MA 114-002 Test 1, SSI 2007
Solution

1. (21 pts) Given the following matrices

$$A=\begin{bmatrix}1&0&2&0\\3&0&1&2\\0&2&1&4\end{bmatrix},\ B=\begin{bmatrix}1\\x\\1\\0\end{bmatrix},\ C=\begin{bmatrix}0&0\\1&0\end{bmatrix},\ D=\begin{bmatrix}3&2&1\\0&1&0\end{bmatrix},\ E=\begin{bmatrix}2&0&y&1\end{bmatrix}.$$

Find the following

a. (3 pts) The size of matrix A
b. (3 pt) a_{21}
c. (3 pts) C + 3D (if it exists)
d. (3 pts) EB (if it exists)
e. (3 pts) DC (if it exists)
f. (3 pts) a matrix X such that XC = CX = C
g. (3 pts) a matrix Y such that B + Y = Y + B = B

Solution:

a. 3 x 4
b. 3
c. does not exist
d. $EB=\begin{bmatrix}2&0&y&1\end{bmatrix}\begin{bmatrix}1\\x\\1\\0\end{bmatrix}=[2\times1+0\times x+y\times1+1\times0]=[2+y]$
e. does not exist.
f. $X=\begin{bmatrix}1&0\\0&1\end{bmatrix}$
g. $Y=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$

2. (6 pts/ea) Solve the system of equations by matrix method. Show all steps and label all row operations.

a. x + 3y = 2
3x + 6y + 3z = 3
2y + 8z = 2

b. 2x + 4y – 8z = 12
-x –2y +10z = -12

Solution:

a.

$$\left[\begin{array}{ccc|c}1&3&0&2\\3&6&3&3\\0&2&8&2\end{array}\right]\xrightarrow[\frac{1}{2}R_3]{\frac{1}{3}R_2}\left[\begin{array}{ccc|c}1&3&0&2\\1&2&1&1\\0&1&4&1\end{array}\right]\xrightarrow{R_2-R_1}\left[\begin{array}{ccc|c}1&3&0&2\\0&-1&1&-1\\0&1&4&1\end{array}\right]\xrightarrow{R_2\leftrightarrow R_3}\left[\begin{array}{ccc|c}1&3&0&2\\0&1&4&1\\0&-1&1&-1\end{array}\right]$$

$$\xrightarrow[R_3+R_2]{R_1-3R_2}\left[\begin{array}{ccc|c}1&0&12&-1\\0&1&4&1\\0&0&5&0\end{array}\right]\xrightarrow{\frac{1}{5}R_3}\left[\begin{array}{ccc|c}1&0&12&-1\\0&1&4&1\\0&0&1&0\end{array}\right]\xrightarrow[R_2-4R_3]{R_1-12R_3}\left[\begin{array}{ccc|c}1&0&0&-1\\0&1&0&1\\0&0&1&0\end{array}\right],\quad\Rightarrow\begin{cases}x=-1\\y=1\\z=0\end{cases}$$

b.

$$\left[\begin{array}{ccc|c} 2 & 4 & -8 & 12 \\ -1 & -2 & 10 & -12 \end{array}\right] \xrightarrow{R_1+R_2} \left[\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ -1 & -2 & 10 & -12 \end{array}\right] \xrightarrow{R_2+R_1} \left[\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 0 & 0 & 12 & -12 \end{array}\right] \xrightarrow{\frac{1}{12}R_2} \left[\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$\xrightarrow{R_1-2R_2} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right], \quad \Rightarrow \begin{cases} x+2y=2 \\ z=-1 \end{cases}, \quad \Rightarrow \begin{cases} x=2-2y \\ z=-1 \end{cases}$$

3. (6 pts/ea) Find the inverse of these matrices if it exists.

a. $A=\begin{bmatrix} 2 & -1 & 1 \\ 2 & 1 & 0 \\ 1 & 2 & -1 \end{bmatrix}$ b. $A=\begin{bmatrix} 3 & 4 & -1 \\ 8 & 12 & 3 \end{bmatrix}$ c. $A=\begin{bmatrix} 8 & 4 \\ 6 & 3 \end{bmatrix}$

Solution:

a. $$\left[\begin{array}{ccc|ccc} 2 & -1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 1 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 2 & -1 & 1 & 1 & 0 & 0 \end{array}\right] \xrightarrow{R_3-R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 0 & -2 & 1 & 1 & -1 & 0 \end{array}\right]$$

$$\xrightarrow{R_2-2R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 0 & 0 & 1 \\ 0 & -3 & 2 & 0 & 1 & -2 \\ 0 & -2 & 1 & 1 & -1 & 0 \end{array}\right] \xrightarrow[R_2-2R_3]{R_1+R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & -2 & 3 & -2 \\ 0 & -2 & 1 & 1 & -1 & 0 \end{array}\right] \xrightarrow{R_3+2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & -2 & 3 & -2 \\ 0 & 0 & 1 & -3 & 5 & -4 \end{array}\right]$$

b. A is not a square matrix. So the inverse of A does not exist.

c. $$\left[\begin{array}{cc|cc} 8 & 4 & 1 & 0 \\ 6 & 3 & 0 & 1 \end{array}\right] \xrightarrow[\frac{1}{3}R_2]{\frac{1}{4}R_1} \left[\begin{array}{cc|cc} 2 & 1 & \frac{1}{4} & 0 \\ 2 & 1 & 0 & \frac{1}{3} \end{array}\right] \xrightarrow{R_2-R_1} \left[\begin{array}{cc|cc} 2 & 1 & \frac{1}{4} & 0 \\ 0 & 0 & -\frac{1}{4} & \frac{1}{3} \end{array}\right]$$

Inverse of A does not exist.

4. (4 pts) Given the equation AX=B, where B is given below, using the inverse of A to find X.

$$A^{-1}=\begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 2 \\ 1 & 2 & -2 \end{bmatrix}, \quad B=\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

Solution:

$$A^{-1}(AX)=A^{-1}B \Rightarrow (A^{-1}A)X=A^{-1}B \Rightarrow X=A^{-1}B=\begin{bmatrix} -1 \\ 3 \\ 8 \end{bmatrix}$$

5. (5 pts) Find all values of x and y that satisfy the matrix equation.

$$[x \quad y]\begin{bmatrix} .8 & .2 \\ .4 & .6 \end{bmatrix}=[x \quad y]$$

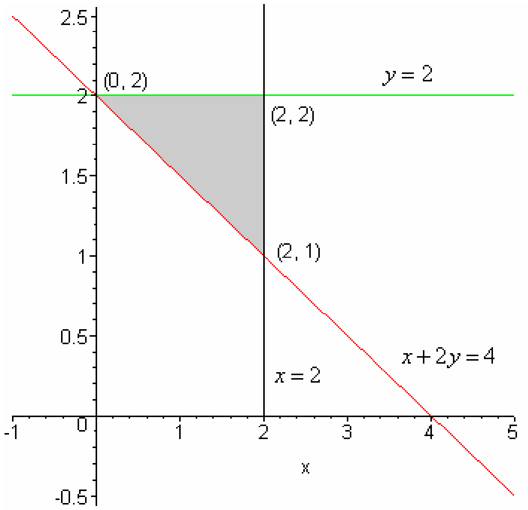
Solution:

$$[.8x+.4y \quad .2x+.6y]=[x \quad y] \Rightarrow \begin{cases} .8x+.4y=x \\ .2x+.6y=y \end{cases} \Rightarrow \begin{cases} .4y=.2x \\ .2x=.4y \end{cases} \Rightarrow x=2y,\ \forall y (y \text{ is arbitrary}).$$

6. (12 pts) Graph the region determined by the following system of inequalities and find all intersection points.

$$x+2y \ge 4, \quad y \le 2, \quad x \le 2$$

Solution:



$$x+2y=4,\quad y=2,\quad \Rightarrow\quad x=0,\quad \Rightarrow\quad (0,2)$$
$$x+2y=4,\quad x=2,\quad \Rightarrow\quad y=1,\quad \Rightarrow\quad (2,1)$$
$$x=2,\quad y=2,\quad \Rightarrow\quad \Rightarrow\quad (2,2)$$

7. (12 pts) Set up the following linear programming problem. In other words, state the variables involved, state all constraints, and state the objective function. Do NOT try to solve the problem.
A factory makes couches, tables, and chairs. Each couch requires 9 hours to fabricate, 7 hours to assemble and 4 hours to finish and makes a profit of $16. Each table takes 2 hours to fabricate, 6 hours to assemble, and 3 hours to finish and makes a profit of $12. Each chair requires 3 hours to fabricate, 1 hour to assemble and 4 hours to finish and makes a profit of $7. The factory has 280 staff-hours each day available for fabrication, 440 staff-hours for assembling, and 300 hours for finishing. How many of each should be made to maximize the profit? And what is the maximum profit?

Solution:

Let $x=\#$ of couches; $y=\#$ of tables; $z=\#$ of chairs.

Objective function: Maximize the profit, $P=16x+12y+7z$

Constraints:

Time to fabricate:	$9x+2y+3z\le 280$
Time to assemble:	$7x+6y+z\le 440$
Time to finish:	$4x+3y+4z\le 300$
Non-negative:	$x\ge 0,\quad y\ge 0,\quad z\ge 0$

8. (16 pts) Solve the following linear programming problem.
A dietitian wants to design a breakfast menu for certain patients. The menu is to include two items A and B. Suppose that each ounce of A provides 3 units of Vitamin C and 2 units of iron and each ounce of B provides 2 units of Vitamin C and 2 units of iron. Suppose the cost of A is 6 cents/ounce and cost of B is 5 cents/ounce. If the breakfast menu must provide at least 12 units of Vitamin C and 10 units of iron, how many ounces of each item should be provided in order to meet the Vitamin C and iron requirement for the least cost? What will this breakfast cost?

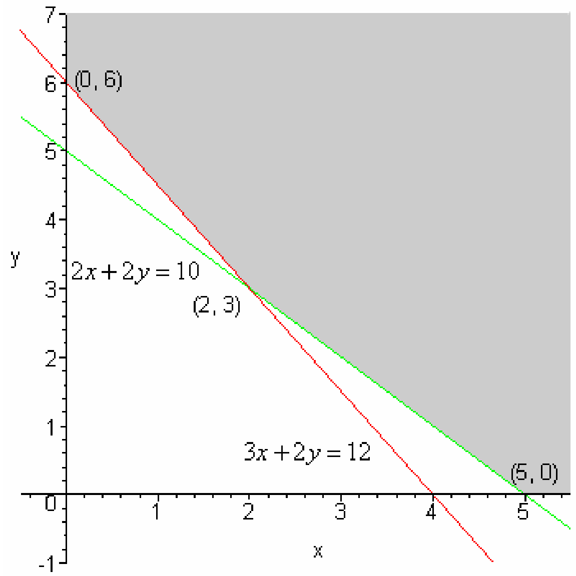
Solution:

Let $x=\#$ ounces of A; $y=\#$ ounces of B; .

Objective function: Minimize the cost, $C=6x+5y$

Constraints:

Vitamin C:	$3x+2y\ge 12$
iron:	$2x+2y\ge 10$
Non-neg.:	$x\ge 0,\quad y\ge 0$



$$3x+2y=12,\ \ x=0,\ \Rightarrow\ y=6,\ \Rightarrow\ (0,6)$$
$$3x+2y=12,\ \ y=0,\ \Rightarrow\ x=4,\ \Rightarrow\ (4,0)$$
$$2x+2y=10,\ \ x=0,\ \Rightarrow\ y=5,\ \Rightarrow\ (0,5)$$
$$2x+2y=10,\ \ y=0,\ \Rightarrow\ x=5,\ \Rightarrow\ (5,0)$$
$$3x+2y=12,\ 2x+2y=10,\ \Rightarrow$$
$$\left[\begin{array}{cc|c} 3 & 2 & 12 \\ 2 & 2 & 10 \end{array}\right] \xrightarrow{R_1-R_2} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 2 & 2 & 10 \end{array}\right] \xrightarrow{R_2-2R_1}$$
$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 2 & 6 \end{array}\right] \xrightarrow{\frac{1}{2}R_2} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \end{array}\right],\ \Rightarrow\ \begin{cases} x=2 \\ y=3 \end{cases}$$
$$i.e., (2,3)$$

So

Corner Point	$C=6x+5y$
(5, 0)	30
(0, 6)	30
(2, 3)	27

Hence, the minimum cost of this breakfast is 27 cents when 2 ounces A and 3 ounces B are provided.

Extra Credit: (5 pts)

In general for matrices, AB=BA is not a true statement. If $A=\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$, find ALL matrices B that make AB=BA true.

Solution:

Let $B=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. $AB=\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}=\begin{bmatrix} c & d \\ 2a & 2b \end{bmatrix}, BA=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}=\begin{bmatrix} 2b & a \\ 2d & c \end{bmatrix}$

By the equality of matrices, following is obtained:

$c=2b;\quad a=d$

So the matrices B must have the following form that make AB=BA true:

$B=\begin{bmatrix} a & b \\ 2b & a \end{bmatrix}$, a and b are arbitrary #s .