F).

Example: Determine whether $(2x + y)dx + (x - 2y)dy = 0$ is exact. If so, solve it.

Example: If $\left(\frac{2}{\sqrt{1-x^2}} + y\cos(xy)dx\right) + (x\cos(xy) - y^{-1/3})dy = 0$ is exact, solve it.

Example: Use the "exact" method to solve the IVP $(e^t x + 1)dt + (e^t - 1)dx = 0 \quad x(1) = 1$

Example: Solve $xydx + xydy = 0$