Endogenous Budget Constraints in Auctions

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Abstract

In prior literature, bidders’ budget constraints have been shown to change revenue and efficiency rankings among auction formats. These results, however, are based on the implicit assumption that the nature of the budget constraint is unaffected by auction rules. I extend the standard symmetric model of auctions for a single good to include principals responsible for deciding on the bidder’s budget. Each principal optimally constrains the bidder to mitigate an agency problem between the two. I show that the outcomes of the first and second price auctions generally agree with those from auction models without budget constraints.

Bidders likely face budget constraints in many real-world auctions, especially in the sale of valuable assets such as wireless spectrum (Cramton, 1995; Salant, 1997; Bulow et al., 2009), and these constraints potentially have important strategic effects on the outcomes of auction models that cannot be captured in standard frameworks.

Current literature on the subject (most notably Che and Gale, 1998) argues that incorporating budget constraints into the standard independent private values model invalidates some well-known results like the revenue equivalence theorem (Riley and Samuelson, 1981; Myerson, 1981). For example, in a model where bidders valuations are private and i.i.d. Che and Gale (1998) show that

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the first price auction both raises more revenue and is more efficient than
the second price auction with budget constraints. Further work extends these
results to show that the all-pay auction dominates the first price auction in
terms of revenue and efficiency (e.g., Che and Gale, 1996; Maskin, 2000; Pai
and Vohra, 2008).

The earlier literature offers various explanations for the underlying cause
of the budget constraints, including imperfect capital markets and agency
problems (Che and Gale, 1998). However, these papers derive their results
from models where the budget constraint is treated as an exogenous random
variable. A potential advantage of this approach is that it allows one to be
agnostic about the source of the budget constraints and focus on the strategic
effects introduced by the constraints, but it ignores the possibility that the
process generating the budget constraints may be affected by a change in
auction rules.

If one tries to describe explicitly an agency problem that generates budget
constraints for the bidders, it seems that a description of the auction rules
should be included as well. Otherwise, one would have to assume that the
parties funding the auction take no interest in the auction design. Explicitly
including a description of the auction rules in the agency problem would allow
the budget to vary according to the rules, an effect that cannot adequately
be captured in a model that treats the budget constraints as a primitive.
The purpose of this paper is to explore how budget constraints might vary
between different auction formats when the mechanism generating the budget
constraint is made explicit.

I develop a model where the bidder’s budget constraint is the endogenous
result of an agency problem between the bidder and a principal responsible
for funding the bidder’s bid. Results from the model suggest that budget
constraints do not invalidate the standard results from the auction literature

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1The bidders’ types are two-dimensional, including a valuation and a budget, distributed
according to some commonly known prior distribution.

2Strictly speaking, the distribution of the budget constraints could be specified differently
for each auction format, but without an explicit description of the mechanism generating
the budget constraint it is not clear how to do this.
when they are treated as endogenous choices. For example, when I restrict
attention to the case of independent signals between bidders, I find no differ-
ence in the expected revenue or efficiency between the first and second price
auctions. Although a special case, the independence case corresponds to much
of the existing literature on budget constraints.

In the more general case where bidders’ information is allowed to be af-
filiated I characterize symmetric, equilibrium strategies for the bidders and
principals. I also partially characterize the relative performance of the first
and second price auctions, showing that the first price auction must be more
efficient. Under affiliation the principal prefers the first price auction to the
second price auction and reacts by relaxing the budget constraint. The effect
on revenue is less clear due to the complicated nature of the model and two
counteracting effects. However, I am able to solve for the equilibrium in an ex-
ample and show that the expected revenue in that case is higher in the second
price auction. This agrees with the revenue ranking from classic symmetric,
affiliated values model (Milgrom and Weber, 1982).

Motivated by existing explanations for the existence of budget constraints,
I model the budget constraint as the outcome of a principal-agent problem
between the bidder and some principal responsible for funding the bidder’s
bid. I believe that this basic setup covers many possible explanations for the
origin of budget constraints. For example, there is a large corporate finance
literature suggesting that capital market imperfections are the results of agency
problems (Shleifer and Vishny, 1997).

The details of the model are easily summarized by the following situation.
An item is auctioned to one of \( N \) firms. Within each firm there is a manager
interested in purchasing the asset, but the manager must get funding approval
from the firm’s board of directors.\(^3\) An agency problem arises because the
board of directors knows that the manager will tend to overpay for the asset
relative to its true value to the firm because the manager has an empire-
building motive or simply receives some private payoff from managing the

\(^3\)Throughout the paper I use the convention that the manager is male and the represen-
tative of the board is female.
Both the manager and the board observe signals about the value of the asset to the firm upon which they base their choices. The board observes its signal first and decides on a budget for the auction, after which the manager observes an additional signal and decides on a bid to place at the auction that is consistent with the budget constraints.

I consider versions of the model with “hard” and “soft” budget constraints. In the hard budget constraint case, the budget is a fixed value which the bid may never exceed. In the soft budget constraint case the board is able to provide a price list to the manager for bids of different sizes. The hard budget constraint case is more common in the existing literature, so I will primarily focus on this case here (Sections 3 and 4). It also may be the appropriate model if we impose a limited liability condition on the principal, rendering schemes that require the bidder to pay a fee to the principal infeasible. With soft budget constraints the principal is able to perfectly manipulate the bidder’s objective and incentivize the bidder to behave exactly as the principal would given the same information (Section 5). One may interpret these two alternatives as representing two extreme descriptions of the form of a budget constraint. Cramton (1995) suggests that in practice budget constraints fall somewhere in between the two (i.e., they take the form of step functions).

1 Related Literature

Following early work on the effect of budget constraints on auction outcomes when valuations are known (Che and Gale, 1996), Che and Gale (1998) are the first to compare revenue between auction formats when both budgets and valuations are treated as private information. A further extension of this work is given in Che and Gale (2006) which develops techniques for comparing

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4A modern reference for this description of managerial motives is Jensen (1986), but the idea can be traced back as far as Schumpeter (1934).

5In other words, the bidder is allowed to submit a bid of any size but must incur a cost of \( c(b) \) to submit the bid, so that in the event that he wins the item he receives \( v - c(b) \), where \( v \) is his valuation.
revenues between auction formats when types are multidimensional and independent. Several papers (Fang and Perreiras, 2002, 2003; Kotowski, 2010; Kotowski and Li, 2011) extend the model of Che and Gale (1998) to allow for valuations to be affiliated, but they retain the assumption that the bidders budgets are determined exogenously. Another direction of research has examined the effect of exogenous budget constraints on auctions of multiple goods (Brusco and Lopomo, 2008; Hafalir et al., 2011).

Several related papers make the financing decision endogenous in an auction. Benoit and Krishna (2001) consider a multiple-object auction setup in complete information where bidders are allowed to choose their own budgets (at zero marginal cost). They show that at least one bidder chooses to restrict his budget in every equilibrium of their game. Zheng (2001) describes a model where bidders are able to supplement an existing cash position by borrowing at some fixed interest rate (common to all bidders) prior to the auction to finance their bids in a first price auction. In addition, the bidders may choose after the auction to default on their bids. These two features are shown to have strong impacts on bidding behavior, such as low-budget bidders bidding more than high-budget bidders for some values of the interest rate.

Rhodes-Kropf and Viswanathan (2005) consider a model similar to the one of Zheng (2001) where bidders supplement a cash position by going to a competitive financing market before and/or after a first price auction takes place. They consider a variety of financing schemes and ask the question of which schemes lead to efficient outcomes in the auction. In a similar vein Zheng (2010) considers a situation where a social planner auctioning a good to cash constrained bidders has a choice of financing schemes to offer bidders and would like to choose the one that maximizes the efficiency of the outcome.

Another branch of literature considers the mechanism design problem, in the spirit of Myerson (1981), of auctioning an item (or items) to bidders with exogenous budget constraints. Various approaches to the difficult problems of a seller maximizing revenue or efficiency have been taken, each imposing different restrictions (Laffont and Robert, 1996; Che and Gale, 1999, 2000; 6

Another variation of this idea is presented in Hyde and Vercammen (2002).
There is a similar line of work in the computer science literature.\textsuperscript{8}

Finally, this paper is similar to work in the industrial organization literature that examines the incentives of owner-manager pairs in oligopoly models (Fershtman and Judd, 1987).

2 Model

The model extends the Milgrom and Weber (1982) model of auctions with affiliated values to allow for a budgeting stage prior to a sealed-bid auction for a single good. To each of $N$ bidders I add a principal responsible for setting their bidder’s budget constraint prior to the auction (there are a total of $2N$ players in the game). Each bidder plays the role of the manager in the firm and each principal plays the role of the board of directors. Given their expected valuations of the asset, both are interested in maximizing the expected profit of the firm at the auction (expected valuation minus expected payment). This could be modeled by making both equity holders in the firm, so that they each receive a constant fraction of the firm’s profits. The agency problem is the result of a systematic difference in how principals and bidders value the asset.

Temporarily ignoring the principals and the budget constraints, the relation between the bidders from opposing firms is as it is in the Milgrom and Weber (1982) model. So I am assuming that the bidders are symmetric and have valuations for the good which may incorporate both private and common value components. In the notation of Milgrom and Weber (1982), bidder $i$ values the object according to $u^B_i(T_i, \{T_j\}_{j \neq i})$, where $u^B$ is symmetric in its last $N - 1$ arguments,\textsuperscript{9} increasing in $T_i$, nondecreasing in $T_j$ ($j \neq i$), nonnegative and continuous.\textsuperscript{10} I use $T_i$ to represent the signal of bidder $i$. Capital letters will

\textsuperscript{7}For example, Laffont and Robert (1996) restrict all bidders to have the same budget constraint, while Maskin (2000) assumes that the budget constraint is common knowledge.

\textsuperscript{8}See Kotowski (2010) for a list of papers in this area.

\textsuperscript{9}In other words, if $\pi(T_{-i})$ is any permutation of the vector of opposing signals then I am assuming that $u^B_i(T_i, T_{-i}) = u^B_i(T_i, \pi(T_{-i}))$.

\textsuperscript{10}Milgrom and Weber (1982) also allow for the utility function to depend on a vector of signals (labeled $S_1, S_2, \ldots$ in the paper) that are not specific to any bidder. I do not make
be used for random variables, while bold typeface indicates a vector. Note that I am implicitly assuming here that the bidders’ valuations only depend on bidder-specific information.

The bidders’ signals are assumed to be affiliated and symmetrically distributed on \([t, \bar{t}]^N\) according to some bounded, atomless density \(f(t)\) with respect to the Lebesgue measure. Affiliation is equivalent to the density being log-supermodular almost everywhere.\(^{11}\) The signals are symmetrically distributed if for any permutation \(\pi(t)\) of \(t\), \(f(t) = f(\pi(t))\).

The bidders (or agents) have no capital with which to make their bids, so they must rely entirely on the funding decision of their principal. In the first stage of the game, each of the principals privately observes some signal about the value of the asset to the company, and based on that signal sets a budget constraint for the bidder at auction. I first consider the case where the principal may only select a hard budget constraint (i.e., the principal allocates a fixed amount of funds which the bidder must not exceed). I briefly consider the case of a soft budget constraint at the end of the paper (Section 5).

Both the bidder and the principal are assumed to receive a zero payoff from losing or not participating in the auction. The principal may set a budget constraint that is low enough to prevent the bidder from winning.

Two conditions on the relationship between each principal and bidder motivate the principal’s use of a budget constraint in the model. The first condition is that given the same information the bidder would be willing to bid more for the asset than the principal would. As mentioned in the introduction this could be the result of the bidder’s empire-building motives or gaining some private payoff from controlling more assets irrespective of the payoff to the firm.

The second is that the bidder is better informed about the value of the asset to the firm than the principal. This could be the result of the bidder’s specialized knowledge about the market that the firm operates in. The assumption use of these signals here, so they are omitted.

\(^{11}\)Milgrom and Weber (1982) discuss the affiliation property in detail. The affiliation property is also known outside economics as multivariate total positivity of order 2 (Karlin and Rinott, 1980).
motivates the principal to employ the bidder to decide on a bid rather than submitting a bid directly herself.

These conditions are formalized in the model by specifying that the relationship between the principal $i$’s valuation, $u_i^P(T_i, \{T_j\}_{j \neq i})$, and bidder $i$’s valuation is such that for all realizations $t$ the principal’s valuation is smaller. To simplify the solution to the model, I use a linear relationship between the two given by

\[ u_i^B(t) = u_i(t) > \delta u_i(t) = u_i^P(t) \]  \hspace{1cm} (1)

with $0 < \delta < 1$ representing how the principal discounts the bidder’s assessment of the value of the good to the firm. Equivalently one could call $1/\delta$ the amount by which the bidder overstates the value of the asset due to his interest in empire-building.\(^\text{12}\)

The principal and bidder are assumed to receive a constant fraction of the payoff to the firm, and thereby are interested in maximizing the firm’s payoff directly. Specifically, let $\sigma_P$ and $\sigma_B$ be the principal’s and bidder’s shares of firm stock and suppose that firm $i$ wins the item for a payment of $p$. Then bidder $i$ receives $\sigma_B(u_i(t) - p) = \sigma_B(1 - \delta)u_i(t) + \sigma_B(\delta u_i(t) - p)$ and principal $i$ receives $\sigma_P(\delta u_i(t) - p)$, where I let $\delta u_i(t)$ be the value to the firm. Note that the term $\sigma_B(1 - \delta)u_i(t)$ is the bidder’s private gain from empire-building. Because the shares, $\sigma_B$ and $\sigma_P$, are assumed to be constant and non-zero they cannot affect the decisions of the principals and bidders, so they are omitted from the rest of the paper.

Principal $i$ receives a signal, $S_i$, in the first stage which is informative because it is affiliated with the bidder $i$’s signal. Affiliation between $S_i$ and $T_i$ and the previous assumptions made on the distribution of $T$ imply that all of the signals in the model are affiliated; however, allowing for arbitrary dependencies between the principals’ signals turns out to be problematic for

\[^{\text{12}}\text{All of the results in the paper go through if one assumes that } u_i^B(t) > u_i^P(t) \text{ for all } t \text{ and that the principal’s valuation can be written as } u_i^P(t) = d(u_i^B(t)) \text{ where } d(x) < x \text{ for all } x \text{ and } d(x) \text{ is increasing and convex. These conditions preserve the assumptions made in Theorem 1. The results on soft budget constraints (Section 5) can also be extended to allow for such a relationship between the bidder’s and principal’s valuations.}\]
the specification of the equilibrium described here. So I make the further assumption that conditional on $T_i = t_i$, $S_i$ is independent of the other signals. In other words, principal $i$’s signal provides no additional information about other signals in the model once bidder $i$’s signal is known. I believe this assumption complements the assumption that the bidders are better informed about the valuation of the good. The bidders in the model generally have better information about the environment than the principals, and this seems natural if bidder is a specialist. Formally, one may write the joint distribution of $(s, t)$ as $f(s, t) = f_T(t) \prod_{i=1}^N f_{S_i | T_i}(s_i | t_i)$.

Figure 1 illustrates the information structure of the model with $N = 4$. The nodes in the graph represent the signals of all of players. Bidders’ signals are labeled $B_i$, and principals’ signals are labeled $P_i$. The edges represent the statistical dependence between the signals.

Finally, I assume that information is private to each of the principal-agent pairs throughout the first and the second stages. In other words, nothing is
learned by firm \(i\) about the signal or budget of firm \(j\) after the conclusion of the first stage.

3 Equilibrium

I describe symmetric equilibrium strategies in this model for the first and second price auctions. As in Milgrom and Weber (1982) I consider the second price auction first. The strategies consist of a function mapping principals’ signals into budgets, \(w(s)\), called the budget function, and a bid function for the bidders, \(B(s,t)\), that depends on signals received by the bidder and the principal. I distinguish between the “constrained” bid function \(B(s,t)\) that incorporates the budget into the bid and the “unconstrained” bid function \(b(t)\) representing the bid that each bidder would make in the absence of a budget constraint.\(^{13}\) In considering the problem facing the principal and bidder of a particular firm it is helpful to recognize that I can treat the strategies (bids and budgets) of the opposing firms as fixed throughout the first and the second stage. This is a consequence of the assumption that firms do not observe the actions of the other firms until the auction is finished.

Taking the perspective of bidder \(i\), the basic idea behind the equilibria of the first and second price auction is to recognize that if the opposing principals use symmetric, increasing strategies and the opposing bids take a form that is increasing in both signals, the bids that bidder \(i\) faces are affiliated (see footnote 14). If one then appropriately modifies the definitions of the key objects in the Milgrom and Weber (1982) paper, the unconstrained choice of bidder \(i\) takes the same form as the equilibrium strategy in Milgrom and Weber (1982), and quasi-concavity of the objective implies that \(B(s,t)\) takes the form \(B(s,t) = \min\{b(t), w(s)\}\). Using this conclusion, I can then provide a result for the existence of a symmetric equilibrium for the principals.

\(^{13}\)The unconstrained bid function is not a function of the principal’s signal, because the principal’s signal does not enter into the value of the asset. Also due to the conditional independence of the principal’s signal, the bidder safely disregards this information.
3.1 Second Price Auction

Consider the decision of bidder $i$ given some arbitrary budget constraint. Suppose the equilibrium $b(t)$ is increasing and continuous. Then the selection of a budget constraint in bid space is equivalent to the selection of a “cutoff” type in type space. Define the function $\hat{t}(s)$ as $w(s) = b(\hat{t}(s))$, so that a selection $\hat{t}(s)$ is equivalent to the selection $w(s)$.

The description of equilibrium strategies is simplified if I define the random variable $\tilde{T} = \min\{T, \hat{t}(S)\}$. In equilibrium $\tilde{T}$ will represent the information that is revealed by an opposing bidder’s bid. To describe the equilibrium I start by assuming that the opposing principals all use the increasing strategy $\hat{t}(s)$ and that the opposing bidders’ strategies take the form

$$B(s, t) = \min\{b(t), b(\hat{t}(s))\} = b(\min\{t, \hat{t}(s)\}) = b(\tilde{t}) \quad (2)$$

where $b(t)$ is some increasing function. The second equality follows because $b(t)$ is increasing and min is nondecreasing. If the opposing bids take this form then the highest opposing bid is submitted by the opposing bidder with the highest realization of $\tilde{T}$. In this sense, one can think of $\tilde{T}$ as representing the type of an opposing bidder.

The distribution of $\tilde{T}$ is a function of the signals of the principal and the bidder, along with the strategy used by the principals. Each $\tilde{T}$ is a composition of nondecreasing functions of affiliated random variables. A straightforward application of Theorem 23 in Milgrom and Weber (1982) shows that $T_1, \tilde{T}_2, \ldots, \tilde{T}_N$ are affiliated.14 Also, affiliation between $T_i$ and $S_i$ and a similar argument implies that $S_i, T_i, \tilde{T}_{-i}$ are affiliated. It follows that the first order statistic of $\tilde{T}_{-i}$ (denoted $\tilde{T}_{(1)}$) and $T_i$ are affiliated (Milgrom and Weber, 1982,

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14 By Theorem 23, $Z = (S, T)$ being affiliated is equivalent to the following inequality holding for any nondecreasing $g$, increasing set $A$, and sublattice $S$,

$$E[g(Z) \mid AS] \geq E[g(Z) \mid S] \geq E[g(Z) \mid \hat{A}S]$$

Let $\tilde{Z}_1 = T_1$ and $\tilde{Z}_i = \tilde{T}_i$ for $i \neq 1$. For some nondecreasing function $h$, $h(\tilde{Z})$ is the composition of two nondecreasing functions of $Z$ and hence $h(\tilde{Z}) = g(Z)$ for some $g$. Therefore, the above inequality holds for any $h$ and $\tilde{Z}$ is affiliated.
Theorem 2).

Now define the function $v(x, y) = E[u(T) \mid T_i = x, \tilde{T}_{i(1)} = y]$.\textsuperscript{15} Given the assumptions on $u$ and affiliation, $v(x, y)$ is increasing in $x$ and nondecreasing in $y$. If the opposing bidders adopt the strategy $b(\tilde{t})$, and bidder $i$ makes a bid according to $b(t')$, then bidder $i$ wins in the event that $t' > \tilde{T}_{i(1)}$. Assuming that the opposing bids take the form $b(\tilde{t}) = v(\tilde{t}, \tilde{t})$, bidder $i$’s payoff from bidding according to $b(t')$ when his true signal is $t$ is

$$
\int_{t}^{t'} (v(t, x) - v(x, x)) g(x \mid t) \, dx
$$

where $g(x \mid t)$ is used for the density of $\tilde{T}_{i(1)}$ conditional on $t$.\textsuperscript{16} By direct analogy with Milgrom and Weber (1982), this function is maximized by choosing $t' = t$. For $t' < t$, it is increasing. Therefore, a choice of $\tilde{t} = \min\{t, \hat{t}\}$ satisfies the Karush-Kuhn-Tucker conditions for an optimum.

**Lemma 1.** Assume the $N - 1$ other bidders use the strategy $b_{-i}(x) = v(x, x)$ and that the other principals use the same increasing budget function given by $\hat{t}_{-i}(s)$, then the unconstrained best response of bidder $i$ is $b(t) = v(t, t)$. Given a budget $\hat{t}(s)$, the constrained best response is to bid $\min\{b(t), b(\hat{t}(s))\} = b(\hat{t})$.

The unconstrained best response defined above is the bidder’s expected payoff conditional on his own information and the information contained in the event that his bid is just equal to the second highest bid. In that sense, this equilibrium is analogous to the one prescribed by Milgrom and Weber (1982).

The budget constraint manifests itself in the above strategies by forcing to the bidder to bid as if his signal were lower than it is. That is, there is a sense in which the bidder is not conditioning on the “correct” information when he makes his bid. However, this is a natural consequence of treating the budget constraints for the principal as a choice in type-space rather than bid-space.

\textsuperscript{15}This function will play a role analogous to the $v(x, y)$ defined in Milgrom and Weber (1982).
\textsuperscript{16}Due to the principals’ signals being independent condition on the realizations of the bidders’ signals $g(x \mid t, s) = g(x \mid t)$. 

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Given the description of second stage play, the principal’s problem is to select a budget constraint that has the effect of lowering the bid of her bidder on average. Left unconstrained the bidders will always select bids that are higher than the principals would like him to.\footnote{Consider, for example, the case where the principals observe their bidders’ signals directly and decide on bids directly. In this case, the symmetric equilibrium is for the principals to bid according to the function $b(t) = \delta v(x, x)$ where $v(x, x)$ is defined as in Milgrom and Weber (1982).}

Letting $f(t \mid s)$ be the density of the bidder’s signal conditional on the principal’s signal, the payoff to a principal receiving a signal of $s$ and choosing a budget constraint consistent with $\hat{t}$ is given by

$$
\int_{-\infty}^{\hat{t}} \int_{-\infty}^{\hat{t}} (\delta v(t, x) - v(x, x)) g(x \mid t) f(t \mid s) \, dx \, dt \\
+ \int_{\hat{t}}^{\infty} \int_{-\infty}^{\hat{t}} (\delta v(t, x) - v(x, x)) g(x \mid t) f(t \mid s) \, dx \, dt
$$

The first and second terms correspond to the payoff to the principal when the bidder is unconstrained and constrained. After cancelling terms and rewriting, the first order condition for this problem can be written as follows

$$
0 = \int_{\hat{t}}^{\infty} (\delta v(t, \hat{t}) - v(\hat{t}, \hat{t})) g(\hat{t} \mid t) f(t \mid s) \, dt
$$

$$
= E[\delta u(T_i, T_{-i}) - u(\hat{t}, \hat{t}) \mid T_i \geq \hat{t}, \tilde{T}_{(1)} = \hat{t}, s] \\
\times P(T_i \geq \hat{t} \mid \tilde{T}_{(1)} = \hat{t}, s)
$$

where the second line replaces the function $v$ with the utility function $u$. The equation is satisfied by setting $\hat{t}$ so that the expected payoff to the principal conditional on the principal’s information, the budget constraint binding, and the highest opposing bidder having an effective type of $\hat{t}$ is zero. For some values of $s$, there may be no $\hat{t}$ in the support of $T_i$ for which the right hand side of (4) is positive, in which case the principal optimally sets $\hat{t} = t$.\footnote{Consider, for example, the case where the principals observe their bidders’ signals directly and decide on bids directly. In this case, the symmetric equilibrium is for the principals to bid according to the function $b(t) = \delta v(x, x)$ where $v(x, x)$ is defined as in Milgrom and Weber (1982).}
prevents the bidder from participating.

To prove that an equilibrium exists I take advantage of work on the existence of monotone pure strategy equilibria in games of incomplete information (Athey, 2001; Reny, 2011). Using results from these papers I am able to show that existence follows from a single crossing condition being satisfied. A sufficient condition for a single crossing condition to hold in this model is that the right-hand side of equation (4) be increasing in $s$. A sufficient condition for this to be true is that $\delta u(t_i, t_{-i}) - u(t_i, t_{-i})$ is nondecreasing in $(t_i, t_{-i})$, since affiliation between $S_i$ and $(T_i, T_{-i})$ then implies that the expectation is increasing in $s$ (Milgrom and Weber, 1982, Theorem 5). The second term, $P(T_i \geq \hat{t} | \tilde{T}(i) = \hat{t}, s)$, is the expected value of a nondecreasing function of $T_i$, so it also must be nondecreasing in $s$.

**Theorem 1.** If the right-hand side of equation (4) is increasing in $s$, a symmetric equilibrium in the Second Price Auction described above is given by $b(t_i) = v(t, t)$ and $w(s) = b(\hat{t}(s))$, where $\hat{t}(s)$ solves (4) when $\hat{t}(s) > \bar{t}$.

**Proof.** Given Lemma 1, the principals may be thought of as competing against each other in an odd sort of auction, where each is asked to name a value for $\hat{t}$ and their bid is calculated according to $\min\{v(t, t), v(\hat{t}, \hat{t})\}$ so that there bid is random from their perspective. Note that conditional on $t$ the bid is not increasing in $\hat{t}$ for all $\hat{t}$, but the expected bid is increasing for all $\hat{t} \in (t, \bar{t})$. I show that this game satisfies the conditions in Athey (2001) including her single crossing condition, and hence possesses an equilibrium in increasing strategies.

The right-hand side of equation (4) being an increasing function of $s$ implies that if $\pi$ is the principal’s objective we have $\pi_{ts} > 0$. This in turn implies the single crossing condition.

I need to verify assumptions A1-A3 in Athey (2001) to apply Theorem 6 in that paper. A1 and A2 follow by assumptions made on the joint density and the utility function. A3 in this case requires that $E[\delta u_i(T_i) | s_i, W_i(\hat{t}_i', \hat{t}_{-i})]$ be increasing in $s$ and nondecreasing in $\hat{t}'$, where $W_i(\hat{t}_i, \hat{t}_{-i})$ represents the event that the principal $i$ wins with $\hat{t}_i$ when the other principals set budget
constraints according to \( \hat{t}_i \). Employing Theorem 5 from Milgrom and Weber (1982) again, this expression is increasing in \( s_i \) and nondecreasing in \( \hat{t}'_i \).

Since the random variable \( T \) is not a primitive of the model, I also need to verify that in equilibrium its distribution satisfies the assumptions of the Athey (2001) paper. In particular, it should not have any mass points except at the lower bound of the support. But as long as \( \hat{t}(s) \) is increasing this is true.

Therefore, Theorem 6 applies and an equilibrium in increasing strategies exists. Furthermore, a symmetric equilibrium, which is required for the arguments made above, also exists. This model satisfies the requirements of Theorem 4.5 in Reny (2011), which guarantees the existence of a symmetric equilibrium in increasing strategies when the bids are restricted to a finite action space. Following the arguments in Athey (2001) but considering only sequences of symmetric equilibria as the bid space becomes finer, it follows that a symmetric equilibrium exists when the bid space is a continuum.

\[ \Box \]

3.2 First Price Auction

The analysis of the first price auction proceeds in the same way, beginning with the problem faced by the bidder. As in the second price auction, I am able to make use of the equilibrium identified in the Milgrom and Weber (1982) paper for the first price auction. With \( v(x, y) \) and \( g(x \mid t) \) defined as above, define the following unconstrained bid function

\[
 b(t) = \int_t^y v(y, y) \, dL(y \mid t)
\]

where

\[
 L(y \mid t) = \exp \left\{ - \int_y^t \frac{g(z \mid z)}{G(z \mid z)} \, dz \right\}
\]

Now suppose that the opposing bidders bid according to \( b(t) \) when they are unconstrained. Again, since \( b(t) \) is an increasing, continuous function,
the principal’s choice of a budget constraint in bid space is equivalent to the choice of a cutoff type in type space. So if the opposing principals adopt the increasing strategy $\hat{t}(s)$, the opposing bids take the form

$$B(s, t) = \min\{b(t), b(\hat{t}(s))\} = b(\min\{t, \hat{t}(s)\}) = b(\tilde{t})$$

Again, define the random variable $\tilde{T} = \min\{T, \hat{t}(S)\}$ with conditional density $g(x \mid t)$. I can now represent the payoff to bidder $i$ of bidding $b(t')$ as

$$\int_{\tilde{T}}^{t'} (v(t, x) - b(x))g(x \mid t)\,dx$$

where I am again using the fact that a bid of $b(t')$ against opposing bids of $b(\tilde{t})$ wins in the event that $t' > \tilde{T}$. The idea behind the following lemma is exactly the same as in the second price auction. The new objects $v(x, y)$ and $g(x \mid t)$ have the same properties as their analogues in Milgrom and Weber (1982), so the proof that it is optimal for the unconstrained bidder to select $t' = t$ proceeds in the same way. The constrained bidder cannot select $t$, but as Milgrom and Weber (1982) show the objective is increasing for $t' < t$, so it must be optimal to select $t' = \hat{t}$. Finally, the argument for why the bid function must be continuous goes through in the same way.$^{18}$

**Lemma 2.** Let $b(t) = \int_{\tilde{T}}^{t} v(x, x)\,dL(x \mid t)$ where $L(x \mid t) = \exp\left\{-\int_{x}^{t} \frac{g(z \mid x)}{G(z \mid x)}\,dz\right\}$. If the opposing bidders bid according to $b_{-i}(\hat{t})$ with $\hat{t}(s)$ increasing, then the unconstrained best response of bidder $i$ is $b(t)$. Due to the quasi-concavity of bidder $i$’s objective, the choice $\min\{b(t), b(\hat{t}(s))\} = b(\tilde{t})$ is a constrained best response.

The principal’s payoff in the first price auction can be written as the payoff in the second price auction with the payment term replaced

$^{18}$The continuity argument in Milgrom and Weber (1982) depends on the symmetry of the signals. For differentiability, one may either rescale the signals to make the bid function differentiable (Milgrom and Weber, 1982) or notice that if the bid function is monotonic it must be differentiable almost everywhere. Combined with continuity these two properties imply that the differential equation defines the behavior of the bidder almost everywhere which is all that is required for a Bayesian Nash Equilibrium.
\[
\int_t^i \int_L \left( \delta v(t, x) - b(t) \right) g(x \mid t) f(t \mid s) \, dx \, dt \\
+ \int_t^\hat{t} \int_L \left( \delta v(t, x) - b(\hat{t}) \right) g(x \mid t) f(t \mid s) \, dx \, dt
\]

The first order condition of the principal’s problem in this case can be written as

\[
0 = \int_t^\hat{t} \left\{ \{ \delta v(t, \hat{t}) - b(\hat{t}) \} g(\hat{t} \mid t) - b'(\hat{t}) G(\hat{t} \mid t) \right\} f(t \mid s) \, dt
\]

\[
= \int_t^\hat{t} \left\{ \delta v(t, \hat{t}) - b(\hat{t}) - b'(\hat{t}) \frac{G(\hat{t} \mid t)}{g(\hat{t} \mid t)} \right\} g(\hat{t} \mid t) f(t \mid s) \, dt
\]

\[
= E \left[ \delta u(T_i, T_{-i}) - b(\hat{t}) - b'(\hat{t}) \frac{G(\hat{t} \mid T_i)}{g(\hat{t} \mid T_i)} \mid T_i \geq \hat{t}, \tilde{T}_{(1)} = \hat{t}, s \right] \times P(T_i \geq \hat{t} \mid \tilde{T}_{(1)} = \hat{t}, s)
\]

An equilibrium in monotone pure strategies exists here as long as I can show that the same single crossing condition is satisfied. For the first price auction the single crossing condition is satisfied immediately without the additional assumptions made in the second price auction case. The integrand in the expectation in equation 6 is increasing in T because the utility function is increasing by assumption and the term \( G(\hat{t} \mid t)/g(\hat{t} \mid t) \) is decreasing in t by affiliation.

**Theorem 2.** A symmetric equilibrium in the First Price Auction described above is given by \( b(t) \) and \( w(s) = b(\hat{t}(s)) \), where \( b(t) \) is defined in Lemma 2 and \( \hat{t}(s) \) solves (6) when \( \hat{t}(s) > t \).

**Proof.** The discussion preceding the proof shows that the Athey (2001) single crossing condition is satisfied here. The remainder of the proof proceeds exactly as in Theorem 1. \( \square \)
4 Revenue and Efficiency

The primary focus of the existing literature on budget constraints in auctions has been on their effect on expected revenue and to a lesser extent on expected efficiency (or social surplus). For the first and second price auctions, Che and Gale (1998) find that the first price auction dominates the second price auction both in terms of expected revenue and expected efficiency.

The strongest results comparing the revenue and efficiency in this model come from considering a special case of the model where an independence condition is satisfied between bidders. In this case we are able to show that the first and second price auctions perform equivalently in terms of both efficiency and revenue. Later in the section I return to the question of the relative performance of the first and second price auctions with affiliated values. In that case I am able to rank the auctions on efficiency and provide a partial result for revenue. There are two counteracting effects on revenue so it is difficult to determine the relative ranking in general. However, I am able to give an example where the second price auction raises more revenue than the first price auction.

4.1 Independent Signals

I make use of the affiliation property twice in the model described above. The first use of affiliation relates the signals of the bidders, while the second use of affiliation relates the signals of a particular principal to her corresponding bidder. In the independence case I make the stronger assumption that the bidders’ signals are statistically independent. In other words I may write the joint distribution of random variables in the model as $f(s, t) = \prod_{i=1}^{N} f_{S_i, T_i}(s, t)$, where $f_{S_i, T_i}(s, t)$ is affiliated.

Since independent signals are trivially affiliated, the strategies described in Theorems 1 and 2 are also equilibria with independent signals. The important effect of independence is to simplify the principal’s decision. The key step is to recognize that the distribution of the effective types of the opposing bidders, represented by $g(x | t)$ in the previous expressions, no longer depends on $t$
when the independence condition is satisfied. Therefore, one may rewrite the principal’s payoff in the first price auction as

\[
\int_\hat{t}^t \int_\hat{t}^t (\delta v(t, x) - b(t)) g(x) f(t \mid s) \, dx \, dt + \int_i^\hat{t} \int_i^t (\delta v(t, x) - b(\hat{t})) g(x) f(t \mid s) \, dx \, dt
\]

\[
= \int_\hat{t}^t \int_\hat{t}^t (\delta v(t, x) - v(x, x)) g(x) f(t \mid s) \, dx \, dt + \int_i^\hat{t} \int_i^t (\delta v(t, x) - v(x, x)) g(x) f(t \mid s) \, dx \, dt
\]

where I have used the fact that under independence \( b(t) = \frac{1}{G(t)} \int_\hat{t}^t v(x, x) \, dx \). By inspection, the principal’s payoff function is the same under both auctions when the independence condition holds implying that the principal must make the same choice for \( \hat{t} \) in both auctions.

To complete the equivalence argument, observe that due to the way the equilibrium strategies are defined the resulting bids are equivalent to the bids that would be observed in an independent private values auction without budget constraints with signals distributed according to \( \tilde{T} = \min\{T, \hat{t}(S)\} \). Therefore, the revenue equivalence theorem applies.

Since the principals and the bidders disagree about the value of the asset to the firm, the efficiency of the auction is potentially ambiguous. The auction may be called efficient if it always allocates to the bidder with the highest valuation, but one may also define efficiency in terms of the valuation of the principals. In this model, however, there is no ambiguity because the bidder with the highest valuation is always paired with the principal with the highest valuation.

With independence between bidders, the two auctions perform the same in terms of efficiency. With the same budget constraint function in both auctions, the same realization of signals must yield the same winner, and the two auctions cannot differ in their allocations.
**Theorem 3.** When signals are independent across bidders, the first and second price auctions with budget constraints are equivalent, both in terms of expected revenue and expected efficiency.

The equivalence in terms of efficiency and revenue also implies that the two auctions are payoff equivalent for both the principal and the bidder. In fact, the bidder’s bidding behavior makes the type of auction (first or second price) and the distribution of the opposing bidder’s signals irrelevant to the principal. The principal’s decision is completely determined by the valuation function and the joint distribution of her signal and the bidder’s, \( f(t \mid s) \). This is a direct consequence of treating the principal’s decision in type-space though. The actual budget constraints (i.e., \( b(\hat{t}) \)) do vary between the auction formats because the bid functions differ.

Note that we will not have full efficiency here because the budget constraint will occasionally bind, so it is possible that the bidder with the highest realization of \( T \) loses to another bidder because he is budget constrained.

To illustrate these results it is helpful to consider a linear example.

**Example 1.** There are 2 firms. Each principal receives a signal, \( S \), distributed uniformly on \([0, 1]\). Conditional on the principal’s signal, \( s \), the bidder’s signal is uniformly distributed on \([s, s + 1]\), so the relation between the two signals is \( T = S + \varepsilon \) where \( \varepsilon \sim U[0, 1] \).\(^{19}\) The bidder has a private value for the item given by his signal, \( t \), and the principal values the item at \( \delta t \).

For either auction, principal \( i \) chooses \( \hat{t} \) to solve

\[
\mathbb{E} [\delta T_i - \hat{t} \mid T_i \geq \hat{t}, s] = 0
\]

which has the solution \( \hat{t}(s) = \frac{\delta}{2-\delta}(s + 1) \). So the bidder is budget constrained when \( \varepsilon > \frac{\delta}{2-\delta} - \frac{2-2\delta}{2-\delta} s \). Note that as \( \delta \) approaches 1 the principal relaxes the budget constraint eventually leaving the bidder unconstrained. When \( \delta = 1 \) the principal can trust the bidder to bid exactly as she would if she were to observe \( T \).

\(^{19}\)The joint density of \( T \) and \( S \) can be written as \( f(t, s) = 1 \{ s \leq t \leq s + 1 \} 1 \{ 0 \leq s \leq 1 \} \) which is affiliated. The bidder’s signals are clearly independent of one another.
In the second price auction, a unconstrained bidder with signal \( t \) has a dominant strategy to bid his value, \( t \). From the seller’s perspective (or the perspective of an opponent) individual bids are therefore distributed according to the random variable \( W \) where \( W = \min\{S + \varepsilon, \frac{\delta}{2-\delta}(S + 1)\} \).

In the first price auction, the unconstrained best response of a bidder with signal \( t \) is

\[
b(t) = \mathbb{E}[W \mid W \leq t]
\]

and to the seller bids are distributed according to \( b(W) \).

In both auctions, the seller’s expected revenue is the expected value of \( W_{(2)} \) where \( W_{(2)} \) is the second order statistic of \( W \). Note that the auction from the seller’s perspective is equivalent to an independent, private values auction where values are drawn according to the distribution of \( W \).

Finally, either auction allocates the good inefficiently if, for example, \( t_1 < t_2 \) but \( W_1 > W_2 \) which occurs with positive probability.

### 4.2 Affiliated Signals

When signals between bidders are affiliated, the first and second price auction are no longer equivalent from the perspective of the principal. In fact, given a signal \( s \) if the principal were to choose the same budget constraint in both auctions, she would earn a higher payoff from the first price auction when bidder’s signals are affiliated. The reason for this is that the principal expects to pay less in the first price auction when she sets the same budget constraint because affiliation between the bidders causes the average bid in the second price auction to be higher (Milgrom and Weber, 1982).

The principal’s preference for the first price auction leads the principal to relax the budget constraint in the first price auction relative to the second price auction. Specifically, I show below that for a given \( s \), the principal chooses a greater \( \hat{t} \) in the first price auction.

**Theorem 4.** In first and second price auctions with the same distribution of signals, let \( \hat{t}^F(s) \) and \( \hat{t}^S(s) \) be equilibrium strategies of the principals in the
first and second price auctions. It must be that $\hat{t}^F(s) \geq \hat{t}^S(s)$ for all $s$.

Proof. Let $\gamma(x \mid y) = \frac{g(x \mid y)}{G(x \mid y)}$ and note that affiliation implies that $\gamma(x \mid y)$ is increasing in $y$ (Milgrom and Weber, 1982, Lemma 1). Using this and replacing $b'(\hat{t})$ the right hand side of (6) can be written as

$$
\int_{\hat{t}}^{t} (\delta v(t, \hat{t}) - v(\hat{t}, \hat{t})) g(\hat{t} \mid t) f(t \mid s) \, dt \\
+ \int_{\hat{t}}^{t} (\gamma(\hat{t} \mid t) - \gamma(\hat{t} \mid \hat{t})) (v(\hat{t}, \hat{t}) - b(\hat{t})) G(\hat{t} \mid t) f(t \mid s) \, dt \quad (7)
$$

The first term corresponds to the first order condition for the second price auction, (4). The second term must be positive for $t > \hat{t}$ because $\gamma$ is increasing in its second argument and $v(\hat{t}, \hat{t}) > b(\hat{t})$. This is true for any symmetric, increasing strategy used by the opposing principals.

Next, suppose that there exists equilibria such that $\hat{t}^F(s) < \hat{t}^S(s)$ for some $s$ where the inequality holds on a set of nonzero measure. I show that the first order condition in the first price auction at such a $\hat{t}^F$ cannot hold. Let $\phi(\hat{t}(s), s, \hat{t})$ be the first order condition for the principal in the second price auction when she receives signal $s$, sets budget $\hat{t}$ and the opposing principals set budgets according to $\hat{t}$. So we have $\phi(\hat{t}^S(s), s, \hat{t}) = 0$. For some $s' < s$, $\phi(\hat{t}^F(s), s', \hat{t}^F) = 0$ where $\hat{t}^F(s) = \hat{t}^S(s')$. Since the single crossing condition holds strictly, $\phi(\hat{t}^F(s), s, \hat{t}^F) > 0$. This and the argument in the previous paragraph imply that (7) must be positive for such an $\hat{t}^F$.

Relaxing the budget constraint must improve the efficiency of the auction because the allocation is more likely to depend on the bidders’ signals and not the principals’. Since it is the bidders’ signals that determine the value of the asset in the model, this must improve efficiency.

**Corollary 1.** The first price auction with budget constraints is more efficient than the second price auction with budget constraints.
Proof. I show that Theorem 4 implies for a given realization of signals if the first price auction allocates inefficiently the second price auction must too. The idea is similar to the one used in Theorem 1 of Che and Gale (1998).

Suppose that bidder 1 has the highest signal, $t_1$, but bidder 2 wins in the first price auction so $\min\{t_1, \hat{t}^F(s_1)\} < \min\{t_2, \hat{t}^F(s_2)\}$. First, bidder 1’s budget constraint must bind (otherwise bidder 1 would win). There are two cases, (i) $\hat{t}^F(s_1) < t_2 < \hat{t}^F(s_2)$ and (ii) $\hat{t}^F(s_1) < \hat{t}^F(s_2) < t_2$. In the second price auction, each of the principals chooses a lower $\hat{t}$ (Theorem 4), so bidder 1’s budget binds and by examining (i) and (ii) it is clear that bidder 2 must be the winner.

The first price auction, however, might allocate efficiently when the second price auction does not. This occurs if, for example, $t_2 < t_1 < \hat{t}^F(s_1) < \hat{t}^F(s_2)$ (so bidder 1 wins in the first price auction) but $\hat{t}^S(s_1) < t_2 < t_1 < \hat{t}^S(s_2)$ (so bidder 2 wins in the second price auction).

Interestingly, Che and Gale (1998) also find that the first price auction is more efficient in their model. Recall that in their model the budget constraint is treated as an exogenous random variable for each bidder which does not depend on the type of auction. The result of this is that the incentive for bidders to shade their bids in the first price auction also causes the budget constraints to bind less often relative to the second price auction. In contrast, here the budget constraints are binding less often because the principals are choosing to relax the budget constraint in response to the first price auction offering a higher payoff with affiliated signals. Also note that when bidders receive independent signals there is no efficiency advantage to the first price auction in this model, and the Che and Gale (1998) model makes an analogous assumption about independence of bidders’ signals.

The efficiency result is likely restricted to the symmetric environment I consider here, as the second price auction tends to outperform the first price auction on efficiency in standard models when the environment is asymmetric (see for example Proposition 2 in Maskin (1996)).

When considering expected revenue, there are two counteracting effects
here and without an explicit solution for the equilibria it is difficult to determine which one dominates. The first effect identified by Milgrom and Weber (1982) is also known as the linkage principal and has been shown to lead the second price auction to raise more revenue. This effect is certainly present in the model, but so is the effect of the principal relaxing the budget constraint in the first price auction which favors the first price auction in terms of revenue.

Example 2 shows that for at least some distributions the Milgrom and Weber (1982) ranking holds. Note that this is the opposite ranking of the one found in Che and Gale (1998). I leave the question of whether or not this ranking always holds in the model for future research.

**Example 2.** Let \( N = 2 \) and suppose that the joint distribution of \((t_1,t_2) \in [0,1]^2\) is given by \( f(t_1,t_2) = 3 \min\{t_1,t_2\} \).\(^{20}\) Assume that \( S_i \) and \( T_i \) are perfectly correlated, and the bidders have private values with \( u(t) = t_i \) for bidder \( i \).

In the second price auction, the solution to the principal’s problem is to set \( \hat{t}(s) = \delta s \). The result is that the bidder is always budget constrained, bidding according to \( \delta s \). Since the budget constraints always bind in this case, bidder 1 and principal 1 win when \( s_1 > s_2 \) or \( t_1 > t_2 \). The principal’s expected payment is then \( \delta E[S_2 \mid S_2 < s_1] = \frac{2}{3} \delta s_1 \).

The solution to the first price auction is more involved. I start by assuming that the strategy of the opposing principal is linear (i.e., \( \hat{t}(s) = s/\alpha \)). From this I calculate the following

\[

g(\hat{t} \mid t) = \frac{2\alpha}{2 - t} \\
G(\hat{t} \mid t) = \frac{2\alpha \hat{t} - t}{2 - t} \\
b(t) = \frac{2\alpha}{4\alpha - 1} t
\]

*Using the assumption that \( \hat{t}(s) = s/\alpha \), (6) becomes*

\(^{20}\)This is the distribution from Example 6.2 in Krishna (2002) truncated to \([0,1]^2\).
\[
\left( \delta s - \frac{2s}{4\alpha - 1} \right) \frac{2\alpha}{2 - s} - \frac{2\alpha}{4\alpha - 1} \frac{s}{2 - s} = 0
\]

which has the solution \( \alpha = \frac{3 + \delta}{48} \). Again, the budget constraint always binds so principal 1 wins when \( s_1 > s_2 \). Principal 1’s expected payment can be written as

\[
b(\hat{t}(s_1))P(S_2 < s_1 \mid s_1) = \frac{2}{3} \delta s_1 P(S_2 < s_1 \mid s_1) = \frac{2}{3} \delta s_1 \frac{s_1}{2 - s_1}
\]

Since \( \frac{s_1}{2 - s_1} < 1 \) for \( s_1 \in (0, 1) \), the principal’s expected payment is lower in the first price auction.

5 Soft Budget Constraints

Up to this point, the principal has been constrained to hard budget constraints. An alternative would be to allow the principal to setup a cost schedule for the bidder that assigns a charge to the bidder for each bid he might select in the auction, so that a bid of \( b \) might cost the bidder \( c(b) \) to place. One way for the principal to implement this might be to allow the bidder to borrow any amount from her but to apply a financing charge to each bid, \( c(b) - b \).

There turns out to be a natural choice for this cost schedule (in the first and second price auctions at least) that induces the bidder to bid exactly as the principal would were she to have the same information as the bidder. This implies that the game is equivalent to a one stage game where the principals observe both \( S \) and \( T \) (although having observed \( T \) the information contained in \( S \) is irrelevant) and decide on bids directly.

Consider the second price auction, and suppose that the principals observe \( T \) and may decide on a bid directly. Define \( w(x, y) = \text{E}[u(T) \mid T_i = x, T_{(1)} = y] \). Then in the symmetric equilibrium each principal bids according to \( b(t) = \delta w(t, t) \) (Milgrom and Weber, 1982). Now consider the cost schedule given by \( c(b) = b/\delta \), so that if the bidder wins the item his payoff is \( u(t) - b/\delta \)

\(^{21}T_{(1)} \) is the first order statistic of the other bidders’ signals.
where $b$ is the payment made. In this case the bidder’s equilibrium strategy is to set $b/\delta = w(t, t)$, which is exactly the bid that principal would make given the same information as the bidder.

The cost schedule $c(b) = b/\delta$ has the same effect on the first price auction. This is easily seen by observing that the bidder’s objective function can be written as

$$
\int_{\text{t}}^{t'} (v(t, x) - b(x)/\delta)h(x \mid t) \, dx
$$

$$
= \frac{1}{\delta} \int_{\text{t}}^{t'} (\delta v(t, x) - b(x))h(x \mid t) \, dx
$$

where to avoid confusion I use $h(x \mid t)$ to represent the density of $T(1)$ given $t$. So the bidder’s objective is a monotonic transformation of what the principal’s objective would be if the principal observed $t$ and submitted a bid in the auction directly. If every principal imposes the same cost schedule, then the game is equivalent to the one where the principals all observe their bidders’ signals and participate directly in the auction, because the objective functions coincide.

**Theorem 5.** When the principal’s are allowed to use arbitrary cost schedules to finance bids in the first and second price auctions, it is an equilibrium for the principals to choose the cost schedule $c(b) = b/\delta$. The result is that the bids submitted coincide with hypothetical bids submitted by principals participating in an auction with all of the information available to them.

Because the bids submitted coincide with the hypothetical game involving only principals, the following corollary is immediate.

**Corollary 2.** In the game with soft budget constraints, the revenue and efficiency of the first and second price auctions are identical to that of the hypothetical games where the principals observe $T$ and submit bids directly.

In other words, in this model introducing soft budget constraints does not change the revenue rankings of Milgrom and Weber (1982). Also note that the
auction is fully efficient when the budget constraints are soft, because the firm with the highest value of $T$ must win the auction in the symmetric equilibrium.

6 Conclusion

Given the classic symmetric auction models (the standard independent private values model and the Milgrom and Weber (1982) model) an important question for auction theory has been how robust are these results to changes in the environment. Budget constraints are cited as an example of a feature of real-world auctions that would lead to failure of the revenue equivalence theorem (Krishna, 2002). However, results from models that incorporate budget constraints largely treat them as exogenous. In the model presented here I show that incorporating budget constraints endogenously can reverse this conclusion and restore revenue equivalence between the first and second price auction.

Taken together, the results from the hard and soft budget constraint versions of the model suggest that incorporating endogenous budget constraints into auction models may not significantly change the qualitative results on revenue and efficiency from the classic (symmetric) auction models.

In the hard budget constraint case I find the following. When signals are independent between bidders, revenue and efficiency equivalence holds between the first and second price auctions. When signals are allowed to be affiliated as in Milgrom and Weber (1982), the auctions are no longer equivalent in terms of efficiency but the revenue ranking of Milgrom and Weber (1982) holds in at least some cases. Whether this ranking always holds or not is a question left for future research.

In the soft budget constraint case, there is no difference between the auctions with budget constraints and auctions without budget constraints where the principal bids directly. That is, the principal is able to perfectly control the behavior of the bidder.

The previous literature on the effect of budget constraints on auctions suggests that the first price auction and even the all-pay auction should be
preferred to the second price auction by a revenue or efficiency maximizing seller. However, in auctions where budget constrained bidders are likely to exist (e.g., spectrum auctions) the formats tend to resemble second price auctions. For example, the U.S. Federal Communications Commission uses a Simultaneous Multiple Round Ascending auction. My results favor the second price auction in many respects and hence seem better aligned with observed practice.

References


