Section 8.5: Power Series

- A **power series** is a series of the form $\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1x + c_2x^2 + \ldots$, where x is a variable and the c_n's are constants called the coefficients of the series.
- Note that a power series may converge for some values of x and diverge for other values of x.
- The sum of the power series is a function $f(x) = c_0 + c_1x + c_2x^2 + \ldots + c_nx^n + \ldots$ whose domain is the set of all x such that the series converges. Although f resembles a polynomial, note that it has infinitely many terms.
- **Example:** Let $c_n = 1$ for all n, and consider the power series $\sum_{n=0}^{\infty} c_n x^n$. What other type of series is this? When will the series converge?
- A series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

where x is a variable, where $\{c_n\}$ is a sequence and a is a constant, is called a **power series about *a* (or centered at *a*)**. It is also called a **power series in $(x - a)$**.

- **Example:** For what values of x does the series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ converge?

- **Example:** For what values of x does the series $\sum_{n=1}^{\infty} \frac{x^n}{n^3}$ converge?

- **Theorem:** For a given power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ there are only three possibilities:

  1. The series converges only when $x = a$.
  2. The series converges for all x.
  3. There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

  Note: The number R is called the **radius of convergence** of the power series. For case 1, we say $R = 0$, and for case 3, we say $R = \infty$.

  Note: The **interval of convergence** of a power series is the interval that consists of all values of x for which the series converges.

- **Example:** Find the radius of convergence and interval of convergence of the series $\sum_{n=1}^{\infty} \left(\frac{5^n (x-2)^n}{8 n^7} \right)$.

- **Example:** Find the radius of convergence and interval of convergence of the series $\sum_{n=0}^{\infty} \frac{n (x+2)^n}{3^{(n+1)}}$.

Section 8.6: Representations of Functions as Power Series

Goal: Represent certain types of functions as sums of power series by manipulating geometric series (or by integrating or differentiating such a series)

Question: Why express a known function as a sum of infinitely many terms?

Answer:

Why?

Recall: $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $|x| < 1$. Thus, we can express the function $f(x) = \frac{1}{1-x}$ as a sum of power series with domain $x \in (-1,1)$.

- **Examples:** Find a power series representation for the following functions and determine the interval of convergence for each.
  - $f(x) = \frac{1}{1+x}$
  - $g(x) = \frac{1}{1-x^3}$
  - $h(x) = \frac{1}{4+x^2}$
  - $j(x) = \frac{x^3}{x-5}$

- **Theorem:** If the power series $\sum c_n(x-a)^n$ has radius of convergence $R > 0$, then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a - R, a + R)$ and

(i) $$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \ldots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$$

(ii) $$\int f(x)dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \ldots = C + \sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1}$$

The radii of convergence for both is R.

Note: Although the radius of convergence remains the same when a power series is differentiated or integrated, this does not mean that the *interval* of convergence remains the same. The original series might converge at an endpoint when the differentiated series diverges at the endpoint.

- **Examples:** Find a power series representation for the following functions and determine the radius of convergence for each.
  - $f(x) = \ln(5 - x)$
  - $g(x) = \int \frac{x}{1+x^5}\,dx$
  - $h(x) = \tan^{-1}(x)$