School Choice with Controlled Choice Constraints: Hard Bounds versus Soft Bounds*

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Abstract

Controlled choice over public schools attempts giving parents options while maintaining diversity. In practice, diversity constraints are often enforced by setting hard upper bounds and hard lower bounds for each student type. We demonstrate that, with hard bounds, there might not exist assignments that satisfy standard fairness and non-wastefulness properties; and only constrained non-wasteful assignments which are fair for same type students can be guaranteed to exist. We introduce the student exchange algorithm with hard bounds (SEAHB) that finds a Pareto optimal assignment among such assignments. To achieve fair (across all types) and non-wasteful assignments, we propose the control constraints to be interpreted as soft bounds–flexible limits that regulate school priorities. In this setting, the deferred acceptance algorithm with soft bounds (DAASB) finds an assignment that is Pareto optimal among fair assignments while eliciting true preferences. Thus, we demonstrate DAASB has clear benefits over SEAHB.

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1 Introduction

*School choice* policies are implemented to grant parents the opportunity to choose the school their child will attend. In order to create a diverse environment for students, school districts often implement *controlled school choice* programs providing parental choice while maintaining the racial, ethnic or socioeconomic balance at schools. Before school choice policies were in effect, children were assigned a public school in their immediate neighborhood. However, neighborhood-based assignment eventually led to socioeconomically segregated neighborhoods, as wealthy parents moved to the neighborhoods of their school of choice. Parents without such means had to send their children to their neighborhood schools, regardless of the quality or appropriateness of those schools for their children. As a result of these concerns, *controlled school choice programs* have become increasingly more popular across the United States.

There are many examples of controlled public school admission policies in the United States. To name just a few, the Jefferson County School District has an assignment plan that requires elementary schools to have between 15 and 50 percent of their students coming from a particular geographic area inside the district that harbors the highest concentration of designated beneficiaries of the affirmative action policy.\(^1\) Similarly, in New York City, “Educational Option” (EdOpt) schools have to accept students across different ability ranges. In particular, 16 percent of students that attend an EdOpt school must score above grade level on the standardized English Language Arts test, 68 percent must score at grade level, and the remaining 16 percent must score below grade level (Abdulkadiroğlu et al., 2005a).\(^2\)

As it is apparent from above two examples, in practice, controlled school choice programs are often enforced by setting feasibility constraints with hard upper bounds and hard lower bounds for different student types.\(^3\) In the first part of our paper, we analyze controlled

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\(^1\)More details on this policy are present on the “No Retreat” brochure on Jefferson Country School District’s website ([http://www.jefferson.k12.ky.us/Pubs/NoRetreatBro.pdf](http://www.jefferson.k12.ky.us/Pubs/NoRetreatBro.pdf)).

\(^2\)There are similar constraints in other countries as well. For example in England, City Technology Colleges are required to admit a group of students from across the ability range and their student body should be representative of the community in the catchment area (Donald Hirch, 1994, page 120).

\(^3\)There are many other examples of controlled school choice. A Racial Imbalance Law that was passed in 1965 in Massachusetts, prohibits racial imbalance and discourages schools from having student enrollments that are more than 50 percent minority. After a series of legal decisions, the Boston Public Schools (BPS) was ordered to implement a controlled choice plan in 1975. Although BPS has been relieved of legal monitoring, it still tries to achieve diversity across ethnic and socioeconomic lines at city schools (Abdulkadiroğlu et al., 2005b, 2006). Likewise, St. Louis and Kansas City, Missouri, must observe court-ordered racial desegregation guidelines for the placement of students in city schools. In contrast, the White Plains Board of Education employ their nationally recognized Controlled Parents’ Choice Program voluntarily. Miami-Dade County Public Schools control for the socioeconomic status of students in order to diminish concentrations of low-income students at certain schools. Similarly, Chicago Public Schools diversify their student bodies by enrolling students in choice options at schools that are not the students’ designated neighborhood schools.
school choice with hard bounds and demonstrate serious problems as a consequence of this approach. In particular, we show that there might not exist assignments that satisfy standard fairness and efficiency (non-wastefulness) properties with hard bounds approach.

In general, a crucial feature of most school choice programs (not only controlled choice programs) is to give some students priority at certain schools. For example, some state and local laws require that students who live in the attendance area of a school must be given priority for that school over students who do not live in the school’s attendance area; siblings of students already attending a school must be given priority; and students requiring a bilingual program must be given priority in schools that offer such programs. All these priority altering decisions, including the controlled choice, should be implemented while preserving the notion of fairness.\footnote{Following Abdulkadiroğlu and Sönmez (2003), we can define an assignment to be \textit{fair} if there is no unmatched student-school pair where the student prefers the school to her assignment and she has higher priority than some other student who is assigned a seat at the school (this is the same notion as stability in the two-sided matching context).}

Abdulkadiroğlu and Sönmez (2003) and Abdulkadiroğlu (2005) consider a relaxed controlled choice problem by employing type-specific quotas (upper bounds). Control is imposed on the maximum number of students from each racial/ethnic group that a school can enroll. This extension does not capture \textit{controlled choice} to the fullest extent because they do not exclude segregated schools in fair assignments. For example, consider a school that can enroll 100 students with hard upper bound of 50 Caucasian students. In this case, a student body of 50 Caucasian students would not violate the maximum quota, yet the school is fully segregated. Such an assignment would be unacceptable for many school districts.

In order to provide a foundation for controlled school choice programs, a thorough analysis of fairness and controlled choice requires a substantial generalization of the model. Following the laws of a state or the policies of a school choice program (or of the school district), an assignment is legally \textit{feasible (under hard bounds)} (or politically acceptable) if both (i) every student is assigned to a public school and (ii) at each school the desegregation guidelines are respected. We incorporate these constraints in the definition of fairness. The nature of controlled choice imposes that a student-school pair can cause a \textit{justified envy} (or blocks) only if matching this pair does neither result in any unassigned student nor violate the controlled choice constraints at any school.

This raises the question of existence of fair and legally feasible assignments in the controlled school choice with hard bounds. We show that feasible student assignments which are fair may not exist (Theorem 1). Due to this impossibility, either fairness needs to be weakened in order to respect legal constraints, or the interpretation of the legal constraints must be changed. We first focus on the case where we relax the notion of fairness (while
maintaining the hard bounds). In this setting, a natural route is to allow envy only among students of the same type. Then, for example, only Caucasian students can justifiably envy other Caucasian students (but not any African-American students). It turns out that legally feasible assignments, which are fair for same types, may not exist if we require additionally non-wastefulness (Theorem 1). These two negative results demonstrates the serious problems associated with the hard bounds approach.

Since hard bounds are implemented in practice, it is still important to investigate what the best mechanism in this context is. In this regard, we show that a positive result emerges if non-wastefulness property is constrained: students can claim empty seats only if the resulting assignment does not cause any envy among students of the same type. In particular, our proposed student exchange algorithm with hard bounds (SEAHB) finds a legally feasible assignment which is both fair for same types and constrained non-wasteful. The assignment is also Pareto efficient among such assignments (Theorem 2). A significant advantage of SEAHB is that, as an input, it can take any feasible matching that is feasible and fair for same types. Therefore, it can be easily adapted by school districts that have controlled school choice since SEAHB can take the assignment produced by the school district as an input. Furthermore, adapting it only improves the welfare of students without violating feasibility or fairness for same types. Unfortunately, SEAHB is not (dominant strategy) incentive compatible, i.e., students may find it preferable to misreport their preferences. Indeed, we show that it is impossible to elicit true preferences in dominant strategies while maintaining fairness and the legal constraints (Theorem 3). Thus we establish that in addition to achieving only weak fairness and non-wastefulness properties, lack of incentive compatibility is another shortcoming of the hard bounds approach.

Instead of relaxing the fairness notion, we can also re-interpret the legal constraints, which are reflected as upper and lower bounds (floors and ceilings, respectively) for each student type in the controlled school choice context. Most school districts administer floors and ceilings as hard bounds, so a theoretical analysis of such policies is inarguably important. However, applications of these hard bounds are quite paternalistic in the sense that assignments can be forced despite student preferences. That is, with this specification school districts end up not allowing students to take some available seats, even if there are no physical limitations. In contrast, we provide an alternative interpretation of these constraints as soft bounds. To be more explicit, school districts may adapt a dynamic priority structure, giving highest priority to student types who do not fill their floors, medium priority

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5Non-wastefulness is a mild efficiency criterion (Balinski and Sönmez, 1999). In our context, this condition requires that empty seats should not be wasted if students claim them while the legal constraints maintained.
to student types who fill their floors and not their ceilings, and lowest priority to student types who fill their ceilings. Yet, schools can still admit fewer students than their floor or more than their ceiling as long as students with higher priorities do not veto this match. In Section 4, we consider this soft bounds view: control policies promote the desired balancing at schools, only when student preferences allow them to do so. In other words, soft bounds policies give the parents an opportunity to establish desired balancing at schools, but do not force them to accept an undesired balance.

With hard bounds, an assignment that is fair and non-wasteful might not exist even if fairness is restricted to students with same types. However, with soft bounds the existence of an assignment that is fair across types and non-wasteful is guaranteed. To show this, we consider the student-proposing deferred acceptance algorithm with soft bounds (DAASB), in which schools tentatively admit the set of students at each step with the dynamic priority structure implied by the soft bounds view of floors and ceilings: highest priorities to the student types who have not filled their floors, and lowest priorities to student types who have filled their ceilings. Furthermore, we provide the following connection between soft bounds and hard bounds. We show that all students weakly prefer the outcome of this algorithm (under soft bounds) to any assignment that is strongly fair across types and non-wasteful under hard bounds (Theorem 6). In this sense, all students are better off with soft bounds.

To summarize, we demonstrate that with hard bounds approach, the best mechanism, namely SEAHB, can only achieve weak fairness (fairness for same types) and efficiency (constrained non-wasteful) properties and lack incentive compatibility. On the other hand, with the soft bounds view, all the desirable properties of the deferred acceptance algorithm are restored: DAASB produces a fair and non-wasteful assignment (which is Pareto optimal among fair and non-wasteful assignments) and, furthermore, DAASB is group incentive compatible (Theorems 4 and 5). Hence, DAASB has clear benefits over SEAHB. A possible drawback of DAASB is that schools cannot guarantee diverse student bodies (but that can only occur if student preferences are not in line with goals of the controlled school choice).

Although we focus on controlled school choice, all of our results equally apply to centralized matching programs where diversity constraints are wished to be implemented. For instance, a college admissions office that wants to avoid completely segregated student bodies may also use controlled policies. Other examples are entry-level labor markets where we may wish to exclude gender segregated worker groups meaning that for each firm there are both female and male workers among its hires. In labor markets we may even desire to control for both race and gender.
Related Literature  In a seminal paper, Abdulkadiroğlu and Sönmez (2003) proposed *the student proposing deferred acceptance algorithm* (also known as Gale-Shapley student optimal algorithm) as an alternative to some popular school choice mechanisms. This mechanism finds the fair assignment which is preferred by every student to any other fair assignment. Moreover, revealing preferences truthfully is a weakly dominant strategy for every student in the preference revelation game in which students submit their preferences over schools first, and then the assignment is determined via the students proposing deferred acceptance algorithm (DAA) using the submitted preferences (Dubins and Freedman, 1981; Roth 1982).\(^6\) Abdulkadiroğlu and Sönmez also study control constraints only with upper bounds and extend the Gale and Shapley algorithm to this setting. Similarly, Budish et al. (forthcoming) consider expected assignments satisfying and determine when such expected assignments can be implemented by a lottery over deterministic assignments satisfying the control constraints.

In a recent paper, Kojima (2010) considers a model where there are two kinds of students (minority and majority) and only a quota for majority students. He investigates the consequences of such affirmative action policies and shows that these policies may hurt minority students, the purported beneficiaries. To overcome this shortcoming, Hafalir et al. (forthcoming) propose *affirmative action with minority reserves* in which schools give higher priority to minority students up to the point that the minorities fill the reserves. They consider both deferred acceptance and top trading cycles algorithms. They also perform simulations and conclude that minorities are on average better off with minority reserves while adverse effects on majorities are mitigated. In this paper, we generalize their interpretation of the affirmative action policies to the case of arbitrary number of student types, and when schools may have ceilings as well as floors for each student type.

In a related paper, Abdulkadiroğlu (2010) considers the same control environment as in this paper but proposes different feasibility and fairness concepts. In particular, due to the non-existence of feasible and fair student assignments, he relaxes feasibility by not requiring that all students are enrolled at a school and then looks for fair assignments which are not dominated by any other fair assignment.

Finally, Kamada and Kojima (2010) study entry-level medical markets with regional caps: hospitals (or schools) are partitioned into regions and each region is controlled by a cap (or ceiling) determining the maximal number of students that can be assigned to the hospitals in that region. Similar to our context, they propose different stability notions like “strong

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\(^6\)Although for schools it is not a weakly dominant strategy to truthfully reveal their preferences in DAA, Kojima and Pathak (2009) have recently shown under some regularity conditions that in DAA the fraction of participants that can gain from misreporting approaches zero as the market becomes large.
stability” and “stability”. Some of their results have a similar flavor like ours: (i) strongly stable assignments do not exist (like fairness (for same types) and non-wastefulness are incompatible under hard bounds for school choice with control) and (ii) stable assignments exist (like fairness and non-wastefulness are compatible under soft bounds) and (iii) their “flexible deferred acceptance algorithm” finds a stable assignment and is incentive compatible (like DAASB finds a fair and non-wasteful assignment under soft bounds and is incentive compatible).

The paper is organized as follows. Section 2 formalizes controlled school choice problem. In Section 3 we consider controlled school choice with hard bounds. We first introduce the desirable criteria, namely fairness and non-wastefulness notions, then we show that there may not exist any feasible assignment which is fair or is both fair for same types and non-wasteful. Therefore, we constrain non-wastefulness and show that SEAHB always finds a feasible assignment which is both fair for same types, constrained non-wasteful and Pareto efficient among assignments with the same properties. Section 3 also studies incentive compatibility with hard bounds approach and shows that there may not exist any incentive compatible mechanism which is both fair for same types and constrained non-wasteful. In Section 4, we consider controlled school choice problem with soft bounds and show that an adaptation of the deferred acceptance algorithm achieves fairness and non-wastefulness under soft bounds, which is also incentive compatible. Section 5 concludes. All omitted proofs are given in the Appendix B.

2 Controlled School Choice

A controlled school choice problem or simply a problem consists of the following:

1. a finite set of students \( S = \{s_1, \ldots, s_n\} \);

2. a finite set of schools \( C = \{c_1, \ldots, c_m\} \);

3. a capacity vector \( q = (q_{c_1}, \ldots, q_{c_m}) \), where \( q_c \) is the capacity of school \( c \in C \) or the number of seats in \( c \in C \);

4. a students’ preference profile \( P_S = (P_{s_1}, \ldots, P_{s_n}) \), where \( P_s \) is the strict preference relation of student \( s \in S \) over \( C \), i.e., \( cP_sc' \) means that student \( s \) strictly prefers school \( c \) to school \( c' \);

5. a schools’ priority profile \( \succ_C = (\succ_{c_1}, \ldots, \succ_{c_m}) \), where \( \succ_c \) is the strict priority ranking of school \( c \in C \) over \( S \); \( s \succ_c s' \) means that student \( s \) has higher priority than student \( s' \) to be enrolled at school \( c \);
6. a type space $T = \{t_1, \ldots, t_k\}$;

7. a type function $\tau : S \rightarrow T$, where $\tau(s)$ is the type of student $s$;

8. for each school $c$, two vectors of type specific constraints $q^T_c = (q^t_1, \ldots, q^t_k)$ and $\overline{q}^T_c = (\overline{q}^t_1, \ldots, \overline{q}^t_k)$ such that $q^t_1 \leq \overline{q}^t_1 \leq q_c$ for all $t \in T$, and $\sum_{t \in T} q^t_1 \leq q_c \leq \sum_{t \in T} \overline{q}^t_1$.

Here, $q^t_c$ is the minimal number of slots that school $c$ must by law allocate to type $t$ students, called the floor for type $t$ at school $c$, whereas $\overline{q}^t_c$ is the maximal number of slots that school $c$ is allowed by law to allocate to type $t$ students, called the ceiling for type $t$ at school $c$.

In summary, a controlled school choice problem is given by

$$\left( S, C, (q_c)_{c \in C}, P_S, \succ_C, T, \tau, (q^T_c, \overline{q}^T_c)_{c \in C} \right).$$

When all other parameters except $P_S$ remain fixed, we simply refer to $P_S$ as a controlled school choice problem.

The set of types may represent different characteristics of students such as: (i) race; (ii) socioeconomic status (determined by free or reduced-price lunch eligibility); or (iii) the district where the student lives. Controlled choice constraints are either enforced by law or state policies (via desegregation orders) or voluntarily adapted by the school district to increase the diversity at schools.\textsuperscript{7}

An assignment $\mu$ is a function from the set $C \cup S$ to the set of all subsets of $C \cup S$ such that

i. $\mu(s) \in C$ for every student $s$;

ii. $|\mu(c)| \leq q_c$ and $\mu(c) \subseteq S$ for every school $c$;

iii. $\mu(s) = c$ if and only if $s \in \mu(c)$.

In words, $\mu(s)$ denotes the school that student $s$ is assigned to; $\mu(c)$ denotes the set of students that are assigned to school $c$; and student $s$ is assigned to school $c$ if and only if school $c$’s assignment contains student $s$. Furthermore, let $S_t \equiv \{s \in S : \tau(s) = t\}$ denote the set of type $t$ students and $\mu^t(c) \equiv \mu(c) \cap S_t$ denote the students of type $t$ that are assigned to school $c$.

\textsuperscript{7}Ehlers (2010) also considers the case when students have multi-dimensional types and when control constraints are imposed in terms of percentages. He demonstrates that these extensions are easily accommodated to controlled school choice problems.
Given two assignments $\mu$ and $\mu'$, we say that $\mu$ **Pareto dominates** $\mu'$ if all students weakly prefer $\mu$ to $\mu'$ and $\mu \neq \mu'$. Similarly, we say that an assignment $\mu$ is **Pareto optimal** (or **Pareto efficient**) among the assignments satisfying certain properties if there is no assignment which both satisfies these properties and Pareto dominates $\mu$.

### 3 Controlled School Choice with Hard Bounds

In this section, the controlled choice constraints are taken as feasibility constraints that have to be implemented in practice. To be more explicit, a set of students $S' \subseteq S$ respects (capacity and controlled choice) constraints at school $c$ if $|S'| \leq q_c$ and for every type $t \in T$, $q^t_c \leq |\{s \in S' : \tau(s) = t\}| \leq \bar{q}^t_c$. An assignment $\mu$ respects constraints if for every school $c$, $\mu(c)$ respects constraints at $c$, i.e., for every type $t$ we have

$$q^t_c \leq |\mu^t(c)| \leq \bar{q}^t_c.$$

As outlined before, the law of many states in the United States requires students to be assigned to schools such that (i) at each school the constraints are respected and (ii) each student is enrolled at a public school. An assignment $\mu$ is (legally or politically) **feasible (under hard bounds)** if $\mu$ respects constraints and every student is assigned to a school.\(^8\)

According to law, every student has a right to attend a public school. Hence, we assume that all students are acceptable to every school. Moreover, we consider the case when students have to give a full ranking of all schools. This is because if students are allowed to give shorter lists and the admissions process requires them to be assigned to a school in their lists, students could simply include only their favorite schools. This clearly may result in non-existence of feasible assignments.\(^9\) We would like to emphasize that students can still prefer their outside options (going to a private school, or being home-schooled) to their assigned schools. Nonetheless, they are required to rank all schools.\(^10\)

Obviously, a controlled school choice problem does not have a feasible solution if there are not enough students of a certain type to fill the minimal number of slots required by law for that type of students in all schools. There are other cases in which controlled choice

\(^8\)Later in Section 4, we are going to re-interpret assumption (i) and study controlled choice with soft bounds.

\(^9\)Indeed, in many school districts some students are always left unassigned at the end of the main assignment and ad-hoc methods are adapted to assign these students to schools with unfilled capacity. For example, the New York City student assignment system has such a second stage (Abdulkadiroğlu et al. (2005a)).

\(^10\)Boston school district states the following in their website (http://www.bostonpublicschools.org/assignment)”If you don’t receive one of your school choices, .... we will assign the student to the school closest to home that has a seat.”
constraints cannot be satisfied. To avoid this issue, from now on we assume that the legal constraints at schools are such that a feasible assignment exists. If no feasible assignments exist, then the laws are not compatible with each other and either they need to be modified (and this issue is out of this paper’s scope) or we may reconsider the controlled choice constraints (which we discuss in Section 4).

What are desirable properties of feasible assignments in controlled school choice problems? The following notions are the natural adaptations of their counterparts in standard two-sided matching literature (without type constraints).

The first requirement is that whenever a student prefers an empty slot in another school to the one assigned to her, the legal constraints must be violated when assigning the empty slot to this student while keeping all other assignments unchanged.\footnote{This requirement is in the spirit of the property “non-wastefulness” introduced by Balinski and Sönmez (1999).}

We say that student \( s \) justifiably claims an empty slot at school \( c \) under the feasible assignment \( \mu \) if

\[
\begin{align*}
\text{(nw1)} \quad & c \notin P_s \mu(s) \text{ and } |\mu(c)| < q_c, \\
\text{(nw2)} \quad & q_{\tau(s)}^\mu(s) < |\mu^\tau(s)(\mu(s))|, \text{ and} \\
\text{(nw3)} \quad & |\mu^\tau(s)(c)| < q_{c}^\tau(s).
\end{align*}
\]

Here (nw1) means student \( s \) prefers an empty slot at school \( c \) to the school assigned to her; (nw2) means that the floor of student \( s \)’s type is not binding at school \( \mu(s) \); and (nw3) means that the ceiling of student \( s \)’s type is not binding at school \( c \). Hence, under (nw1-3) student \( s \) can be assigned to the better school \( c \) without changing the assignments of the other students and without violating the constraints at any school. A feasible assignment \( \mu \) is \textbf{non-wasteful} if no student justifiably claims an empty slot at any school.

Another well-studied requirement in the literature is fairness or no-envy (Foley, 1967).\footnote{See for example Tadenuma and Thomson (1991), for an excellent survey, also see Thomson (forthcoming), Thomson (2000) and Young (1995).} In school choice, student \( s \) envies student \( s' \) when \( s \) prefers the school at which \( s' \) is enrolled, say school \( c \), to her school. However, the nature of controlled school choice imposes the following (legal) constraints: Envy is justified only when

\[
\begin{align*}
\text{(i)} \quad & \text{student } s \text{ has higher priority to be enrolled at school } c \text{ than student } s', \\
\text{(ii)} \quad & \text{student } s \text{ can be enrolled at school } c \text{ without violating the controlled choice constraints (at all schools) by removing } s' \text{ from } c, \text{ and}
\end{align*}
\]
(iii) student $s'$ can be enrolled at another school without violating the constraints at that school.

Formally, we say that student $s$ justifiably envies student $s'$ at school $c$ under the feasible assignment $\mu$ if there exists another feasible assignment $\mu'$ such that

\((f1)\) $\mu(s') = c$, $cP_s \mu(s)$ and $s \succ_c s'$,

\((f2)\) $\mu'(s) = c$, $\mu'(s') \neq c$, and $\mu'(\hat{s}) = \mu(\hat{s})$ for all $\hat{s} \in S \setminus \{s, s'\}$.

Because $\mu'$ is feasible, $(f2)$ simply says that $(\mu(c) \setminus \{s'\}) \cup \{s\}$ respects the controlled choice constraints at school $c$ and student $s'$ can be enrolled at school $c' = \mu'(s')$ such that $(\mu(c') \setminus \{s\}) \cup \{s'\}$ respects the controlled choice constraints at $c'$. In other words, assigning $s$ a slot at $c$, $s'$ a slot at $c'$, and keeping all the other assignments intact does not violate any controlled choice constraint at any school. A feasible assignment $\mu$ is fair across types (or fair) if no student justifiably envies any student.

In what follows, we also consider a weaker version of envy (and fairness) where envy is justified only if both the envying student and the envied student are of the same type. If this is the case, then (ii) and (iii) are always true since then the envying student and the envied student can simply exchange schools. More formally, we say that student $s$ justifiably envies student $s'$ of the same type at school $c$ under the feasible assignment $\mu$ if

\((f1^*)\) $\mu(s') = c$, $cP_s \mu(s)$ and $s \succ_c s'$, and

\((f2^*)\) $\tau(s) = \tau(s')$.

In $(f1^*)$, student $s'$ is enrolled at school $c$ and both student $s$ prefers school $c$ to his assigned school $\mu(s)$ and student $s$ has higher priority to be enrolled at school $c$ than student $s'$. By $(f2^*)$, student $s$ and student $s'$ are of the same type. Then we obtain a feasible assignment when students $s$ and $s'$ exchange their slots, i.e., choose $\mu'$ as follows: $\mu'(s) = \mu(s')$, $\mu'(s') = \mu(s)$, and $\mu'(\hat{s}) = \mu(\hat{s})$ for all $\hat{s} \in S \setminus \{s, s'\}$. The assignment $\mu'$ is feasible because $s$ and $s'$ are of the same type and $\mu$ is feasible. A feasible assignment $\mu$ is fair for same types if no student justifiably envies any other student of the same type.

Our first result shows the difficulty in finding assignments that satisfy the legal constraints together with other desirable properties such as fairness and non-wastefulness by establishing two benchmark incompatibility results (even under the assumption that a feasible assignment exists).

Theorem 1  
(i) The set of feasible assignments that are fair across types may be empty in a controlled school choice problem.
(ii) The set of feasible assignments that are both fair for same types and non-wasteful may be empty in a controlled school choice problem.

The proof of Theorem 1 is provided in the Appendix; and it is by means of examples. Note that in contrast to the literature on matching, our impossibility result is not obtained by violating the responsiveness condition (or “substitutability”) of schools’ preferences over sets of students, but by controlled choice. Even though the upper bounds alone do not seem to cause any problems (Abdulkadiroğlu and Sönmez, 2003), the lower bounds yield the negative results. Intuitively, the lower bounds create complementarities between students for schools, which is problematic. For example, if school c has a lower bound of 2 for students of type t, then student s1 of type t or student s2 of type t alone cannot be admitted to school c without any other type t student, even though students s1 and s2 together can be assigned to school c. Complementarities are known to cause problems in auctions (Cramton et al., 2006) and assignment problems (Budish, 2011).

Even though non-wastefulness is incompatible with fairness for same types, it can be replaced with a slightly weaker property. We say that a feasible assignment μ is constrained non-wasteful if a student s claims an empty slot at school c under μ, then the assignment μ′ (where μ′(s) = c and μ′(s′) = μ(s′) for all s′ ∈ S\{s}) is not fair for same types.

If the feasible assignment μ is fair for same types and constrained non-wasteful, then the above definition is equivalent to the requirement that whenever a student s of type t justifiably claims an empty slot at school c under μ, then some other student s′ of same type t justifiably envies student s at school c under the assignment μ′ (where μ′ is defined above). The idea of feasible assignments that are both fair for same types and constrained non-wasteful is similar to the one of “bargaining sets”: if a type t student s has an objection to μ because s claims an empty slot at c, then there will be a counter-objection once s is assigned to c since some other type t student will then justifiably envy s at c. Roughly speaking, an outcome belongs to the “bargaining set” if and only if for any objection to the outcome there exists a counter-objection.13

We now show that there exists a feasible assignment that is both fair for same types and constrained non-wasteful in a controlled school choice problem. To show this, we introduce a new mechanism that we call student exchange algorithm with hard bounds (SEAHB). The algorithm takes any feasible assignment that is fair for same types as input (we later introduce a simple mechanism that finds a feasible assignment that is fair for same types), and produces a feasible assignment that is not only fair for same types but also Pareto efficient among such

assignments. Therefore, the outcome is also constrained non-wasteful (Lemma 3). In the algorithm, schools exchange students to improve the welfare of students such that feasibility and fairness for same types are preserved. Hence, the new assignment is better for all exchanged students. The main difficulty in the algorithm is finding the exchange cycles so that the new assignment satisfies fairness for same types and it is also feasible. To overcome this issue, for each assignment $\mu$ we introduce the associated application graph $G(\mu)$.

For any assignment $\mu$, define the directed application graph $G(\mu) = (V(\mu), E(\mu))$, where $V(\mu)$ is the set of nodes and $E(\mu) \subseteq V(\mu) \times V(\mu)$ is the set of directed edges, as follows. For ease of notation, we sometimes suppress the dependence on assignment $\mu$ and denote the application graph by $G = (V, E)$.

First, $V(\mu) = \bigcup \{c(t) : t \in T, \mu^t(c) \neq \emptyset\} \cup \{c(t_0) : |\mu(c)| < q_c\}$. In words, for each school $c$, we create a copy of $c$ for all student types that are present in $\mu(c)$. Moreover, if there is an empty seat in school $c$, then we create an additional node $c(t_0)$ to represent the empty seats.

Second, $E(\mu)$ consists of the following directed edges. For each student type $t$ and school $c$, we consider all type $t$ students who would prefer to be matched with $c$ rather than their current assignments, i.e., $\{s \in S_t : cP_s\mu(s)\}$. If this set is empty, then we do nothing. Otherwise, if this set is non-empty, then we consider the student with the highest priority according to $\succ_c$ in this set. Let $\hat{s}$ be this student.\(^{14}\) Assume $\hat{s}$ is assigned to school $\hat{c} \equiv \mu(\hat{s})$. Then,

(i) $(\hat{c}(t), c(t)) \in E(\mu)$ if $c(t) \in V(\mu),$

(ii) $(\hat{c}(t), c(t')) \in E(\mu)$ if $t' \neq t$, $|\mu^{t'}(c)| > q_c^{t'}$ and $|\mu^t(c)| < q_c^t,$

(iii) $(\hat{c}(t), c(t_0)) \in E(\mu)$ if $c(t_0) \in V(\mu)$ and $|\mu^t(c)| < q_c^t,$

(iv) $(c(t_0), c'(t'))$ if $t' \in T, c' \neq c$, and $|\mu^{t'}(c')| > q_{c'}^{t'}$.

In (i), $\hat{c}(t)$ points to $c(t)$, if there is a type $t$ student in $\mu(c)$. In (ii), $\hat{c}(t)$ points to $c(t')$ if $s$ can be admitted to $c$ by replacing a student of type $t'$ without violating the feasibility constraints in $c$ (type $t'$ can leave $c$ and type $t$ can join $c$) In (iii), $\hat{c}(t)$ points to $c(t_0)$ if $c$ has an empty seat in $\mu$ and $s$ can take that seat without violating the feasibility constraints in $c$. Finally, in (iv), $c(t_0)$ points to $c'(t')$ where $c' \neq c$ if a student of type $t'$ can be expelled from $c'$ without violating the feasibility constraints in $c'$.

Intuitively, in student exchange algorithm with hard bounds, we search for “trading cycles” in which we improve the matches of the students included in cycles while making sure

\(^{14}\)The reason that we only consider $\hat{s}$, and not other students with lower priorities according to $\succ_c$ is to be able to satisfy fairness for same types throughout the mechanism.
that the controlled choice constraints are not violated. Conditions (i) – (iv) is implemented to ensure that trades are feasible according to controlled choice constraints.

A more formal definition of the algorithm is in order.

**Student Exchange Algorithm with Hard Bounds**

Step 0. Consider an assignment $\mu_0$ that is fair for same types.

Step $\ell$. Construct $G(\mu_{\ell-1})$. If there are no cycles in $G(\mu_{\ell-1})$, then stop. Otherwise, consider a cycle in the graph, $c_1' (t_1') \rightarrow c_2' (t_2') \rightarrow \ldots \rightarrow c_p' (t_p') \rightarrow c_1' (t_1')$. Rematch students associated with each node in the cycle as follows. For each $c_i' (t_i')$, $i \in \{1, \ldots, p\}$, there exists $s_i \in \mu_{\ell-1} (c_i')$ such that $s_i \succeq c_{i+1}$ for all $s' \in \{s_i \in \mu_{\ell-1} (c_i') : c_{i+1} P_s \mu (s)\}$ where $c_{p+1} \equiv c_1$. Let $\mu^{\ell} (s_i) = c_{i+1}$ for all such students, otherwise let $\mu^{\ell} (s) = \mu^{\ell-1} (s)$. Go to step $\ell + 1$.

In the student exchange algorithm with hard bounds, we start with a feasible assignment that is fair for same types. Then we improve the assignments of students in each step such that this property is preserved. The algorithm has to end in a finite number of steps since in each step at least one student’s assignment is improved. We are now ready to state our main result for this section.

**Theorem 2** For any controlled school choice problem the student exchange algorithm with hard bounds yields a feasible assignment $\mu$ that is fair for same types and constrained non-wasteful. Moreover, $\mu$ is Pareto optimal among such assignments.

The proof of Theorem 2 is provided in the Appendix. A key lemma in the proof, Lemma 5, shows that if an assignment is fair for same types but not Pareto efficient among such assignments, then there exists a cycle in its application graph. Therefore, such an assignment cannot be the termination point of the algorithm. In other words, the algorithm produces a feasible assignment that is fair for same types, which is also Pareto optimal among such assignments. But any such assignment also has to be constrained non-wasteful (Lemma 3).

Erdil and Ergin (2008) also deploy cycles to improve the welfare of students while preserving fairness. However, their model does not have the controlled-choice constraints. In contrast, the main difficulty in our setup is finding the exchange cycles to improve the welfare of students in a feasible way. To this end, we use the application graph to find the exchange cycles, which is the main crux of our proof.

The input to the student exchange algorithm with hard bounds is any feasible assignment that is fair for same types. Such an assignment can be found in a number of ways, for example,
by using the controlled deferred acceptance algorithm of Ehlers (2010). Here, we provide a simpler and more intuitive algorithm of finding such an assignment that we call the deferred acceptance algorithm with fixed type allotments (DAAFTA).

Deferred acceptance algorithm with fixed type allotments (DAAFTA)

Roughly, in DAAFTA the capacities of schools are feasibly divided to type specific seats, and each type of student is allocated a seat in pseudo-schools that admit their types according to deferred acceptance algorithm. More specifically, to produce an assignment that is fair for same types, we consider a feasible assignment $\mu$ (which we know exists). For each school $c$ and type $t$, we create a pseudo school $c(t)$ with a capacity of $|\mu^t(c)|$. For each student type $t$, we create a school choice problem without the controlled-choice constraints in which only type $t$ students and schools $c(t)$ participate. Each school $c(t)$ ranks type $t$ students according to $\succ_t$ and each student $s$ ranks schools according to $\succ_s$. In each school choice problem, we run the student-proposing deferred acceptance algorithm (Gale and Shapley, 1962). When we aggregate all of the outcomes these problems, the assignment that we find is feasible and fair for same types. We provide a formal definition in the Appendix A.

To demonstrate how the SEAHB followed by DAAFTA works, we provide the following example.

Example 1. An illustration of SEAHB. Assume that there are six students $\{s_1, s_2, s_3, s_4, s_5, s_6\}$, four schools $\{c_1, c_2, c_3, c_4\}$ and three student types $\{t_1, t_2, t_3\}$ such that $\tau(s_1) = \tau(s_3) = t_1$, $\tau(s_2) = \tau(s_5) = t_2$, and $\tau(s_4) = \tau(s_6) = t_3$. Schools $c_1$, $c_3$, $c_4$ have capacities of two and $c_2$ has a capacity of one. The only effective control constraints are $q_{t_3}^{c_1} = 1$ and $q_{t_2}^{c_3} = q_{t_2}^{c_4} = 0$ (all other floors are zero and all other ceilings are equal to quotas). For all schools, student priorities are the same and given as follows: for all $c \in C$,

$$s_3 \succ_c s_5 \succ_c s_1 \succ_c s_2 \succ_c s_4 \succ_c s_6.$$ 

For students $s \in \{s_1, s_4, s_5, s_6\}$ the preferences are $c_1P_sc_2P_sc_3P_sc_4$; whereas for students $s \in \{s_2, s_3\}$ the preferences are $c_2P_sc_1P_sc_3P_sc_4$. This information is summarized in Table 1.
TABLE 1.

<table>
<thead>
<tr>
<th></th>
<th>$P_{s_1} = P_{s_4} = P_{s_5} = P_{s_6}$</th>
<th>$P_{s_2} = P_{s_3}$</th>
<th>$\succ_{c_1} \succ_{c_2} \succ_{c_3} \succ_{c_4}$</th>
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<tbody>
<tr>
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<td>$c_2$</td>
<td>$s_3$</td>
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<td></td>
<td>$c_2$</td>
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<td>$s_6$</td>
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</tbody>
</table>

capacities

|      | $q_{c_1} = 2$               | $q_{c_2} = 1$   | $q_{c_3} = 2$               | $q_{c_4} = 2$               |

effective ceilings

|      | $q_{c_3}^{t_3} = 0$ | $q_{c_3}^{t_3} = 0$ |

effective floors

Consider a feasible assignment in which $c_1$ admits one type $t_1$ student and one type $t_3$ student; $c_2$ admits one type $t_2$ student; $c_3$ admits one type $t_1$ student; and $c_4$ admits one type $t_2$ student and one type $t_3$ student. More specifically, consider $\mu$ as:

$$\mu = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ {s_1, s_4} & s_2 & s_3 & {s_5, s_6} \end{pmatrix}.$$ 

We first apply DAAFTA to $\mu$ to get a feasible assignment that is fair for same types: $s_1$ and $s_3$ exchange seats, and $s_2$ and $s_5$ exchange seats. Hence, the following assignment $\mu^0$ is the input to SEAHB:

$$\mu^0 = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ {s_3, s_4} & s_5 & s_1 & {s_2, s_6} \end{pmatrix}.$$ 

In step 1 of the SEAHB, we first construct $G(\mu^0) = (V(\mu^0), E(\mu^0))$. School $c_1$ is assigned $s_3$ and $s_4$, whose types are $t_1$ and $t_3$, respectively: $c_1(t_1), c_1(t_3) \in V(\mu^0)$. School $c_2$ is assigned $s_5$, whose type is $t_2$: $c_2(t_2) \in V(\mu^0)$. School $c_3$ is assigned $s_1$, whose type is $t_1$, and has an empty seat: $c_3(t_1), c_3(t_0) \in V(\mu^0)$. Finally, school $c_4$ is assigned $s_2$ and $s_6$ whose types are $t_2$ and $t_3$, respectively: $c_4(t_2), c_4(t_3) \in V(\mu^0)$.

Similarly the set of edges $E(\mu^0)$ is constructed. For example, consider type $t_1$ and school $c_2$. The set of type $t_1$ students who would like to switch to $c_2$ is $\{s_1, s_3\}$. Since $s_3 \succ_{c_2} s_1$, we get $(c_1(t_1), c_2(t_2)) \in E(\mu^0)$ and $(c_3(t_1), c_2(t_2)) \notin E(\mu^0)$. We construct all of the edges in this way to get $G(\mu^0)$ depicted in Figure 1.

The only cycle in this graph is $c_1(t_1) \rightarrow c_2(t_2) \rightarrow c_1(t_1)$. Hence, we rematch students associated with each node in the cycle, so $s_3$ is matched to $c_2$ and $s_5$ is matched with $c_1$. 

16
Note that both $s_3$ and $s_5$ prefer their new schools to old schools. The new assignment $\mu^1$ is given by:

$$
\mu^1 = \begin{pmatrix}
    c_1 & c_2 & c_3 & c_4 \\
    \{s_4, s_5\} & s_3 & s_1 & \{s_2, s_6\}
\end{pmatrix}.
$$

In step 2 of the algorithm, we construct $G(\mu^1)$. In $G(\mu^1)$, there are no cycles, so the algorithm outputs $\mu^1$. By Theorem 2, $\mu^1$ is fair for same types, constrained non-wasteful, and Pareto efficient among such assignments.

### 3.1 Incentives

Apart from preferences of the students, all other components of a controlled school choice problem are exogenously determined (like the capacities of the schools) or set by law (like the priority rankings and the controlled choice constraints). On the other hand, each student’s preference is his/her private information. Therefore, student preferences should be revealed by students to the school choice program. Since students must be assigned to schools for any possible preference profile, the matching mechanism of the program should be compatible with each possible profile while respecting the legal constraints imposed by the state. We define a mechanism to be (legally) feasible if it selects a feasible assignment for any reported student profile. A feasible mechanism is fair for same types if it selects an assignment that is fair for same types for any controlled school choice problem. Analogously we define non-wastefulness and constrained non-wastefulness, respectively, for a mechanism.

Any program would like to elicit the true preferences from students. If students choose to misreport, then the assignment chosen by the program is based on false preferences and
may be highly unfair given the true preferences. Avoiding this problem means constructing a mechanism where no student has ever an incentive to strategically misrepresent her true preference. Any mechanism which makes truthful revelation of preferences a dominant strategy for each student is called (dominant strategy) incentive compatible. Formally, an assignment mechanism \( \phi \) is dominant strategy incentive compatible if for any \( s \in S \), and for any profile \( P_S \), there exists no \( P'_s \) such that \( \phi_s(P'_s, P_{-s}) \neq \phi_s(P_S) \).

In contrast to the school choice problems studied in the previous literature, it turns out that it is impossible to construct a mechanism that is incentive compatible, fair for same types and constrained non-wasteful while respecting the diversity constraints given by law.

**Theorem 3** In controlled school choice there is no feasible mechanism that is dominant strategy incentive compatible, fair for same types and constrained non-wasteful.

The proof of Theorem 3 is provided in the Appendix, where we provide an example to prove the non-existence of such mechanisms.

The non-existence of feasible mechanisms, which are incentive compatible, fair for same types and (constrained) non-wasteful, shows that controlled school choice is not equivalent to the famous college admissions problem. In all models of school choice studied so far it was possible to build parallels between the school choice problem and the college admissions problem and to show that the deferred acceptance algorithm, which is non-wasteful, fair and incentive compatible, can be applied to the problem in hand since there were no diversity constraints (the floors) which are present in the controlled choice.

In college admissions, any incentive compatible mechanism which for students chooses the unique stable assignment that students prefer over any other stable assignment. In the controlled school choice, however, there is not always a unique candidate for a feasible assignment that is fair for same types and (constrained) non-wasteful.\(^\text{15}\) This provides some intuition for Theorem 3, i.e., for the non-existence of feasible mechanisms that are incentive compatible, fair for same types and (constrained) non-wasteful.

**Remark 1** Theorem 3 implies that SEAHB is not incentive compatible. Due to this fact, students may misrepresent their preferences over schools. Now if the students play a Nash equilibrium (NE), what are the properties of the assignment of any NE? It is easy to see that the outcome of any NE must be constrained non-wasteful.\(^\text{16}\) Unfortunately, the outcome of

\(^{15}\)In the example used to prove (i) of Theorem 1, \( \mu_1 \) and \( \mu_3 \) are the only feasible assignments which are fair for same types and constrained non-wasteful. Student \( s_1 \) strictly prefers \( \mu_3 \) to \( \mu_1 \) whereas student \( s_2 \) strictly prefers \( \mu_1 \) to \( \mu_3 \).

\(^{16}\)Otherwise a student would justifiably claim an empty slot and after assigning her to this empty slot, the resulting assignment is fair for same types. Then this student would benefit from alternating her preferences such that she proposes to the school with empty slot before proposing to the school to which she is assigned to.
4 Controlled School Choice with Soft Bounds

Some school districts administer floors and ceilings as hard bounds, so a theoretical analysis of such policies is inarguably important. In the previous sections, we accommodate this constraint by considering an assignment infeasible if it assigns less than $q^t_c$ or more than $q^t_c$ number of type $t$ students to school $c$. However, applications of these hard bounds are quite paternalistic in the sense that assignments may be enforced despite student preferences. In contrast, in this section we view these constraints as soft bounds. In the controlled school choice with soft bounds, school districts adapt a dynamic priority structure: giving highest priorities to the student types who have not filled their floors; medium priorities to the student types who have filled their floors, but not filled their ceilings; and lowest priorities to student types who have filled their ceilings. Yet, schools can still admit fewer students than their floors or more than their ceilings as long as students with higher priorities do not veto this assignment. With this view, there are no feasibility constraints as long as school quotas are not exceeded (and our approach below can be used for situations where no feasible assignment exists). All controlled choice concerns are embedded in the choice functions of the schools that we describe below.

The choice function for school $c$ depends on quota $q_c$, floors $q^T_c$, and ceilings $q^T_c$ as described above. However, we are going to take these parameters as given and simplify the notation by omitting them. To define the choice function more formally, given $\tilde{S} \subseteq S$, let $Ch_c(\tilde{S}, q_c, (q^t_c)_{t \in T})$ be the subset of students $\tilde{S}' \subseteq \tilde{S}$ that includes the highest ranked students in $\tilde{S}$ according to $\succ_c$ such that there are no more than $q_c$ students in total and $q^t_c$ students of type $t$. In addition, let

\[
Ch_c^{(1)}(\tilde{S}) \equiv Ch_c(\tilde{S}, q_c, (q^t_c)_{t \in T}),
\]
\[
Ch_c^{(2)}(\tilde{S}) \equiv Ch_c(\tilde{S} \setminus Ch_c^{(1)}(\tilde{S}), q_c - |Ch_c^{(1)}(\tilde{S})|, (q^t_c - q^t_c')_{t \in T}), \text{ and}
\]
\[
Ch_c^{(3)}(\tilde{S}) \equiv Ch_c(\tilde{S} \setminus (Ch_c^{(1)}(\tilde{S}) \cup Ch_c^{(2)}(\tilde{S})), q_c - |Ch_c^{(1)}(\tilde{S}) \cup Ch_c^{(2)}(\tilde{S})|, (q_c - q^t_c')_{t \in T}).
\]

Intuitively, $Ch_c^{(1)}(\tilde{S})$ is the set of students chosen with the highest priorities among $\tilde{S}$ without exceeding the floor of each student type, $Ch_c^{(2)}(\tilde{S})$ is the set of remaining students chosen from $\tilde{S}$ with the highest priorities without exceeding the ceilings, and $Ch_c^{(3)}(\tilde{S})$ is the

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17Ehlers (2010) provides an explicit example.
18In school choice problems without control and legal constraints, Ergin and Sönmez (2006) consider revelation games induced by the Boston school choice mechanism and the deferred acceptance algorithm.
set of students chosen above the ceilings. Finally, \( Ch_c(\tilde{S}) \equiv Ch_c^{(1)}(\tilde{S}) \cup Ch_c^{(2)}(\tilde{S}) \cup Ch_c^{(3)}(\tilde{S}) \) is the set of students chosen from \( \tilde{S} \). It is apparent from this formulation that schools dynamically give highest priorities to the student types who have not filled their floors; medium priorities to the student types who have filled their floors, but not filled their ceilings; and lowest priorities to student types who have filled their ceilings.

Now we define the desirable properties under the soft bounds approach. In controlled school choice with soft bounds, any assignment is feasible. An assignment \( \mu \) is non-wasteful under soft bounds if for any student \( s \) and any school \( c \), \( cP_s \mu(s) \) implies \( |\mu(c)| = q_c \). Previously, non-wastefulness required student \( s \) to be matched with \( c \) without violating ceilings and floors, which is not required anymore. Furthermore, an assignment \( \mu \) removes justifiable envy under soft bounds if for any student \( s \) and any school \( c \) such that \( cP_s \mu(s) \) with \( \tau(s) = t \), we have both \( |\mu^t(c)| \geq q^t_c \) and \( s' \succ_c s \) for all \( s' \in \mu^t(c) \), and either

(i) \( |\mu^t(c)| \geq q^t_c \) and \( s' \succ_c s \) for all \( s' \in \mu(c) \) such that \( |\mu^{\tau(s')}(c)| > q^{\tau(s')}_c \), or

(ii) \( q^t_c > |\mu^t(c)| \geq q^t'_c \), and

(a) \( |\mu^t'(c)| \leq q^t'_c \) for all \( t' \in T \setminus \{t\} \), and

(b) \( s' \succ_c s \) for all \( s' \in \mu(c) \) such that \( q^{\tau(s')}_c \geq |\mu^{\tau(s')}(c)| > q^{\tau(s')}_c \).

Less formally, an assignment removes justifiable envy under soft bounds if a student \( s \) of type \( t \) cannot attend a favorable school \( c \), then type \( t \) students fill their floor in \( c \) and \( c \) prefers all type \( t \) students that it has been assigned to \( s \). In other words, either \( c \) has admitted more type \( t \) students than its ceiling, and all students with types exceeding their ceilings are preferred to \( s \); or \( c \) has admitted more type \( t \) students than its floor, but not more than its ceiling, yet there are no students with types exceeding their ceilings, and all students with types exceeding their floors are preferred to \( s \).

Finally, an assignment \( \mu \) is fair under soft bounds if it removes justifiable envy under soft bounds.

With hard bounds, no assignment that is fair and non-wasteful exists even if fairness is restricted to students with the same types (Theorem 1). However, with soft bounds we guarantee the existence of an assignment that is non-wasteful and fair under soft bounds. To show this, we consider the student-proposing deferred acceptance algorithm with soft bounds, defined below.

Deferred Acceptance Algorithm with Soft Bounds (DAASB)
Step 1  Start with the assignment in which no student is matched. Each student $s$ applies to her first-choice school. Let $S_{c,1}$ denote the set of students who applied to school $c$. School $c$ accepts the students in $Ch_c(S_{c,1})$ and rejects the rest.

Step $k$  Start with the tentative assignment obtained at the end of step $k-1$. Each student $s$ who got rejected at step $k-1$ applies to her next-choice school. Let $S_{c,k}$ denote the set of students who either were tentatively matched to $c$ at the end of step $k-1$, or applied to school $c$ in this step. Each school accepts the students in $Ch_c(S_{c,k})$ and rejects the rest. If there are no rejections, then stop.

DAASB terminates when there are no rejections. At each step of the algorithm, there is at least one student who gets rejected. Hence, the algorithm ends in finite time.

Below, we establish that the well-known properties of the deferred acceptance algorithm continue to hold with DAASB.

Theorem 4  For any controlled school choice problem, DAASB yields an assignment that is fair and non-wasteful under soft bounds. Moreover, in the assignment produced by DAASB each student is matched with the best school among the set of all such assignments.

Even though the school priorities are determined dynamically using the floors and ceilings, the choice functions of schools satisfy a desirable property called substitutability. Choice function $Ch_c$ satisfies substitutability if for any group of students $\tilde{S}$ that contains students $s$ and $s'$ ($s \neq s'$), $s \in Ch_c(\tilde{S})$ implies $s \in Ch_c(\tilde{S} \setminus \{s'\})$. Substitutability was introduced by Kelso and Crawford (1982) for matching markets with transfers and adapted to traditional matching markets by Roth (1984).

It is easy to see that schools’ choice functions satisfy substitutability: if $s \in Ch_c^{(1)}(\tilde{S})$, then $s \in Ch_c^{(1)}(\tilde{S} \setminus \{s'\})$. Otherwise, if $s \in Ch_c^{(i)}(\tilde{S})$ for $i = 2$ or $i = 3$, then either $s' \notin \bigcup_{j=1}^{i} Ch_c^{(j)}(\tilde{S})$ and $Ch_c^{(i)}(\tilde{S}) = Ch_c^{(i)}(\tilde{S} \setminus \{s'\})$ or $s' \in \bigcup_{j=1}^{i} Ch_c^{(j)}(\tilde{S})$ and $(\bigcup_{j=1}^{i} Ch_c^{(j)}(\tilde{S})) \setminus \{s'\} \subseteq \bigcup_{j=1}^{i} Ch_c^{(j)}(\tilde{S} \setminus \{s'\})$. Therefore, $Ch_c$ is substitutable for every $c$. The proof of Theorem 4 follows immediately from Theorem 6.8 in Roth and Sotomayor (1990) since choice functions are substitutable.

To demonstrate how DAASB works, we provide the following example.

Example 2.  An illustration of DAASB. Assume that the school choice problem is the same as the environment in Example 1, with six students $\{s_1, s_2, s_3, s_4, s_5, s_6\}$, four schools $\{c_1, c_2, c_3, c_4\}$ and three student types $\{t_1, t_2, t_3\}$ such that $\tau(s_1) = \tau(s_3) = t_1$, $\tau(s_2) = \tau(s_5) = t_2$, and $\tau(s_4) = \tau(s_6) = t_3$. School priorities, student preferences and school quotas
are summarized in Table 1 above, but here constraints are soft bounds instead of hard bounds.

In the first round students apply to schools:

\[
\begin{array}{ccccccc}
    s_1 & s_2 & s_3 & s_4 & s_5 & s_6 \\
    c_1 & c_2 & c_2 & c_1 & c_1 & c_1 \\
\end{array}
\]

The relevant choice sets are for:

\[
Ch_{c_1}(1)(\{s_1, s_4, s_5, s_6\}) = \{s_4\}
\]
\[
Ch_{c_1}(2)(\{s_1, s_5, s_6\}) = \{s_5\}
\]
\[
Ch_{c_2}(2)(\{s_2, s_3\}) = \{s_3\}
\]

After first round, \(s_1\) and \(s_6\) are rejected by \(c_1\) and \(s_2\) is rejected by \(c_2\). In the second round students apply to following schools:

\[
\begin{array}{ccccccc}
    s_1 & s_2 & s_3 & s_4 & s_5 & s_6 \\
    c_2 & c_1 & c_2 & c_1 & c_1 & c_2 \\
\end{array}
\]

Corresponding choice sets in the second round are:

\[
Ch_{c_1}(1)(\{s_2, s_4, s_5\}) = \{s_4\}
\]
\[
Ch_{c_1}(2)(\{s_2, s_5\}) = \{s_5\}
\]
\[
Ch_{c_2}(2)(\{s_1, s_3, s_6\}) = \{s_3\}
\]

Therefore, \(s_1\) and \(s_6\) are rejected by \(c_2\) and \(s_2\) is rejected by \(c_1\). In the third round, the students will apply to:

\[
\begin{array}{ccccccc}
    s_1 & s_2 & s_3 & s_4 & s_5 & s_6 \\
    c_3 & c_3 & c_2 & c_1 & c_1 & c_3 \\
\end{array}
\]

The assignments to \(c_1\) and \(c_2\) were finalized in the second round. So, the only relevant school at this stage is \(c_3\) and its choice sets are:

\[
Ch_{c_3}(2)(\{s_1, s_2, s_6\}) = \{s_1\}
\]
\[
Ch_{c_3}(3)(\{s_2, s_6\}) = \{s_2\}
\]

Inevitably \(s_6\) is rejected by \(c_3\). In the last round, \(s_6\) applies to \(c_4\) and gets in. So the final
assignment of DAASB is:

\[ \mu = \left( \begin{array}{cccc}
  c_1 & c_2 & c_3 & c_4 \\
  \{s_4, s_5\} & s_3 & \{s_1, s_2\} & s_6
\end{array} \right). \]

This assignment weakly Pareto dominates the assignment obtained by SEAHB (see Example 1), since the only difference between these assignments is that \( s_2 \) is assigned to \( c_3 \) instead of \( c_4 \), which she prefers.

Even though SEAHB fails to satisfy incentive compatibility, DAASB satisfies a stronger version of incentive compatibility: An assignment mechanism \( \phi \) (choosing for any profile an assignment) is **group (dominant strategy) incentive compatible** if for any group of students \( \hat{S} \subseteq S \), for any profile \( P_S \) there exists no \( P''_S \) such that \( \phi_s(P''_S, P_{S\setminus S}) \neq \phi_s(P_S) \) for all \( s \in \hat{S} \). If a mechanism is group incentive compatible, then there exists no group of students who can jointly change their preference profiles to make each student in the group better off.

**Theorem 5** DAASB is group dominant strategy incentive compatible.

Schools’ choice functions satisfy another desirable property called the law of aggregate demand in addition to substitutability. School \( c \)’s preferences satisfy the law of aggregate demand if for any \( S'' \subseteq S' \subseteq S \), we have \(|Ch_c(S'')| \leq |Ch_c(S')|\). The law of aggregate demand was first introduced in Alkan (2002) and Alkan and Gale (2003) for traditional matching markets as size monotonicity. Later it was adapted to matching with contracts by Fleiner (2003) and Hatfield and Milgrom (2005).

The law of aggregate demand is satisfied in our setup since if \(|Ch_c(S'')| = q_c\) then \(|Ch_c(S')| = q_c\). Moreover, if \(|Ch_c(S'')| < q_c\) then either \(|Ch_c(S')| = |Ch_c(S'')|\) if \( S'' = S' \) or \(|Ch_c(S')| < |Ch_c(S'')|\) if \( |S''| < |S'|\).

Since choice functions satisfy substitutability and the law of aggregate demand, the DAASB is group dominant strategy incentive compatible (Hatfield and Kojima, 2009). Hatfield and Kojima (2009) show, in a many-to-one matching model with contracts, that if the schools’ choice functions satisfy these two properties, then the student proposing deferred acceptance algorithm is group incentive compatible. Our setup can be trivially embedded in the many-to-one matching model with contracts of Hatfield and Kojima (2009), so the conclusion follows.\(^\text{19}\)

Finally, we establish a Pareto dominance relation between the outcome of DAASB and non-wasteful assignments that satisfy another fairness notion which is stronger than fairness

\(^\text{19}\)See also Martínez et al. (2004).
across types. This result gives us an interesting connection between hard bounds and soft bounds for controlled school choice. The fairness notion is defined as follows.

A student $s$ weakly-envies student $s'$ when $s$ prefers the school at which $s'$ is enrolled, say school $c$, to her school. However, the nature of controlled school choice imposes the following (legal) constraints: Weak-envy is justified only when

(i) student $s$ has higher priority to be enrolled at school $c$ than student $s'$,

(ii) student $s$ can be enrolled at school $c$ without violating controlled choice constraints by removing $s'$ from $c$.\(^{20}\)

An assignment is **strongly-fair across types** if no student justifiably weakly-envies any student. Now, we proceed with the formal result.

**Theorem 6** Suppose that $\mu$ is a feasible assignment that is strongly-fair across types and non-wasteful. Then all students weakly prefer the outcome of DAASB to $\mu$.

The proof of Theorem 6 is provided in the Appendix. Since $\mu$ is a feasible assignment that is strongly-fair across types, it is also fair under soft bounds. In addition, if $\mu$ is also non-wasteful under soft bounds, then the result follows from Theorem 4. Otherwise, if $\mu$ is not non-wasteful under soft bounds, then there exist a school $c$ and a student $s$ such that $cP_s \mu(s)$ and $|\mu(c)| < q_c$. Whenever there exists such a pair we can improve the matches of students by reassigning them to empty seats. In the proof, we show that this improvement process delivers an assignment that is fair under soft bounds and non-wasteful under soft bounds. Again, the conclusion is reached by applying Theorem 4. Therefore, if feasible assignments that are strongly-fair across types and non-wasteful exist, then the outcome of DAASB (weakly) Pareto dominates all such assignments. In such situations all students are weakly better off under soft bounds than under hard bounds.\(^{21}\)

Although we have required students to submit full rankings of all schools, it is easy to check that all our results under soft bounds remain true if students are only required to submit partial rankings of the schools which they strictly prefer to their outside options (like private schooling). Hence, the desirable properties of DAASB continue to hold in this case.

\(^{20}\)Recall that envy is justified only with the additional requirement that $s'$ can be enrolled at another school without violating constraints.

\(^{21}\)The corresponding result for assignments that are fair across types and non-wasteful does not hold. An example showing the contrary is available from the authors.
5 Conclusion

Although there is a large literature in education evaluating and estimating the effects of segregation across schools on students’ achievements (Hanushek et al. (2002), Guryan (2004), Card and Rothstein (2005), and others),\textsuperscript{22} on how to measure segregation and determine optimal desegregation guidelines,\textsuperscript{23} none of these papers discusses the problem of \textit{how in practice} to assign students to schools while complying with these desegregation guidelines. This is exactly what the first part of our paper does.

Without controlled choice, the student proposing deferred acceptance algorithm eliminates any justified envy and makes truthful revelation of preferences a dominant strategy for students (Abdulkadiroğlu and Sönmez, 2003). Once controlled choice constraints are imposed, we have shown that as hard bounds it may be impossible to eliminate justified envy. Justified envy can only be eliminated among students of the same type.

Thus we demonstrate that controlled choice comes with a price, especially when the bounds are taken as inflexible hard bounds. The alternative view of soft bounds has clear benefits over hard bounds. It results in much attractive fairness, efficiency, and incentive properties. The downside of it is that desired diversity in schools is achieved only when student preferences are also in line with them. If the school districts’ objectives are not very paternalistic, and only giving an opportunity to parents to achieve diversity is good enough, soft bound policies is the clear winner. Otherwise the choice is hard bounds, along with the prices coming with it.\textsuperscript{24}

Appendix A: Deferred Acceptance Algorithm with Fixed Type Allotments (DAAFTA)

Consider a feasible assignment $\mu$ (which we know exists). For each school $c$ and type $t$, we create a pseudo school $c(t)$ with a capacity of $|\mu'(c)|$. For each student type $t$, we create a school choice problem without the controlled-choice constraints in which only type $t$ students and schools $c(t)$ participate. Each school $c(t)$ ranks type $t$ students according to $\succ_t$ and each student $s$ ranks schools according to $\succ_s$.

\textsuperscript{22}We refer the interested reader to Echenique et al. (2006) for an illuminating account of this literature.

\textsuperscript{23}School segregation can be purely racial or, as in Echenique et al. (2006), school segregation is measured according to the spectral segregation index of Echenique and Fryer (2007) which uses the intensity of social interactions among the members of a group (see also Cutler and Glaeser (1997)).

\textsuperscript{24}One benefit of hard bounds is that, it is straightforward to check whether they are implemented or not. On the other hand, if the districts use soft bounds, schools are not guaranteed to have diverse student bodies. Hence parents can question whether these control policies are appropriately applied.
That is, we consider $K$ standard school choice problems, one for each $t \in T$. In school choice problem for $t$, there are $m$ schools, denoted by $c(t)$ for $c \in C$, and $S_t$ is the set of students participating in this problem. School capacities and priorities, and student preferences are defined as follows. $c(t)$ has a capacity of $|\mu^t(c)|$ (which can be 0). For $s \in S_t$, preferences are defined by $c(t)P^t_s c'(t)$ if and only if $cP_s c'$. For school $c(t)$ priorities are the same as original priorities: for $s,s' \in S_t$, we have $s \succ_{c(t)} s'$ if and only if $s \succ c s'$.

In each different school choice problem, we run the student-proposing deferred acceptance algorithm (Gale and Shapley, 1962). Let us denote the matching obtained by DA algorithm by $\mu^*$. Then we aggregate all of the outcomes these problems. That is, we define $\mu^*$ as follows: $\mu^*(c) = \cup_{t \in T} \mu^*(c(t))$ and for $s \in S_t$, $\mu^*(s) = \mu^t(s)$.

$\mu^*$ is the matching produced by DAAFTA. Now, we can easily argue that $\mu^*$ is feasible and fair for same types. It is feasible since the total number of seats and number of students in problem $t$ are exactly equal to each other, hence we have $|\mu^*(c)| = |\mu^t(c)|$. It is fair because by applying DA algorithm in problem $t$, we are making sure that no student of type $t$ justifiably envies another student of type $t$.

DAAFTA, however, can be wasteful. Consider a student $s$ of type $t$ and her top school $c$. Suppose that the initial feasible assignment $\mu$ is such that $|\mu^t(c)| = 0$ and $|\mu(c)| < q_c$, then an empty seat in $c$ is wasted which can be happily taken by $s$. Hence, DAAFTA is “rigid”: for each type $t$, the slots, which will be filled with type-$t$ students, are exogenously given by the feasible assignment $\mu$. This rigidity is also a source of inefficiency. To see this consider a problem with no control constraints, with two schools: $c_1$ and $c_2$ with quotas of 1 each, and two students: $s_1$ of type $t_1$ and $s_2$ of type $t_2$. Suppose $s_i$ prefers $c_i$ more than $c_j$, and $s_i$ has top priority for $c_i$ ($i \in \{1, 2\}$ and $j = 3 - i$). Consider $\mu$ which matches $s_i$ with $c_j$ and another matching $\hat{\mu}$ which matches $s_i$ with $c_i$. It is easy to see that applying DAAFTA on $\mu$ will not change the matching; yet $\mu$ is dominated by $\hat{\mu}$.

Appendix B: Proofs

In this Appendix, we provide the omitted proofs.

Proof of Theorem 1

The proof for both parts is by means of an example. For part (i) consider the following problem consisting of three schools $\{c_1, c_2, c_3\}$ and three students $\{s_1, s_2, s_3\}$. Each school has a capacity of one ($q_c = 1$ for all schools $c$). The type space consists of two types $t_1$ and $t_2$. Students $s_1$ and $s_2$ are of type $t_1$ whereas student $s_3$ is of type $t_2$. For all types the ceiling
is equal to one at all schools ($\bar{q}_c^t = 1$ for all types $t$ and all schools $c$). School $c_1$ has a floor for type $t_1$ of $q_{t_1}^{c_1} = 1$. All other floors are equal to zero. The schools’ priorities are given by $s_2 \succ c_1 \ s_1 \succ c_1 \ s_3$, $s_2 \succ c_2 \ s_1 \succ c_2 \ s_3$ and $s_1 \succ c_1 \ s_2 \succ c_3 \ s_3$. The students’ preferences are given by $c_2P_s c_3P_s c_1$, $c_3P_s c_2P_s c_1$ and $c_2P_s c_3P_s c_1$. This information is summarized in Table 2.

<table>
<thead>
<tr>
<th>$\succ c_1$</th>
<th>$\succ c_2$</th>
<th>$\succ c_3$</th>
<th>$P_{s_1}$</th>
<th>$P_{s_2}$</th>
<th>$P_{s_3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_2$</td>
<td>$s_2$</td>
<td>$s_1$</td>
<td>$c_2$</td>
<td>$c_3$</td>
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<td>$s_1$</td>
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<td>$s_3$</td>
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<td>$c_1$</td>
<td>$c_1$</td>
<td>$c_1$</td>
</tr>
</tbody>
</table>

Next we determine the set of assignments which are both feasible and fair across types for this problem. Feasibility requires that student $s_1$ or student $s_2$ is assigned school $c_1$ and all students are enrolled at a school. Therefore,

$$
\mu_1 = \begin{pmatrix}
  c_1 & c_2 & c_3 \\
  s_1 & s_2 & s_3 
\end{pmatrix}
$$

$s_2$ envies $s_3$ \quad $\mu_2 = \begin{pmatrix}
  c_1 & c_2 & c_3 \\
  s_1 & s_3 & s_2 
\end{pmatrix}$

$s_2$ envies $s_1$ \quad $\mu_4 = \begin{pmatrix}
  c_1 & c_2 & c_3 \\
  s_2 & s_1 & s_3 
\end{pmatrix}$

$s_1$ envies $s_3$ \quad $\mu_3 = \begin{pmatrix}
  c_1 & c_2 & c_3 \\
  s_2 & s_3 & s_1 
\end{pmatrix}$

are the only assignments which are feasible. Now (as indicated above)

(i) $\mu_1$ is not fair across types because $s_2$ justifiably envies $s_3$ at $c_3$,

(ii) $\mu_2$ is not fair across types because $s_1$ justifiably envies $s_2$ at $c_3$,

(iii) $\mu_3$ is not fair across types because $s_1$ justifiably envies $s_3$ at $c_2$, and

(iv) $\mu_4$ is not fair across types because $s_2$ justifiably envies $s_1$ at $c_2$.

Hence there is no assignment which is both feasible and fair across types.
For part (ii) consider the following problem consisting of three schools \( \{c_1, c_2, c_3\} \) and two students \( \{s_1, s_2\} \). Each school has a capacity of two \( (q_c = 2 \text{ for all schools } c) \). All students are of the same type \( t \). The ceiling of type \( t \) is equal to two at all schools \( (q^t_c = 2 \text{ for all schools } c) \). School \( c_1 \) has a floor for type \( t \) of \( q^t_{c_1} = 1 \). All other floors are equal to zero. The schools’ priorities are given by \( s_2 \succ c_1 \succ s_1, s_2 \succ c_2 \succ s_1 \) and \( s_1 \succ c_3 \succ s_2 \). The students’ preferences are given by \( c_2P_{s_1}c_3P_{s_1}c_1 \) and \( c_3P_{s_2}c_2P_{s_2}c_1 \). This information is summarized in Table 3.

**Table 3.**

<table>
<thead>
<tr>
<th>( \succ_{c_1} )</th>
<th>( \succ_{c_2} )</th>
<th>( \succ_{c_3} )</th>
<th>( P_{s_1} )</th>
<th>( P_{s_2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s_2 )</td>
<td>( s_2 )</td>
<td>( s_1 )</td>
<td>( c_2 )</td>
<td>( c_3 )</td>
</tr>
<tr>
<td>( s_1 )</td>
<td>( s_1 )</td>
<td>( s_2 )</td>
<td>( c_3 )</td>
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</tbody>
</table>

capacities \( q_{c_1} = 2 \), \( q_{c_2} = 2 \), \( q_{c_3} = 2 \)

ceiling for \( t \) \( \overline{q}^t_{c_1} = 2 \), \( \overline{q}^t_{c_2} = 2 \), \( \overline{q}^t_{c_3} = 2 \)

floor for \( t \) \( q_{c_1}^t = 1 \), \( q_{c_2}^t = 0 \), \( q_{c_3}^t = 0 \)

Next we determine the set of assignments which are feasible for this problem. Feasibility requires that student \( s_1 \) or student \( s_2 \) is assigned school \( c_1 \) and all students are enrolled at a school. Therefore,

\[
\mu_1 = \begin{pmatrix} c_1 & c_2 & c_3 \\ s_1 & s_2 & \emptyset \end{pmatrix} \text{ } s_2 \text{ claims } c_3 \quad \text{ } \mu_2 = \begin{pmatrix} c_1 & c_2 & c_3 \\ s_1 & \emptyset & s_2 \end{pmatrix}
\]

\( s_2 \text{ envies } s_1 \uparrow \quad \text{ } \downarrow s_1 \text{ envies } s_2 \)

\[
\mu_4 = \begin{pmatrix} c_1 & c_2 & c_3 \\ s_2 & s_1 & \emptyset \end{pmatrix} \quad \text{ } s_1 \text{ claims } c_2 \quad \mu_3 = \begin{pmatrix} c_1 & c_2 & c_3 \\ s_2 & \emptyset & s_1 \end{pmatrix}
\]

and \( \mu_5 = \begin{pmatrix} c_1 & c_2 & c_3 \\ \{s_1, s_2\} & \emptyset & \emptyset \end{pmatrix} \) are the only assignments which are feasible. Now (as indicated above)

(i) \( \mu_1 \) is wasteful because \( s_2 \) justifiably claims an empty slot at \( c_3 \),

(ii) \( \mu_2 \) is not fair for same types because \( s_1 \) justifiably envies \( s_2 \) at \( c_3 \),

(iii) \( \mu_3 \) is wasteful because \( s_1 \) justifiably claims an empty slot at \( c_2 \),

(iv) \( \mu_4 \) is not fair for same types because \( s_2 \) justifiably envies \( s_1 \) at \( c_2 \); and

(v) \( \mu_5 \) is wasteful because \( s_1 \) justifiably claims an empty slot at \( c_2 \).
Hence there is no feasible assignment which is both fair for same types and non-wasteful.

**Proof of Theorem 2**

Let $P_S$ be a controlled school choice problem. We prove the claim by using the following lemmas.

**Lemma 1** The assignment produced by the student exchange algorithm with hard bounds is feasible.

**Proof.** For $t, t' \in T$, each node $c(t)$ only points to a node $c'(t')$ when a type $t'$ student can be fired from school $c'$ and a type $t$ student can be admitted to $c'$ without violating the feasibility conditions in school $c'$. Thus, when we execute a cycle consisting of such nodes we get a feasible assignment. On the other hand, suppose that we execute a cycle containing $c(t_0)$. Let the cycle include the following path $c'(t) \rightarrow c(t_0) \rightarrow c''(t')$. Since $c'(t)$ is pointing $c(t_0)$, then a type $t$ student can take an empty seat in $c$ without violating feasibility constraints. Similarly, since $c(t_0)$ is pointing $c''(t')$ a type $t'$ student can be fired from school $c''$ without violating the feasibility constraints. Therefore, the assignment produced is feasible. ■

**Lemma 2** The assignment produced by the student exchange algorithm with hard bounds is fair for same types.

**Proof.** Let $\mu$ be assignment that is the input of the algorithm and $\mu'$ be the assignment produced by it. Suppose for contradiction that $\mu'$ is not fair for same types. Therefore, there exist students $s$ and $s'$ of the same type such that $s'$ justifiably envies $s$ at school $c$ under $\mu'$: $\tau(s) = \tau(s')$, $c \equiv \mu'(s)P_s \mu'(s')$, and $s' \succ_c s$. There are two cases depending on whether $\mu(s) = \mu'(s)$.

- $\mu(s) = \mu'(s)$: For student $s'$, let $R_{s'}$ be the weak order associated with $P_{s'}$. Since $\mu$ is fair for same types and $s' \succ_c s$, we have $\mu(s')R_{s'}c$. Since the algorithm improves the match of every student or keeps it the same, we get $\mu'(s')R_{s'}\mu(s')$. This and $\mu(s')R_{s'}c$ imply $\mu'(s')R_{s'}c$.

- $\mu(s) \neq \mu'(s)$: In this case, $s$ must have matched with $c$ in one of the steps of the algorithm. To have a node for type $\tau(s)$ in $\mu(s)$ point to any node for school $c$, $s$ must have the highest priority among type $\tau(s)$ students who prefer $c$ to their current assignments. Since $s'$ wants to switch to $c$ at any step of the algorithm, we get $s \succ_c s'$.

In both cases, we get a contradiction. The conclusion follows. ■
Lemma 3 Suppose that $\mu$ is a feasible assignment that is fair for same types, which is also Pareto efficient among such assignments. Then $\mu$ is constrained non-wasteful.

Proof. Suppose, otherwise, that $\mu$ violates constrained non-wastefulness. Then there exists a student $s$ and school $c$ with an empty seat such that the assignment in which school $c$ admits student $s$ without changing the matches of any other student is fair for same types. This gives a contradiction to Pareto efficiency. ■

Lemma 4 Suppose that $\mu$ is an assignment that is fair for same types and Pareto efficient among such assignments. Then $\mu$ is also Pareto efficient among assignments that are fair for same types and constrained non-wasteful.

Proof. This follows from the fact that the set of assignments that are fair for same types is a superset of the set of assignments that are fair for same types and constrained non-wasteful. The conclusion follows from Lemma 3 and the fact that if $\mu$ is Pareto efficient in a bigger set, then it is also going to be Pareto efficient in a smaller set. ■

Lemma 5 Let $\mu$ be an assignment that is fair for same types, which is not Pareto efficient among such assignments. Then there exists a cycle in $G(\mu)$, and hence the assignment produced by the student exchange algorithm with hard bounds is different than $\mu$.

Proof. Let $\mu'$ be an assignment that is fair for same types, which Pareto dominates $\mu$. We are going to show that there exists a cycle in $G(\mu)$. To do this, we split the analysis whether there exists a school $c$ such that $|\mu(c)| \neq |\mu'(c)|$.

Case 1: (There exists $c$ such that $|\mu(c)| \neq |\mu'(c)|$.) Since the total number of assigned students is the same in both $\mu$ and $\mu'$, there exists $c$ such that $|\mu'(c)| > |\mu(c)|$. Hence, there exists $t_i \in T$ such that $|\mu'^{t_i}(c)| > |\mu^{t_i}(c)|$. Hence, in $G(\mu)$ there exists a school $c^{(1)}$ such that $c^{(1)}(t_i)$ is pointing $c(t_0)$. If the floor of type $t_i$ in $c^{(1)}$ is not binding in $\mu$, then $c(t_0)$ is also pointing $c^{(1)}(t_i)$. Therefore, there exists a cycle and we are done. Suppose otherwise that the floor of type $t_i$ in $c^{(1)}$ is binding at $\mu$. Let $s \in \mu^{t_i}(c^{(1)})$ be the student with the
highest priority according to $\succ_c$ among $\mu^{t_i}(c^{(1)})$. Either $s$ is matched with $c$ in $\mu'$, or $s$ is not matched with it, then $s$ must have been matched with a better school in $\mu'$ since $\mu'$ is fair. Both imply that there exists a student of type $t_i$ who is in $\mu'(c^{(1)})$ but not in $\mu(c^{(1)})$ since $\mu'$ is feasible and that the number of type $t_i$ students in $\mu^{t_i}(c^{(1)})$ is only at the floor level, i.e., $|\mu^{t_i}(c^{(1)})| = q^{t_i}_{c^{(1)}}$. Therefore, there exists a school $c^{(2)}$ such that $c^{(2)}(t_i)$ is pointing to $c^{(1)}(t_i)$ in $G(\mu)$. By a similar argument, we see that either $c(t_0)$ is pointing to $c^{(2)}(t_i)$ or that there exists a school $c^{(3)}$ such that $c^{(3)}(t_i)$ is pointing to $c^{(2)}(t_i)$. Since there is a finite number of schools, by mathematical induction, we see that there exists a positive number $p$ such that $c(t_0)$ is pointing to $c^{(p)}(t_i)$ and for every $l = 1, \ldots, p$ $c^{(l)}(t_i)$ is pointing to $c^{(l-1)}(t_i)$. Hence, there exists a cycle consisting of type $t_i$ nodes and a node for an empty seat in $G(\mu)$.

![Application Graph G(\mu) for Case 2.](image)

**Case 2:** (For all $c$, $|\mu(c)| = |\mu'(c)|$.) In this case, since $\mu \neq \mu'$ there exist student $s \in S_{t_i}$ and school $c$ such that $s \in \mu'(c) \setminus \mu(c)$. Therefore, in $G(\mu)$, $c(t_i)$ is being pointed by $c^{(1)}(t_i)$ for some $c^{(1)} \in C$. If there exists a type $t_i$ student in $\mu'(c^{(1)}) \setminus \mu(c^{(1)})$ then there exists another node $c^{(2)}(t_i)$ pointing to $c^{(1)}(t_i)$. Suppose otherwise that there exists no such student. The type $t_i$ student with the highest priority in $\mu^{t_i}(c^{(1)})$ must have been matched with a new school in $\mu'$ (either $c$ or another one) that she prefers over $c$ since $\mu'$ is fair for same types. Therefore, we get that $|\mu^{t_i}(c^{(1)})| > q^{t_i}_{c^{(1)}}$; and that $|\mu'(c^{(1)}) \setminus \mu(c^{(1)})| > 0$ since $|\mu(c^{(1)})| = |\mu'(c^{(1)})|$. Consider a type $t_j$ such that there exists a student $s'$ of type $t_j$ such that $s' \in \mu'(c^{(1)}) \setminus \mu(c^{(1)})$. Hence, there exists a node $c^{(2)}(t_j)$ pointing to $c^{(1)}(t_i)$ in $G(\mu)$ since the number of type $t_i$ students exceed their floor. We continue in this fashion constructing a path in $G(\mu)$. Since there exists a finite number of nodes in $G(\mu)$, we conclude that this path must be a cycle. This completes the argument.

Now we establish Theorem 2 using these lemmas. Let $\mu$ be the assignment produced by the student exchange algorithm with hard bounds. It is fair for same types (Lemma 2). Since the algorithm produces $\mu$, there are no cycles in $G(\mu)$, so $\mu$ is also Pareto efficient among assignments that are fair for same types (Lemma 5). Therefore, it is also constrained non-wasteful (Lemma 3). Moreover, it is also Pareto efficient among assignments that are fair for same types and constrained non-wasteful (Lemma 4).
Proof of Theorem 3

The proof is by means of an example. Consider the following problem consisting of three schools \( \{c_1, c_2, c_3\} \) and two students \( \{s_1, s_2\} \). Each school has a capacity of two \( (q_c = 2 \text{ for all schools } c) \). The type space consists of a single type \( t \), i.e., both students are of the same type \( t \). The ceiling for type \( t \) is equal to two for each school \( (\bar{q}_c^t = 2 \text{ for all schools } c) \). School \( c_1 \) has a floor for type \( t \) of \( q_{c_1}^t = 1 \) and both other schools have a floor of 0 for type \( t \). Schools \( c_1 \) and \( c_2 \) give higher priority to student \( s_2 \) whereas school \( c_3 \) gives higher priority student \( s_1 \). The students’ preferences are given by \( c_2 P_1 c_1 P_2 s_1 c_3 P_1 s_2 \) and \( c_3 P_1 c_1 P_2 s_2 c_2 \). This information is summarized in Table 4.

<table>
<thead>
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<td>( c_3 )</td>
<td>( c_2 )</td>
</tr>
</tbody>
</table>

Next we determine the set of feasible assignments. Feasibility requires that one of the students is assigned school \( c_1 \) and each student is assigned a school. Then it is straightforward to verify that

\[
\begin{aligned}
\mu_1 &= \left( \begin{array}{ccc} c_1 & c_2 & c_3 \\ s_1 & \emptyset & s_2 \end{array} \right), & \mu_2 &= \left( \begin{array}{ccc} c_1 & c_2 & c_3 \\ s_1 & s_2 & \emptyset \end{array} \right) \\
\mu_3 &= \left( \begin{array}{ccc} c_1 & c_2 & c_3 \\ s_2 & \emptyset & s_1 \end{array} \right), & \mu_4 &= \left( \begin{array}{ccc} c_1 & c_2 & c_3 \\ s_2 & s_1 & \emptyset \end{array} \right), & \mu_5 &= \left( \begin{array}{ccc} c_1 & c_2 & c_3 \\ \{s_1, s_2\} & \emptyset & \emptyset \end{array} \right)
\end{aligned}
\]

is the set of all feasible assignments.

It is easy to check that \( \mu_1 \) and \( \mu_4 \) are the only feasible assignments which are both fair for same types and constrained non-wasteful for this controlled school choice problem. Note that under \( P_S \),

(i) \( \mu_2 \) and \( \mu_5 \) are not constrained non-wasteful since \( s_2 \) justifiably claims an empty slot at

\( c_3 \) under both \( \mu_2 \) and \( \mu_5 \) and the resulting assignment \( \mu_1 \) is fair for same types, and
(ii) \( \mu_3 \) is not constrained non-wasteful since \( s_1 \) justifiably claims an empty slot at \( c_2 \) under \( \mu_3 \) and the resulting assignment \( \mu_4 \) is fair for same types.

Any feasible mechanism which is both fair for same types and constrained non-wasteful must select either the assignment \( \mu_1 \) or the assignment \( \mu_4 \). We will show that in each case there is a student who profitably manipulates the mechanism.

**Case 1:** The mechanism selects \( \mu_1 \).

Under \( \mu_1 \) student \( s_1 \) is assigned school \( c_1 \). We will show that student \( s_1 \) gains by misreporting his true preference. Suppose that student \( s_1 \) states the (false) preference \( P'_{s_1} \) given by \( c_2 P'_{s_1} c_3 P'_{s_1} c_1 P'_{s_1} s_1 \), and student \( s_2 \) were to report his true preference \( P_{s_2} \). Keeping all other components of the above problem fixed, in the new problem the students’ preferences are \( P'_S = (P'_{s_1}, P_{s_2}) \).

In the new problem under \( \mu_1 \) student \( s_1 \) justifiably envies student \( s_2 \) at school \( c_3 \) since (f1) \( \mu_1(s_1) = c_1, c_3 P'_{s_1} c_1 \) and \( s_1 \succ c_3 s_2 \), and (f2) \( \tau(s_1) = \tau(s_2) \). Note that under \( P'_S \),

(i) \( \mu_1 \) and \( \mu_2 \) are not fair for same types, and

(ii) \( \mu_3 \) and \( \mu_5 \) are not constrained non-wasteful since \( s_1 \) justifiably claims an empty slot at \( c_2 \) under both \( \mu_3 \) and \( \mu_5 \) and the resulting assignment \( \mu_4 \) is fair for same types.

Thus, the unique feasible assignment, which is both fair for same types and non-wasteful for the new problem, is \( \mu_4 \). Hence, any feasible mechanism, which is both fair for same types and constrained non-wasteful, must select the assignment \( \mu_4 \) for the new problem. Under \( \mu_4 \) student \( s_1 \) is assigned school \( c_2 \) which is strictly preferred to \( c_1 \) under the true preference \( P_{s_1} \). Thus student \( s_1 \) does better by stating \( P'_{s_1} \) than by stating his true preference \( P_{s_1} \), and the mechanism is not incentive compatible.

**Case 2:** The mechanism selects \( \mu_4 \).

Under \( \mu_4 \) student \( s_2 \) is assigned school \( c_1 \). Similarly as in Case 1 we will show that student \( s_2 \) gains by misreporting his preference. Suppose that student \( s_2 \) states the (false) preference \( P'_{s_2} \) given by \( c_3 P'_{s_2} c_2 P'_{s_2} c_1 P'_{s_2} s_2 \), and student \( s_1 \) were to report his true preference \( P_{s_1} \). Keeping all other components of the above problem fixed, in the new problem the students’ preferences are \( P'_S = (P'_{s_1}, P'_{s_2}) \).

In the new problem under \( \mu_4 \) student \( s_2 \) justifiably envies student \( s_1 \) at school \( c_2 \) since (f1) \( \mu_4(s_2) = c_1, c_2 P'_{s_2} c_1 \) and \( s_2 \succ c_2 s_1 \), and (f2) \( \tau(s_2) = \tau(s_1) \). Note that under \( P'_S \),

(i) \( \mu_4 \) is not fair for same types,
(ii) $\mu_2$ and $\mu_5$ are not constrained non-wasteful since $s_2$ justifiably claims an empty slot at $c_3$ under both $\mu_2$ and $\mu_5$ and the resulting assignment $\mu_1$ is fair for same types, and

(iii) $\mu_3$ is not constrained non-wasteful since $s_1$ justifiably claims an empty slot at $c_1$ under $\mu_3$ and the resulting assignment $\mu_5$ is fair for same types.

The unique feasible assignment, which is both fair for same types and constrained non-wasteful for the new problem, is $\mu_1$. Hence, any feasible mechanism, which is both fair for same types and constrained non-wasteful, must select the assignment $\mu_1$ for the new problem. Under $\mu_1$ student $s_2$ is assigned school $c_3$ which is strictly preferred to $c_1$ under the true preference $P_{s_2}$. Thus student $s_2$ does better by stating $P'_{s_2}$ than by stating his true preference $P_{s_2}$, and the mechanism is not incentive compatible.

\[\Box\]

**Proof of Theorem 6**

Let $\mu$ be a feasible assignment that is strongly-fair across types and non-wasteful. Since $\mu$ is a feasible assignment, for every school $c$ and student type $t$ we have $q_t^c \leq |\mu^t(c)| \leq \pi_t^c$. Together with strong-fairness across types, this implies that $\mu$ is fair under soft bounds. If $\mu$ is also non-wasteful under soft bounds, the conclusion follows from Theorem 4. Suppose otherwise that $\mu$ violates non-wastefulness under soft bounds. This means that there exist a school $c$ and a student $s$ such that $cP_s\mu(s)$ and $|\mu(c)| < q_c$. Whenever there exists such a pair we apply the following algorithm to improve students’ matches. Note that this algorithm is equivalent to the school-proposing deferred acceptance algorithm if $\mu$ is the assignment in which no agent is matched.\(^{25}\)

**Step 1** For school $c$ defined above, find $S^1 \equiv \{s \in S : cP_s\mu(s)\}$. Among the students in $S^1$ first match the highest ranked students according to $\succ_c$ until the ceilings are filled. Then match the best students according to $\succ_c$ up to the capacity or until $S^1$ is exhausted. Define $\mu_1$ to be the new assignment.

**Step k** If there is no school with an empty seat that a student prefers to her match in $\mu_{k-1}$, then stop. Otherwise consider one such school, say $c_k$. Let $S^k \equiv \{s \in S : c_kP_s\mu_{k-1}(s)\}$. Among the students in $S^k$ first match the highest ranked students according to $\succ_{c_k}$ until the floors are filled. Then match the highest ranked students according to $\succ_{c_k}$ until the ceilings are filled. Finally, match the best students if there are more students and seats available. Define $\mu_k$ to be the new assignment.

\(^{25}\)This is similar to the vacancy-chain dynamics studied in Blum et al. (1997).
This algorithm ends in finite time since it improves the match of at least one student at every step of the algorithm. Let $\hat{\mu}$ denote the assignment produced by this algorithm. It is clear that $\hat{\mu}$ is non-wasteful under soft bounds. We further claim that $\hat{\mu}$ removes justifiable envy under soft bounds.

Consider a student $s$ and school $c$ such that $cP_0\hat{\mu}(s)$. Let $\tau(s) = t$. For any student $s' \in \mu'(c)$, either $s'$ was already matched with $c$ in strongly-fair across types assignment $\mu$ which implies $s' >_c s$, or that $s'$ got matched with $c$ in the above algorithm which also implies $s' >_c s$. Now we split the rest of the analysis depending on whether type $t$ students fill their floors or ceilings.

**Case 1 ($|\hat{\mu}'(c)| \geq q_c^t$):** Consider $s' \in \hat{\mu}(c)$ such that $|\hat{\mu}_{\tau(s')}(c)| > q_c^{\tau(s')}$. Since $\mu$ is feasible it must be that some type $\tau(s')$ students got matched with $c$ in the above algorithm. Moreover, such students must have lower priority compared to other type $\tau(s')$ students who were matched with $c$ in $\mu$. In addition, type $\tau(s')$ students who got matched with $c$ in the above algorithm has a descending priority with respect to the order they were matched. The last type $\tau(s')$ student who got matched with $c$ must have a higher priority than $s$ since type $\tau(s')$ has already filled their ceilings and student $s$ is not admitted to $c$ in this step even though she wants to switch to $c$. This implies that every student of type $\tau(s')$ is preferred to $s$.

**Case 2 ($|\bar{\mu}^t(c)| > |\hat{\mu}'(c)|$):** In this case, for any $t' \in \{t\}$ we cannot have $|\hat{\mu}'(c)| > q_c^{\hat{\mu}(c)}$: At least one student of type $t'$ must have been matched with $c$ during the above algorithm since $\mu$ is feasible. Consider the last student of type $t'$ who got matched with $c$. At the stage when this student got matched with $c$, since $s$ is not matched with $c$, it must be that type $t$ students have filled their ceilings. Later on some type $t$ students in $c$ must have matched with other schools, so that type $t$ students do not fill their ceilings in school $c$ at the end of the algorithm. After the step when type $t$ students do not fill their floors anymore, type $t$ students can be admitted without violating school $c$’s quota. Since $s$ is not matched with $c$, and that type $t$ students do not fill their ceilings at the end of the algorithm, we get a contradiction. Therefore, $|\hat{\mu}'(c)| \leq q_c^t$.

To complete the argument for Case 2, consider type $t'$ such that $\bar{\mu}^{t'}_c \geq |\hat{\mu}'(c)| > q_c^{t'}$. Let $s'$ be the student in $\hat{\mu}'(c)$ with the least priority among type $t'$ students. If $\mu(s') = c$ and $|\mu'(c)| > q_c^{t'}$ then $s' >_c s$ since $\mu$ is fair. If $\mu(s') = c$ and $|\mu'(c)| = q_c^{t'}$ then at least one type $t'$ student must be matched with $c$ during the above algorithm. But this gives a contradiction since that student prefers $c$ to her match in $\mu$ and she has a higher priority than $s'$. Finally, if $\mu(s') \neq c$, then $s'$ has been matched with $c$ during the above algorithm. If at the stage when $s'$ is admitted, type $t$ students do not fill their ceilings then $s' >_c s$. Otherwise, if type $t$ students fill their ceiling at that stage, then some of these students must have matched with
other schools later in the algorithm. Since \( s \) is not matched with \( c \), and that type \( t \) students do not fill their ceilings at the end of the algorithm, we get a contradiction. Therefore, in all of the possibilities we conclude \( s' \succ_c s \).

Thus, \( \hat{\mu} \) removes justifiable envy under soft bounds. Hence, \( \hat{\mu} \) is both fair under soft bounds and non-wasteful under soft bounds. Since under DAASB all students are matched to the best outcome among such assignments, the conclusion follows.

References


