Name__Key__ Final Exam- MA 107-003 SS1 2009, Jun 23: 1-4PM Bonus +2pts
(Put your name on both the problem sheet and your answer sheets)

Answer following questions. Show **ALL OF YOUR WORK** to get full credit.

1. Find the vertex of the graph of the quadratic function $f(x) = 2x^2 - 6x + 1$.

2. List all possible rational zeroes of $f(x) = x^3 + 3x - 4$. Find the quotient and remainder if $f(x)$ is divided by $(x-1)$, ie, if we write $f(x) = q(x) \cdot (x-1) + r(x)$, find $q(x)$ and $r(x)$ in the expression, where $r(x)$ is either 0 or a constant in this case.

3. Factor completely: $\frac{3(x+2)(x-3)^2-(x+2)^2(2)(x-3)}{(x-3)^3}$.

4. If $f(x) = 3x - x^2 + 1$, find the difference quotient of $f(x) : \frac{f(x+h)-f(x)}{h}$.

5. Find the inverse function $g^{-1}(x)$ of the one-to-one function $g(x) = \frac{2}{x-3}, x \neq 3$. You don't need to check $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$ here. **(Note $g^{-1}(x) \neq \frac{1}{g(x)}$!)**

6. Write down in order transformations applied to the graph of $y = x^3$ to get the graph of $y = -3(x-4)^3 + 1$. (The order of transformations should be same as the order of your calculations)

7.Given the polynomial $f(x) = 2x^3(x-2)^2(2x-8)$. Find the degree, leading coefficient, end behavior, all zeros and their multiplicity, and touches/crosses the x-axis. Then use this information to sketch the graph of the function, and find out the intervals where $f(x) > 0$.

8.. For the rational function $R(x) = \frac{x^2-4x+3}{x^2-3x+2}$ find its domain, the vertical and horizontal asymptotes and the x-intercept(s) of the function.

9.. How long does it take for an investment to triple in value if it is invested at 10% per annum compounded continuously?

10. Write $\log \frac{a^3}{b^5\sqrt{d}}$ in expanded form.

11. Solve for x:
 (a) $e^{x^2} = (e^x)^3 \cdot \frac{1}{e^2}$;
 (b) $\ln x + \ln(x+1) = \ln(x+9)$;
 (c) $\log_2(x) + 2\log_2(3) = 2$;
 (d) $5^{x-1} = 7^{3x}$.

12. Use special triangles to find exact values of $\sin^2(60°) + \cot(\pi/6) + \cos^2(\pi/4)$.

13. Find the exact value of each of the remaining trig functions if $\cos(\theta) = -\frac{2}{3}$ and $\sin(\theta) > 0$.

14. Find the reference angles for: a. $\theta = 350°$; b. $\theta = -9\pi/4$.

15. Solve the inequality $f(x) = \frac{2x+4}{x-4} \leq 0$.

16. To measure the height of a building, two sightings are taken a distance of 200 feet apart. If the first angle of elevation is 44° and the second is 31°, what is the height of the building?

17. Find the region bounded by following curves (sketch curves and shade the bounded region) and identify all vertices:

$$y = 1, \quad y - 2x = 1, \quad y = 5 - 2x.$$

Formulas you might use

$(a+b)(c+d) = ac + ad + bc + bd$
$(a+b)^2 = a^2 + 2ab + b^2$
$(a-b)^2 = a^2 - 2ab + b^2$
$a^2 + 2ab + b^2 = (a+b)^2$
$a^2 - 2ab + b^2 = (a-b)^2$

Quadratic formula for quadratic equation:

$$ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \text{ (\textbf{if} } b^2 - 4ac \geq 0)$$

Exp/Log Identities:

$a^s \cdot a^t = a^{s+t}$	$\frac{a^s}{a^t} = a^{s-t}$	$(a^s)^r = a^{rs}$
$(ab)^s = a^s b^s$	$(\frac{a}{b})^s = \frac{a^s}{b^s}$	$(\frac{1}{a})^s = a^{-s}$
$a^1 = a$	$a^0 = 1$	
$y = \log_a(x) \Leftrightarrow a^y = x$	$\log_a(MN) = \log_a(M) + \log_a(N)$	$\log_a(\frac{M}{N}) = \log_a(M) - \log_a(N)$
$\log_a(M^r) = r \cdot \log_a(M)$	$\log_a(M) = \frac{\log_b(M)}{\log_b(a)}$	

Trig Identities

$$\tan(\theta) \cdot \cot(\theta) = 1 \; (\tan(\theta) = \frac{1}{\cot(\theta)},\; \cot(\theta) = \frac{1}{\tan(\theta)})$$
$$\sec(\theta) \cdot \cos(\theta) = 1 \; (\sec(\theta) = \frac{1}{\cos(\theta)},\; \cos(\theta) = \frac{1}{\sec(\theta)})$$
$$\csc(\theta) \cdot \sin(\theta) = 1 \; (\csc(\theta) = \frac{1}{\sin(\theta)},\; \sin(\theta) = \frac{1}{\csc(\theta)})$$
$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)};\; \cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$$
$$\sin^2(\theta) + \cos^2(\theta) = 1$$
$$\tan^2(\theta) + 1 = \sec^2(\theta)$$
$$\cot^2(\theta) + 1 = \csc^2(\theta)$$

Transformations

I SHIFTING

Given a function $y = f(x)$ and a constant $c > 0$,

(a) $y = f(x) + c$ **shifts graph UP c unit (add c to y-values)**

(b) $y = f(x) - c$ **shifts graph DOWN c unit (subtract c from y-values)**

(c) $y = f(x + c)$ **shifts graph LEFT c unit (subtract c from x-values)**

(d) $y = f(x - c)$ **shifts graph RIGHT c unit (add c to x-values)**

II STRETCHING/COMPRESSING

Given a function $y = f(x)$ and a constant $c > 1$,

(a) $y = c * f(x)$ **VERTICAL STRETCH by a factor of c (multiply y's by c)**

(b) $y = 1/c * f(x)$ **VERTICAL COMPRESS by a factor of $1/c$ (divide y's by c)**

(c) $y = f(c * x)$ **HORIZONTAL COMPRESS by a factor of $1/c$ (divide x's by c)**

(d) $y = f(1/c * x)$ **HORIZONTAL STRETCH by a factor of c (multiply x's by c)**

III REFLECTING

(a) $y = -f(x)$ **Reflects graph about the X-AXIS**

(b) $y = f(-x)$ **Reflects graph about the Y-AXIS**

① vertex: $(h,k) = \left(\frac{-b}{2a}, \frac{4ac-b^2}{4a}\right)$. $(a=2,\ b=-6,\ c=1)$ ⓡ

$$\frac{-b}{2a} = \frac{-(-6)}{2\cdot 2} = \frac{6}{4} = \frac{3}{2}$$

$$\frac{4ac-b^2}{4a} = \frac{4\cdot 2\cdot 1-(-6)^2}{4\cdot 2} = \frac{8-36}{8} = \frac{-28}{8} = -\frac{7}{2}.$$

So vertex is $\left(\frac{3}{2}, \frac{-7}{2}\right)$

② All possible rational zeros have the form $\frac{p}{q}$.

p is from $a_0 = -4$. $\Rightarrow p = \pm 1, \pm 2, \pm 4$.

q is from $a_n = 1 \Rightarrow q = \pm 1$

$\Rightarrow \frac{p}{q} = \pm 1, \pm 2, \pm 4.$

Find q & r using long division.

$$\begin{array}{r l}
 & x^2 + x + 4 \quad \leftarrow q(x) \\
x-1 \,\big) & x^3 + 0x^2 + 3x - 4 \\
 & x^3 - x^2 \\
\hline
 & x^2 + 3x \\
 & x^2 - x \\
\hline
 & 4x - 4 \\
 & 4x - 4 \\
\hline
 & 0 \leftarrow r(x)
\end{array}$$

$\Rightarrow q(x) = x^2 + x + 4$. $r(x) = 0$.

3

$$\frac{3(x+2)(x-3)^2-(x+2)^2(2)(x-3)}{(x-3)^3}$$

$$=\frac{(x+2)\cancel{(x-3)}[3(x-3)-2(x+2)]}{(x-3)^{\cancel{3}\,2}}$$

$$=\frac{(x+2)[3x-9-2x-4]}{(x-3)^2}$$

$$=\frac{(x+2)(x-13)}{(x-3)^2}$$

4

$$f(x+h)=3(x+h)-(x+h)^2+1$$

$$=3x+3h-(x^2+2xh+h^2)+1$$

$$=3x+3h-x^2-2xh-h^2+1$$

$$f(x+h)-f(x)=3x+3h-x^2-2xh-h^2+1-(3x-x^2+1)$$

$$=\cancel{3x}+3h-\cancel{x^2}-2xh-h^2\cancel{+1}\cancel{-3x}\cancel{+x^2}\cancel{-1}$$

$$=3h-2xh-h^2$$

$$\frac{f(x+h)-f(x)}{h}=\frac{3h-2xh-h^2}{h}=\frac{\cancel{h}(3-2x-h)}{\cancel{h}}=3-2x-h$$

⑤ $y = \frac{2}{x-3} \Rightarrow x = \frac{2}{y-3}$ ③

$(y-3) \cdot x = \frac{2}{y-3} \cdot (y-3)$

$yx - 3x = 2$

$yx = 2 + 3x$

$y = \frac{2+3x}{x}$

⑥ $y = x^3 \xrightarrow[①]{} y = (x-4)^3 \xrightarrow[②]{} y = 3(x-4)^3 \xrightarrow[③]{} y = -3(x-4)^3$

$\xrightarrow[④]{} y = -3(x-4)^3 + 1$

$f(x) \Rightarrow f(x-4)$

①: Shift right 4 units

②: $f(x) \to 3f(x)$: Vertical stretch by a factor of 3.

③: $f(x) \to -f(x)$: reflect about x-axis.

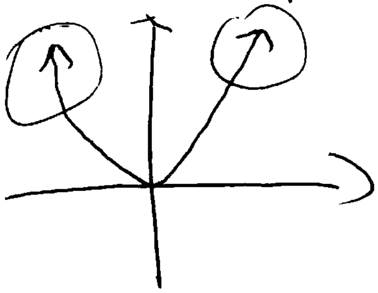
④: $f(x) \to f(x)+1$: Shift up 1 unit.

⑨. $f(x) = 2x^3(x-2)^2(2x-8)$. ④

$n = 3+2+1 = 6$.

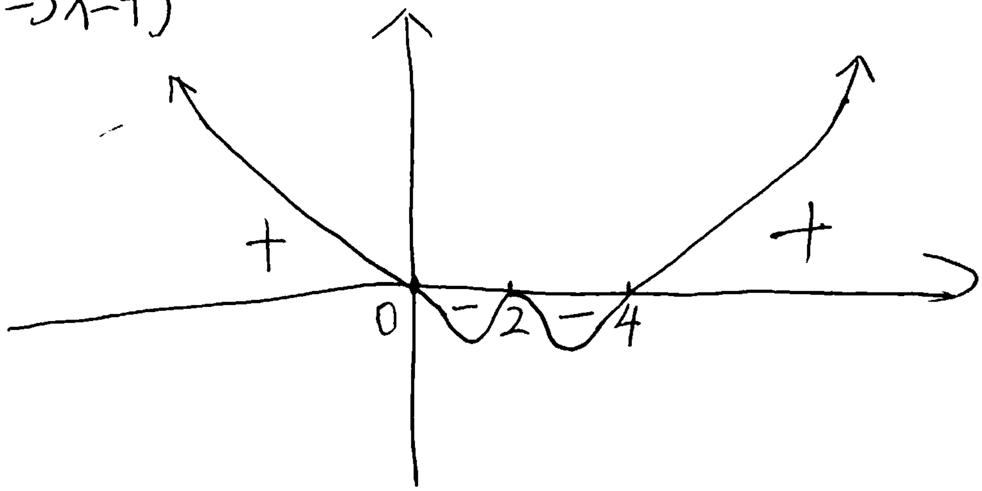
$a_6 = 2 \cdot 2 = 4$.

$\Rightarrow$ the power function (end behavior) : $g(x) = 4x^6$



zeros :	mult.	touches/crosses
$x=0$	3	crosses
$x=2$	2	touches
$x=4$	1	crosses.

$(2x-8=0 \Rightarrow x=4)$



$f(x) > 0 \Rightarrow x < 0$ or $x > 4$.

8. $R(x) = \dfrac{x^2-4x+3}{x^2-3x+2} = \dfrac{p(x)}{q(x)}$.

$q(x)=0 \Rightarrow x^2-3x+2=0$

$(x-2)(x-1)=0$

$x=2$ or $x=1$

$\Rightarrow$ Domain of R: $q(x)\neq 0 \Rightarrow x\neq 2$ and $x\neq 1$

VA: make sure $R(x)$ is in the lowest terms.

$$R(x) = \frac{x^2-4x+3}{x^2-3x+2} = \frac{(x-3)(x-1)}{(x-2)(x-1)} = \frac{x-3}{x-2}.$$

$\Rightarrow$ VA: $x=2$.

HA: $n = \text{degree}(p) = 2$. $m = \text{degree}(q) = 2$.

$m=n \Rightarrow$ HA: $y = \dfrac{\text{L.C. of } p}{\text{L.C. of } q} = \dfrac{1}{1}$

or $y=1$.

x-int: $p(x)=0$. $x^2-4x+3=0$

$(x-3)(x-1)=0$

$x=3$ or $x=1$.

$x=1$ is not in the domain

$\Rightarrow$ $x=3$ is the only x-int R has.

(9). Compounded continuously: (6)

$$A = pe^{rt}$$

$$A = 3p \quad , \quad r = 10\% = .1$$

$$\Rightarrow \frac{3p}{p} = \frac{p}{p} e^{.1t}$$

$$3 = e^{.1t}$$

$$.1t = \ln 3$$

$$t = \frac{\ln 3}{.1} = 10 \cdot \ln 3$$

(10). $$\log \frac{a^3}{b^5 \sqrt{d}} = \log a^3 - \log(b^5 \sqrt{d})$$

$$= 3\log a - (\log b^5 + \log \sqrt{d})$$

$$= 3\log a - (5\log b + \log d^{\frac{1}{2}})$$

$$= 3\log a - (5\log b + \tfrac{1}{2}\log d)$$

$$= 3\log a - 5\log b - \tfrac{1}{2}\log d.$$

⑪ (a) $e^{x^2} = (e^x)^3 \cdot \frac{1}{e^2}$

$e^{x^2} = e^{3x} \cdot e^{-2}$

$e^{x^2} = e^{3x-2}$

$x^2 = 3x-2$

$x^2 - 3x + 2 = 0$

$(x-2)(x-1) = 0$

$x=2$ or $x=1$.

(b) $\ln x + \ln(x+1) = \ln(x+9)$

$\ln(x \cdot (x+1)) = \ln(x+9)$

$x(x+1) = x+9$

$x^2 + \cancel{x} = \cancel{x} + 9$

$x^2 = 9$

$x = -3$ or $x = 3$

check: $x = -3 < 0$ ✗

$x = 3 > 0$, $x+1 = 3+1 = 4 > 0$, $x+9 = 3+9 = 12 > 0$ ✓

$\Rightarrow$ the only solution is $x=3$.

8

11 (c). $\log_2(x) + 2\log_2(3) = 2$

$$\log_2(x) + \log_2(3^2) = 2$$

$$\log_2(x \cdot 3^2) = 2$$

$$x \cdot 3^2 = 2^2$$

$$x \cdot 9 = 4$$

$$x = \frac{4}{9}$$

check: $x = \frac{4}{9} > 0$ ✓

$\Rightarrow x = \frac{4}{9}$ is the solution.

(d). $5^{x-1} = 7^{3x}$

$$\ln(5^{x-1}) = \ln(7^{3x})$$

$$(x-1)\ln 5 = (3x)\ln 7$$

$$x \cdot \ln 5 - \ln 5 = x(3 \cdot \ln 7)$$

$$x(\ln 5) - x \cdot (3\ln 7) = \ln 5$$

$$x(\ln 5 - 3\ln 7) = \ln 5$$

$$x = \frac{\ln 5}{\ln 5 - 3\ln 7}.$$

12. $\sin^2(60^\circ) + \cot(\frac{\pi}{6}) + \cos^2(\pi/4)$

$= (\frac{\sqrt{3}}{2})^2 + \sqrt{3} + (\frac{\sqrt{2}}{2})^2$

$= \frac{3}{4} + \sqrt{3} + \frac{2}{4} = \frac{5}{4} + \sqrt{3}$

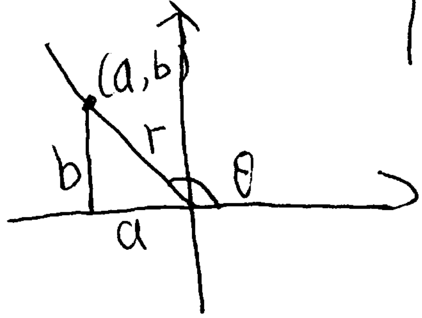
$\cancel{= \frac{3}{4} + \frac{1}{2} + \sqrt{3}}$

$\cancel{= \frac{3}{4} + \frac{2}{4} + \sqrt{3}}$

$\cancel{= \frac{5}{4} + \sqrt{3}}$

13. $\cos\theta = -\frac{2}{3} < 0$ and $\sin\theta > 0$.

$\sin > 0 \Rightarrow \begin{cases} \theta \in QI \\ \theta \in QII \Rightarrow \theta \in QII. \\ \cos\theta < 0 \end{cases}$



$\Rightarrow a < 0,\ b > 0,\ r > 0.$

$\cos\theta = -\frac{2}{3} = \frac{a}{r}$

$\Rightarrow$ assume $a = -2,\ r = 3.$

$\Rightarrow a^2 + b^2 = r^2.$

$(-2)^2 + b^2 = 3^2$

$\Rightarrow b = -\sqrt{5}$ or $b = \sqrt{5}.$

$b > 0 \Rightarrow b = \sqrt{5}.$

$\Rightarrow \sin\theta = \frac{\sqrt{5}}{3}$

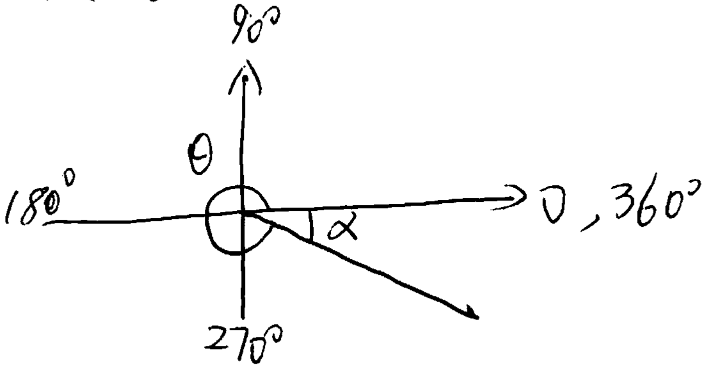
$\tan\theta = \frac{\sqrt{5}}{-2} = -\frac{\sqrt{5}}{2}$

$\cot\theta = \frac{-2}{\sqrt{5}} = -\frac{2\sqrt{5}}{5}$

$\sec\theta = \frac{3}{-2} = -\frac{3}{2}$

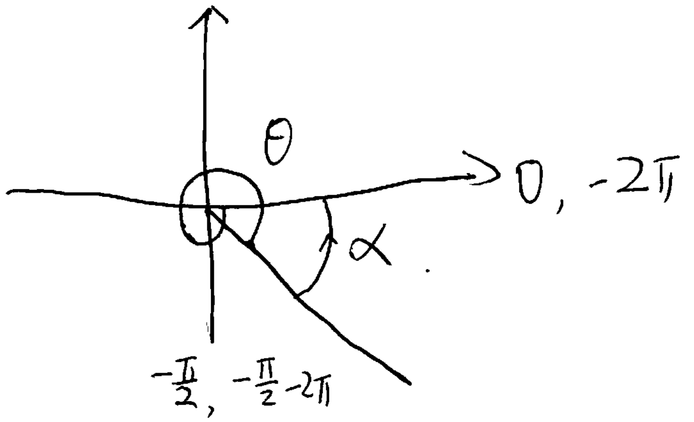
$\csc\theta = \frac{3}{\sqrt{5}} = \frac{3\sqrt{5}}{5}.$

14.(a) $\theta = 350°$ 10



$$\theta + \alpha = 360°$$
$$350° + \alpha = 360°$$
$$\alpha = 10°$$

(b) $\theta = -\frac{9\pi}{4} = -\frac{8\pi + \pi}{4} = -2\pi - \frac{\pi}{4}$

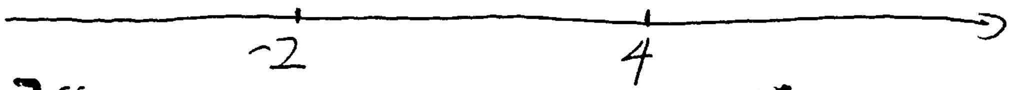


$$\theta + \alpha = -2\pi$$
$$-2\pi - \frac{\pi}{4} + \alpha = -2\pi$$
$$\alpha = \frac{\pi}{4}$$

15. $f(x)=\frac{2x+4}{x-4}\leq 0$. (11)

zero(s): $2x+4=0 \Rightarrow 2x=-4 \Rightarrow x=-2$

points excluded: $x-4=0 \Rightarrow x=4 \Rightarrow x\neq 4$. (Domain of f).



$x=-3\in(-\infty,-2)$ + $f(-3)=\frac{2(-3)+4}{-3-4}=\frac{-6+4}{-7}=\frac{-2}{-7}=\frac{2}{7}>0$

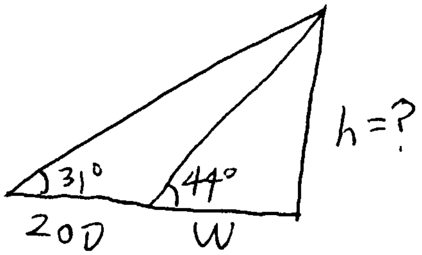
$x=0\in(-2,4)$ − $f(0)=\frac{2\cdot 0+4}{0-4}=\frac{4}{-4}<0$

$x=5\in(4,\infty)$ + $f(5)=\frac{2\cdot 5+4}{5-4}=\frac{14}{1}>0$

$f(x)<0 \Rightarrow -2<x<4$.

$f(x)\leq 0 \Rightarrow -2\leq x<4$. (since we also have zero $x=-2$)

16.



$44°: \begin{cases}\tan 44° = \frac{h}{w}\end{cases}$

$31°: \tan 31° = \frac{h}{200+w}$

$\Rightarrow \begin{cases} h = w\cdot\tan 44° \\ h=(200+w)\cdot\tan 31° \end{cases}$

$\Rightarrow w\cdot\tan 44° = (200+w)\tan 31°$

$w\cdot\tan 44° = 200\cdot\tan 31° + w\cdot\tan 31°$

$w\cdot\tan 44° - w\cdot\tan 31° = 200\cdot\tan 31°$

$w(\tan 44° - \tan 31°) = 200\cdot\tan 31°$

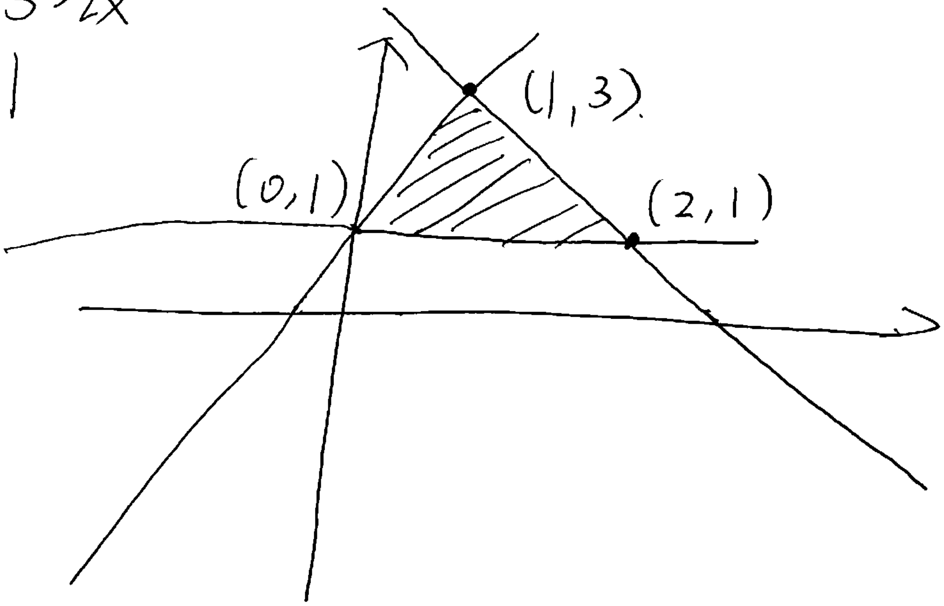
$w=\frac{200\cdot\tan 31°}{\tan 44° - \tan 31°}$

$h = w\cdot\tan 44° = \frac{200\cdot\tan 31°\cdot\tan 44°}{\tan 44° - \tan 31°}$

17. $y=2x+1 \Leftrightarrow y-2x=1.$

$y=5-2x$

$y=1$



$\begin{cases} y=1 \\ y=2x+1 \end{cases} \Rightarrow 1=2x+1 \Rightarrow 2x=0 \Rightarrow x=0 \Rightarrow \begin{cases} x=0 \\ y=1 \end{cases} \Rightarrow (0,1)$

$\begin{cases} y=1 \\ y=5-2x \end{cases} \Rightarrow 1=5-2x \Rightarrow 2x=5-1 \Rightarrow 2x=4 \Rightarrow x=2.$

$\Rightarrow \begin{cases} x=2 \\ y=1 \end{cases} \Rightarrow (2,1)$

$\begin{cases} y=2x+1 \\ y=5-2x \end{cases} \Rightarrow 2x+1=5-2x \Rightarrow 2x+2x=5-1 \Rightarrow 4x=4 \Rightarrow x=1$

$\Rightarrow \begin{cases} x=1 \\ y=2\cdot 1+1=3 \end{cases} \Rightarrow (1,3)$