Section 7.1: Modeling with Differential Equations (continued)

Important Comments:

- When asked to solve a differential equation, one is expected to find ____.

- There _________ systematic technique enabling us to solve all differential equations. Thus, _____________________ ___________________________________.

Definitions:

- Solutions obtained from integrating the DE's are called ________________________. The ______________________ ____ will contain ____ arbitrary constants resulting from _______________ ____________________________.

- ***Particular solutions*** are ____ ____. (For instance, a solution that must satisfy a condition of the form _________________ will be a particular solution.)

- Solutions that cannot be expressed by the ____________________ are called _________________________.

- Constrains that are specified at the initial point, generally time point, are called_____________________. Problems with specified ______________________ are called _____________________________.

- Constrains that are specified at the boundary point, generally space points, are called_________________. Problems with specified ______________________ are called ______________________________.

Section 7.2: Direction Fields and Euler's Method

Discussion: As indicated above, solving a differential equation is not an easy task. In fact, ____________________________ ____. Therefore, learning information about the solution through ____ is very useful.

Graphical Approach – Direction Fields

Definition:

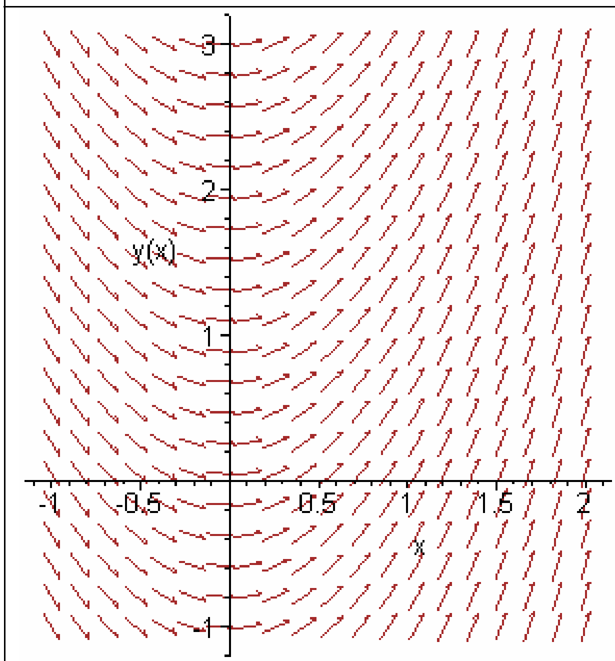
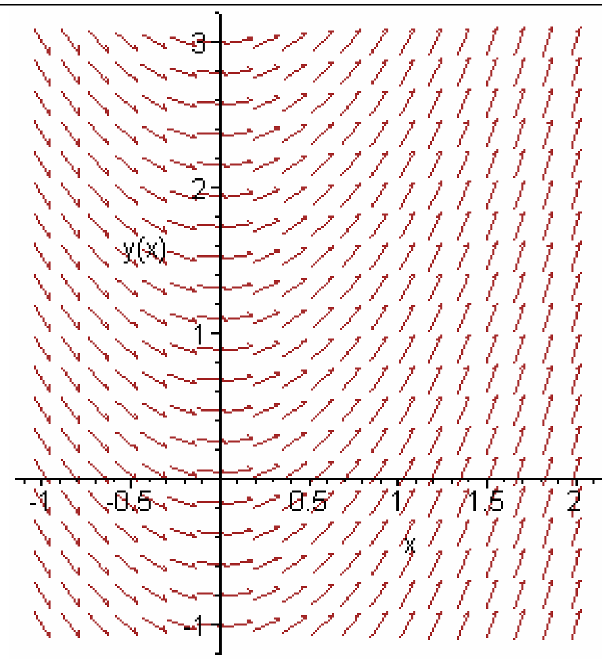
Suppose we have a first-order differential equations of the form $y' = F(x, y)$.

The differential equation says that the ____

If we draw short line segments with slope __________ at several points _______, the result is called a ____________________ (or ________________).

Why bother with direction fields?

Example 1: Sketch the direction field for the DE $\frac{dy}{dx} = 2x$. Then, sketch the solution curves $y = f(x)$ satisfying the initial conditions $y(0) = 0, 1, 2$.

Here's a sketch of the direction field:	Sketch the solution curves satisfying the initial conditions:
	

So, without even knowing how to find the antiderivative of $2x$, we can tell from the direction fields that it should be close to if not exactly $x^2 + C$.

Example 2: Sketch the direction field for the DE $\frac{dy}{dx} = 1 + y$ by hand. Also sketch the solution curves for $y(0) = -2$, $y(0) = -1$, $y(0) = 2$.

Example 3: Sketch the direction field of $y' = y + x^2$. In addition, sketch a solution curve that passes through the point (0,1), (0,-1), and (0,2).

Existence and Uniqueness: Consider a first-order DE $\frac{dy}{dx} = F(x, y)$. For any (x_0, y_0), ________________________________ ____.

Why is this true?

Numerical Approach – Euler's Method

Discussion: Although direction fields can be used to obtain qualitative information aobut solutions to a DE, we might want more detailed information than a direction field can provide. For example, we might be interested in determining how long it will take before a solution is near the limiting value. Linear approximation is a useful way to approximate solutions to differential equations.

Euler's Method:

Consider the general first-order initial value problem $\frac{dy}{dx} = F(x, y)$, $y(x_0) = y_0$.

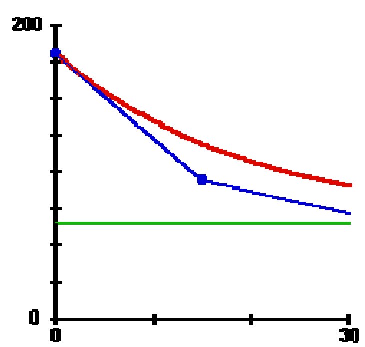
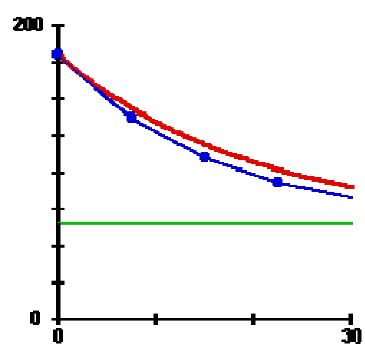
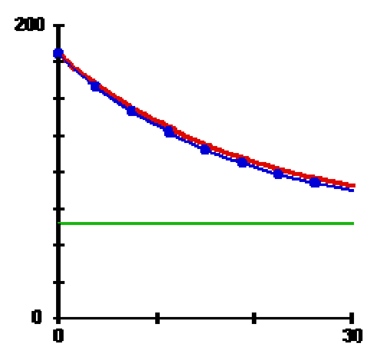
The **goal** is to

The differential equation tells us that the slope at (x_0, y_0) is ___________________, so the approximate solution when ________________ is:

⋮

This technique is called_________________________________.

Example: Consider the initial value problem $\frac{dy}{dx} = 0.05\,(65 - y)$, $y(0) = 180$ on the interval [0, 30].

This graph shows the approximation when h = 15.	Now, h = 7.5.	Here, h is smaller. Notice that the approximation is better.
		

Let's calculuate the approximations when h = 15 to make sure that the above plot is correct.

y_0 =

y_1 =

y_2 =

- **Comment:** If we decrease the step size,____________________________________. Since we know how to use Maple, we can make the step size very small, and let Maple do all the work.

Example: Use Euler's method with step size 0.1 to estimate $y(0.5)$ where $y(x)$ is the solution of the initial-value problem $\frac{dy}{dt} = y,\ y(0)=1$.

Example: Consider the DE $\frac{dy}{dx} = x^3$ with initial condition $y(0) = 0$. Use Euler's method with step size 0.1 to approximate $y(0.5)$. How does this compare with the exact value?

Example: Use Euler's method to compute $y(1)$, where $y(x)$ is the solution of the initial value problem

$$\frac{dy}{dx} + 3x^2y = 6x^2,\ y(0) = 3$$

using step sizes $h = 0.2$. Then, verify that $y = 2 + e^{-x^3}$ is the exact solution of the differential equation.

- The ***error*** in using Euler's method is the difference between the approximate value and the true value. If the number of steps used is n, then the error is ***approximately*** proportional to $^1/_n$.
- ***Absolute error*** =
- ***Relative error*** =
- There are many more methods to use for approximating solution curves (often more accurate). Some examples are:
  - o
  - o

  In this course, we only use Euler's method to calculuate numerical approximations.

Example (continued): Find the Absolute and Relative error for the preceeding example.

Section 7.3: Separable Equations

- **Discussion:** we are now ready to symbolically solve certain differential equations. We will begin with first-order differential equations.
- **Definition:** A first-order differential equation of the form ________________ is said to be _______________ or to have ________________________.
- **Method of ____:**

1. Rewrite _______________________ as _______________________. (If necessary, rewrite y' as $\frac{dy}{dx}$.)
2.
3.
4.

- **Comment:**
- **Why is the method of separation of variables valid?**

Note:

- **Example 1:** Solve $(1+x)\,dy - y\,dx = 0$.

- **Example 2:** Solve the initial-value problem $\frac{dy}{dx} = -\frac{x}{y}$, $y(4) = 3$.

- **Example 3:** Solve the initial-value problem $xe^{-t}\frac{dx}{dt} = t$, $x(0) = 1$.

- **Definition:** When all the curves in a family $G(x, y, C_1) = 0$ ________________________________ all the curves in another family $H(x, y, C_2) = 0$, the families are said to be __________________________of each other. If ________________ is the differential equation of one family, then the differential equation for the ________________________of this family is ______________________________.

- **Example 4:** Find a differential equation for the family of curves $y = kx^2$. Then, find the orthogonal trajectories for the family.

- **Example 5:** Find a differential equation for the family of curves $y = (x + k)^{-1}$. Then, find the orthogonal trajectories for the family.

Section 7.4: Exponential Growth and Decay

Natural Growth or Decay

- If $y(t)$ is the amount of something present at time t and the rate of change of y with respect to t is proportional to its size $y(t)$ at any time, then the initial value problem that models this situation (with constant of proportionality k) is

  where ________________________________.

  This is called the _______________________________ or the _______________________________.

- This is a separable differential equation with solution

Proof:

Population Growth

- Assume **population growth** is proportional to the current size of the population. This leads to the differential equation

  where ____. If the initial population is ___________, then by the solution to the general **law of natural growth**, the population is

- Note that __________________. Since ________________ is the __________________, this means that for the exponential growth model, ____..

- Note that the relative growth rate is ____.

- **Example:** A bacteria culture grows with constant relative groth rate. After 2 hours there are 600 bacteria and after 8 hours the count is 75,000.

  (a) Find the initial population.

  (b) Find an expression for the population after t hours.

  (c) Find the number of cells after 5 hours.

  (d) Find the rate of growth after 5 hours.

  (e) When will the population reach 200,000?

Previous Example's Solution:

- **Example:** The table gives estimates of the world population, in millions, from 1750 to 2000:

Year	*Population (in millions)*
1750	790
1800	980
1850	1260
1900	1650
1950	2560
2000	6070

(a) Use the exponential model and the population figures for 1750 and 1800 to predict the world population in 1900 and 1950. Compare with the actual figures.

(b) Use the exponential model and the population figures for 1900 and 1950 to predict the world population in 2000. Compare with the actual population and try to explain the discrepancy.

Radioactive Decay

- A radioactive substance emits radiation which causes the substance to decay. (Some of the atoms of the radioactive substance decay into another element or another isotope of the same element.)
- Let $m(t)$ be the amount of the radioactive substance at time t.
- It has been shown experimentally that the rate of decay ($\frac{dm}{dt}$) is proportional to the amount of the element present at time t.
- Thus, this situation is modeled by _________________, where ____.
- If __________________, by the ______________________________________, the solution to this problem is
- The rate of decay is usually given as a_______________, the time it takes for____________________________. Note that ____________________________________

Proof:

- **Example:** Carbon-14 has a half-life of 5730 years. How much is left of 500 *mg* after *t* years?
- **Example:** After 3 days a sample of radon-222 decayed to 58% of its original amount. What is the half-life of radon-222?

Continuously Compounded Interest

- **Interest compounded annually**: The amount A_0 is invested in an account paying interest annually at the rate of r. Let $A(t)$ be the amount in an account after t years. Assume that the annual interest rate r is in decimal form. Then,

- **Interest compounded *n* times a year**: If the account above pays interest n times a year instead of annually, then the interest at each interest period is r/n and the number of interest periods in t years is nt. Thus,

- **Continuously Compounded Interest**: By letting $n \to \infty$ in the formula above, we find that if A_0 is invested in an account whose interest is compounded continuously with an interest rate of r (written as a decimal), then

  where t is the number of years the money is invested.

- By differentiating _______________________, we see that ________________________. So, for continuously compounded interest, the rate of increase of the investment is ________________________ to its size.

- **Example:** If $500 is borrowed at 14% interest, find the amount due at the end of 2 years if the interest is compounded continuously.

Additional Worked Example

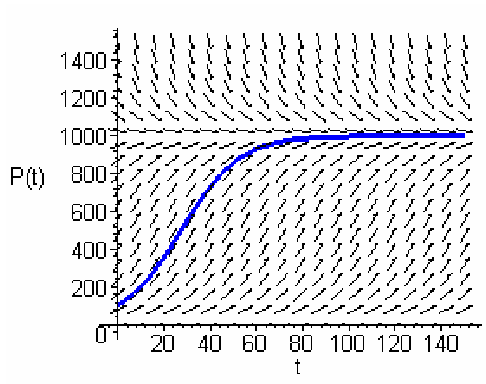
- **Example:** When a cake is removed from an oven, its temperature is measured at 300°F. Three minutes later its temperature is 200°F. How long will it take for the cake to cool off to a room temperature of 70°F?

  Solution:

Section 7.5: The Logistic Equation

- **Discussion:** A population often increases exponentially in its early stages but levels off eventually and approaches its carrying capacity because of limited resources. To model this type of population growth, we consider the logistic differential equation.
- **Logistic Differential Equation:**

  - If P is small compared with L, then ____.
  - If $P \rightarrow L$ (the population approaches its carrying capacity), then ______________________.
  - If $0 < P < L$, then ____.
  - If $P > L$, then ____.
- **Direction Field for a Logistic Equation:** The plot below shows the direction field for a logistic DE $\frac{dP}{dt} = .08P\left(1 - \frac{P}{1000}\right)$, $P(0) = 100$.



- **Analytic Solution:** The logistic equation $\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right)$ is separable, so we can solve it explicitly using separation of

  variables. We obtain the solution ________________, where ________________ and ________

  ____________________. Note that $\lim_{t \to \infty} P(t) =$.

Proof:

- **Example:** Suppose a student carrying a flu virus returns to an isolated college campus of 1000 students. If it is assumed that the rate at which the virus spreads is proportional not only to the number x of infected students but also to the number of students not infected, determine the number of infected students after 6 days if it is further observed that after 4 days $x(4) = 50$.

  Solution:

- **Example:** One model for the spread of a rumor is that the rate of spread is proportional to the product of the fraction y of the population who have heard the rumor and the fraction who have not heard the rumor.
  - Write a differential equation that is satisfied by y.
  - Solve the differential equation.
  - A small town has 1000 inhabitants. At 8 a.m., 80 people have heard the rumor. By noon, half the town has heard it. At what time will 90% of the population have heard the rumor?