Algebraic Topology: A brief introduction
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This chapter is intended to serve as a brief, and far from comprehensive, introduction to Algebraic Topology to help the reading flow of this dissertation. Interested readers may refer to a standard text book such as [1] for a detailed exposition.

0.1 Topology

Topological spaces enable us to generalize the notion of continuity of maps and thereby help us study continuous maps without involving metrics. This is accomplished by viewing the spaces as a collection of open sets and studying how the maps operate on these open sets.

Definition The Topology $T$ of a space $X$ is a collection of open sets $\{U\}$, such that the open sets satisfy the following properties:

\begin{align*}
X, \emptyset & \in T \\
U_1, U_2 & \in T \Rightarrow U_1 \cap U_2 \in T \\
U_i & \in T, (i \in I) \Rightarrow \bigcup_i U_i \in T
\end{align*}

A Topological Space $\{X, T\}$ is a collection of the set and its topology.

The mechanisms by which these open sets may be assigned, is arbitrary as long they satisfy the above properties. In practice, the most common way to define open sets is by way of metrics. However, it is important to note that even in this case, the definitions of continuous maps can be much simplified using open sets especially for large dimensions. For example, the classical definition for continuous maps on the real line is given as:

Definition A mapping $f : \mathbb{R} \rightarrow \mathbb{R}$ is said to be continuous at $x_0$ if for any $\epsilon > 0$, $\exists \delta > 0$, such that $|f(x) - f(x_0)| < \epsilon$ for all $x$ satisfying $|x - x_0| < \delta$. A mapping $f$ is continuous if it is continuous at every $x \in \mathbb{R}$.
Notice the dependence of the above definition on the norm $|\cdot|$ which is induced by the Euclidean metric on $\mathbb{R}$. Although, an equivalent definition [2] for continuity can be given as

**Definition** A mapping $f$ from a topological space $(X, T_1)$ to a topological space $(Y, T_2)$ is continuous if

$$U \in T_2 \Rightarrow f^{-1}(U) \in T_1$$

*By convention, we will use $X$ for $\{X, T\}$, all mappings are continuous mappings, and “spaces” mean Topological spaces in the remainder of this chapter.*

### 0.2 Topological Analysis

Topological analysis [4] may loosely be construed to be the study of global organization of spaces without paying much heed to fine geometrical structure. For a space embedded in $\mathbb{R}^3$, this amounts to analyzing properties such as, is the space connected?, does the space wrap upon itself?, does the surface have holes in it? or does the surface enclose a three dimensional void? and so on. As such, the developed tools provide the right generalization to study organizational features of the network without expending resources on finer details. We have so far defined tools which can generally describe spaces and continuous maps on them. We next describe some tools which help us in the above mentioned analysis.

**Definition** Let $X$ and $Y$ be two spaces. Two maps $f_1, f_2 : X \rightarrow Y$ are said to be **homotopic** ($f_1 \approx f_2$) to each other if $\exists$ a continuous map $F : X \times I \rightarrow Y$ (where $I = [0, 1]$) such that $F(s, 0) = f_1(s)$ and $F(s, 1) = f_2(s)$. Such a function $F$ is called a **Homotopy** between $f_1$ and $f_2$.

**Definition** Two spaces $X$ and $Y$ are said to be **Homotopy Equivalent** if $\exists$ continuous maps $f : X \rightarrow Y$ and $g : Y \rightarrow X$ such that $f \circ g \approx \text{id}$ and $g \circ f \approx \text{id}$. Such a map $f$ is called a **homotopy equivalence**.

The above definition means that if two spaces $X$ and $Y$ are homotopy equivalent, then one can be continuously deformed into the other. Note the requirement for a bijection, that the composition with inverse be equal to identity, has been relaxed to being homotopic with identity. Figure 1 shows an example of Homotopy equivalent spaces.

At this point, we have described tools which provide a generalized equivalence of the classical notion of “Euclidean Invariance” or “Equivalent up to rotation”. As seen in Figure 1, although the two spaces $X$ and $Y$ locally have different geometric properties, we can still say they are equivalent in some sense, i.e., they both have one loop. We can further strengthen this notion of equivalence as given in the following definition.
Definition Two spaces $X$ and $Y$ are said to be **homeomorphic** to each other, if $\exists$ a bijective map $f : X \to Y$ such that the inverse $f^{-1}$ map is also continuous. Such a map $f$ is called a **homeomorphism**.

For example, the open segment $(0, 1)$ is homeomorphic to the real line $\mathbb{R}$. The open disk $E^2 = \{(x, y)|x^2 + y^2 < 1\}$ is homeomorphic to the plane $\mathbb{R}^2$. Homeomorphic spaces have the same topological properties, although, for many such properties, homotopy equivalence would be sufficient.

### 0.3 Representation of Topological Spaces

In order to facilitate analysis of these spaces, we need concise and simple representations. There are many ways to construct these representations [1], namely:

1. CW-complexes
2. Delta Complexes
3. Simplicial Complexes
4. Singular Complexes

These four methods of building the representations have their merits and demerits, but a common feature among them is that they build a "simple" topological space which is either homeomorphic or homotopy equivalent to the original space. They thus offer a simplified analysis which preserves the extracted topological features from these simple spaces. The above enumeration lists methods in the increasing order of complexity. Since the applications in this report only use Simplicial Complexes, we will not attempt to describe all the other methods here, but rather provide a brief comparison.

CW-complexes are easiest to construct and most suitable for “computations by hand” but have little practical (engineering) relevance, i.e., they are not suitable for representation on a digital computer. Delta complexes and Simplicial complexes are both representable on a digital computer but Simplicial complexes reflect finer details (in the sense that they use larger number of simplices) about the original space and are hence most suitable especially when only sample points of the original space are available. It is in this light that we opt for a simplicial complex. While singular complexes greatly simplify theoretical analysis, they too, cannot be represented on a computer.
0.3.1 Simplicial Complexes

Simplicial Complexes are representations of given topological spaces using simplices (simple pieces). A $k^{th}$ order simplex or $k$-simplex $\sigma^k$ is the set of all points given by the convex combination of $k + 1$ linearly-independent points, $\sigma^k = (v_0, ..., v_k)$. Thus, a standard 0-simplex is just a point, a standard 1-simplex is a line segment, a 2-simplex a triangle and so on. Figure 2 shows simplices of order 0 through 4. We can also specify a certain orientation to these simplices which is defined by the ordering of the $k + 1$ points. The arrows shown in Figure 2 represent the orientation of the simplices.

Figure 3 illustrates a topological space being represented by a simplicial complex. Note that the space on the left has the same topology as the simplicial complex, and therefore, its topological invariants may be obtained by analyzing the simplicial complex. A simplicial complex is constructed by taking a collection of simplices, and gluing them together in a certain way. This gluing process is formally defined as follows:

**Definition** Consider a $k$-simplex $\sigma^k = (v_0, ..., v_k)$. The simplex determined by a subset $\sigma^l \subset \sigma^k, l \leq k$ is called a face of the simplex $\sigma^k$. If $l = k - 1$, the face $\sigma^l$ is called a proper face.

**Definition** Two simplices $\sigma^k_i$ and $\sigma^l_j$ can be glued together to form a Simplicial Complex $K$ by just taking the disjoint union of both simplices whose resulting space is a quotient space with respect to an equivalence relation. The equivalence relation is applied to a face from each of the simplices which are then declared equivalent.

$$K = \frac{\sigma^k_i \coprod \sigma^l_j}{\sim} \text{ where } \sigma^m_m \sim \sigma^m_m \text{ and } \sigma^m_m \subset \sigma^k_i, \sigma^m_m \subset \sigma^l_j$$

The above gluing process may be repeated to form any simplicial complex. Given a simplicial complex $K$, a new complex $K'$ may be formed by an addition of a simplex $\sigma_j$ as follows,

$$K' = \frac{K \coprod \sigma_j}{\sim} \text{ where } \sigma^m_m \sim \sigma^m_m \text{ and } \sigma^m_m \subset \sigma_i, \sigma_i \in K, \sigma^m_m \subset \sigma_j$$

With this ability of representing complex topological spaces with manageable structures, we are now in a position to carry our analysis task further. To that end, we first consider some algebraic tools presented in the next section.

0.4 Homological Algebra

**Definition** A group $G$ is called a free group if it does not contain any subgroup of finite order, i.e., no subgroup is cyclic.
Figure 1: Homotopy Equivalent Spaces

Figure 2: Simplices of order 1,2,3,4 depicted in (a),(b),(c) and (d) respectively. Simplices are the building blocks of a simplicial complex.

Figure 3: An illustration of a topological space represented by a simplicial complex. Note that the space on the left has the same topology as the simplicial complex, and therefore, its topological invariants may be obtained by analyzing the simplicial complex.
Definition A **Graded Abelian Group** \( \{C_k, \partial_k\} \) is a sequence of free abelian groups \( \{C_k\} \) together with homomorphisms \( \{\partial_k : C_k \to C_{k-1}\} \) called the boundary operators.

\[
\rightarrow C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_{k+1}} C_k \xrightarrow{\partial_k} C_{k-1} \cdots C_0 \rightarrow 0
\]

Definition A **Chain Complex** is a graded abelian group with the boundary operators satisfying the property

\[
\partial_{k-1} \circ \partial_k = 0 \text{ or } \partial^2 = 0
\]

The groups \( \{C_k\} \) are called **chain spaces** and their elements are called **chains**.

The relation \( \partial_{k-1} \circ \partial_k = 0 \) implies that the image of one boundary operator is a subset of the kernel (or null space) of the next boundary operator, i.e.,

\[
Img(\partial_k) \subset Ker(\partial_{k-1})
\]

As we will see in the next section, chain complexes will be our entry point from topological spaces into algebra. To each topological space, we will assign a chain complex.

Definition An **exact sequence** is a graded abelian group with the boundary operators satisfying the property

\[
Img(\partial_k) = Ker(\partial_{k-1})
\]

It is clear from the above definition that the exact sequence is also a chain complex. The importance of exact sequences is pervasive in Algebraic topology, and we will in fact use them to prove an important theorem in chapter ??.

To each chain complex, we again assign another sequence of abelian groups \( \{H_k\} \) called the homology groups defined as follows

Definition Given a chain complex \( C = \{C_k, \partial_k\} \), the \( k^{th} \) **homology group** \( H_k(C) \) of the chain complex is given as

\[
H_k(C) = ker(\partial_k)/Img(\partial_{k+1})
\]

i.e., the \( k^{th} \) homology group is the quotient group formed by considering all the elements in \( ker(\partial_k) \) which are also in the \( Img(\partial_{k+1}) \) to be equivalent to zero.

Note that homology by the above definition is well defined because \( Img(\partial_k) \subset Ker(\partial_{k-1}) \).

Two cycles \( c_1, c_2 \in ker(\partial_1) \) which are equivalent to each other in the quotient space \( H_1 \), i.e., \( c_1 - c_2 \in Img(\partial_2) \) are said to be **homologous** to each other. As will be seen, these homology groups will be instrumental in uncovering the properties of our topological spaces.

We will give a few important properties of these homology groups after the following basic definitions.
**Definition** Given two chain complexes $C = \{C_k, \partial_k\}$ and $D = \{D_k, \partial_k\}$, a **chain map** $F_\# : C \rightarrow D$ is a sequence of homomorphisms $\{F_k : C_k \rightarrow D_k\}$ such that the diagrams of the following type commute,

\[
\begin{array}{ccc}
C_k & \xrightarrow{F_k} & D_k \\
\downarrow\partial_k & & \downarrow\partial_k \\
C_{k-1} & \xrightarrow{F_{k-1}} & D_{k-1}
\end{array}
\]

i.e., $\partial_k \circ F_k = F_{k-1} \circ \partial_k$.

The above definition states that the chain maps preserve the boundary operation. In other words, applying a chain map and followed by a boundary operator on the range chain complex is equivalent to first applying a boundary operator followed by a chain map. Although we are using the same symbol $\partial_k$ to denote the boundary operators in both chain complexes, which complex it belongs to should be clear from the context. We state the following theorem without proof [5]:

**Theorem 0.4.1** A chain map $f_\# : C \rightarrow D$ induces homomorphisms $f_* : H_*(C) \rightarrow H_*(D)$ on the corresponding homology groups.

The above theorem states that given a chain map between two chain complexes, we can find corresponding homomorphisms between the homology groups of these chain complexes. This may be viewed as analogous to the fact that linear transformations on signals induce corresponding operations on their Fourier transforms. This property is very common in many mathematical topics and is known as functoriality.

**Definition** Let $f, g : C \rightarrow D$ be chain maps. We say that $f$ is **chain homotopic** to $g$ if there are homomorphisms $\{\phi_k\}$ of graded abelian groups $\phi_k : C_k \rightarrow D_{k+1}$ such that

$$\phi_{k-1} \circ \partial_k + \partial_{k+1} \circ \phi_k = f - g$$

The operation of $\phi$ can be represented by the following diagram

\[
\begin{array}{ccc}
C_k & \xrightarrow{\phi_k} & D_{k+1} \\
\downarrow\partial_k & & \downarrow\partial_{k+1} \\
C_{k-1} & \xrightarrow{\phi_{k-1}} & D_k
\end{array}
\]

Chain Homotopy may also be viewed as analogous to homotopy of continuous maps, the difference being that one operates on chain maps while the other operates on topological spaces. As we will see in the subsequent section, we can form chain maps from a topological space at which point we will put all these concepts together. The following theorem is an important consequence of chain homotopies [5].
Theorem 0.4.2 Given two chain complexes $C$ and $D$, if $\exists$ chain maps $f : C \to D$ and $g : D \to C$ such that $f \circ g$ is chain homotopic to $\text{id}$ and $g \circ f$ is chain homotopic to $\text{id}$, then the homology groups of the chain complexes $H_k(C)$ and $H_k(D)$ are isomorphic.

Finally we introduce the laplacian operators which, for our purposes, will be shown to greatly simplify the computational framework of homology groups.

Definition Given a chain complex $C$, the $k^{th}$ Laplacian operator $L_k : C_k \to C_k$ is defined as

$$L_k = \partial_{k+1} \circ \partial_k^* + \partial_k^* \circ \partial_k$$

where $\partial_k^*$ is the adjoint of $\partial_k$

When field coefficients are used, the chain spaces $C_k$ are vector spaces, and the boundary operators have matrix representation for a given basis. The matrix representation for the adjoint operator $\partial_k^*$ is just the transpose of that of $\partial_k$. It can be shown [3] that the kernel of the laplacian operator $L_k$ is isomorphic to the $k^{th}$ homology group, i.e., $\ker(L_k) \cong H_k(C)$.

0.5 Topological Analysis Revisited

Having defined chain complexes, maps on chain complexes and the relation between homologies of chain complexes, we close the loop by constructing a chain complex given a simplicial complex representing a Topological space.

We have seen in Section 0.3 that simplicial complexes are built from a collection of simplices by gluing together their faces. A complete definition of a simplicial complex is as follows,

Definition Given a set of vertices $V$, a simplicial complex $K = \{ \sigma_i \}$ is a collection of simplices. Each simplex is determined by a set of vertices $(v_0, \ldots, v_k)$ in $V$, i.e., $\sigma^k$ is the set of convex combinations of $(v_0, \ldots, v_k)$. The simplices have the following properties:

- $\sigma_i, \sigma_j \in K \Rightarrow \sigma_i \cap \sigma_j \in K$
- All faces of $\sigma_i$ are in $K$.

An important fact to note here, is that each simplex in a simplicial complex can be determined by a unique set of vertices.

0.5.1 Chain Complex from a Simplicial Complex

To construct a chain complex, we require a sequence of abelian groups $\{C_k\}$ (chain spaces) and a sequence of boundary operators $\{\partial_k\}$ such that $\partial^2 = 0$. 
Building chain spaces

Given a simplicial complex $K$, we form an abelian group called a chain space $\{C_k(K)\}$ by considering all the $k$-simplices $\{\sigma^k_i\}$ as generators. Thus if $\sigma^k_i, \sigma^k_j \in K$, then $a_1\sigma^k_i + a_2\sigma^k_j \in C_k(K)$, $a_1, a_2 \in \mathbb{Z}$. 

Boundary Operators

The construction of the chain spaces $C_k(K)$ is purely algebraic. We now relate the algebraic structure in these chain spaces to the combinatorial structure in the simplicial complex $K$ in two specific ways

1. For a given simplex $\sigma^k$, the additive inverse $-\sigma^k$ will be related to the orientation of the simplices.

2. The boundary operator between the chain spaces $C_k(K)$ and $C_{k-1}(K)$ will be related to the relation between the simplices and their faces.

Note that we still have some additional algebraic structure left in the chain spaces, i.e., arbitrary coefficients in $\mathbb{Z}$, to which no meaning was assigned in the combinatorial structure. The additive inverse in terms of the orientation is given as:

$$\text{if } \sigma^k = (v_0, \ldots, v_i, v_{i+1}, \ldots, v_k) \text{ then } -\sigma^k = (v_0, \ldots, v_{i+1}, v_i, \ldots, v_k)$$

(2)

and, the simplices and their faces are linked using the boundary operators. Since $\{C_k(K)\}$ is an abelian group, defining the boundary operator $\partial_k : C_k(K) \to C_{k-1}(K)$ is equivalent to defining it on each of its generators. Therefore, given a generator $\sigma^k = (v_0, \ldots, v_k) \in C_k$, we define the operation of a boundary operator as

$$\partial_k(v_0, \ldots, v_k) = \sum_i -1^i (v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k)$$

(3)

Note that each summand is a $k-1$ simplex, hence the sum lies in $C_{k-1}(K)$. Verification of $\partial^2 = 0$ is a straightforward but lengthy process. Therefore, we will show that it is in fact the case by considering an example. The operation of $\partial_1 \circ \partial_2$ on a 2-simplex $(v_0, v_1, v_2)$ is

$$\partial_1 \circ \partial_2(v_0, v_1, v_2) = \partial_1((v_1, v_2) - (v_0, v_2) + (v_0, v_1)) = v_2 - v_1 - v_2 + v_0 + v_1 - v_0 = 0$$

(4)

Now that we have a sequence of chain spaces and a sequence of boundary operators which satisfy $\partial^2 = 0$, we have a chain complex. We have made our transition from topological spaces into algebra. We now state a few theorems which relate continuous maps on topological spaces to homomorphisms and isomorphisms on algebraic spaces.

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*If we use $\mathbb{R}$ as our coefficients in place of $\mathbb{Z}$, we will a vector space*
Theorem 0.5.1  [5] If $X$ and $Y$ are topological spaces and $K_X, K_Y$ are simplicial complexes representing these spaces, a map $f : X \to Y$ induces a chain map $f_\# : C_*(K_X) \to C_*(K_Y)$ on their corresponding chain complexes. Further by theorem 0.4.1, $f$ induces a sequence of homomorphisms $f_* : H_k(C_*(K_X)) \to H_k(C_*(K_Y))$ on their corresponding homology groups.

The theorem states that if there is a continuous mapping between two topological spaces, we can find a corresponding homomorphism between their homology groups. In other words, any transformation in the topological spaces can be reflected as some transformation on the homology spaces. Again, notice the functorial property of assigning homology groups to topological spaces. An important extension of the above theorem follows.

Theorem 0.5.2  [5] Let $X$ and $Y$ be homotopy equivalent topological spaces, i.e., $\exists$ maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \approx \text{id}$ and $g \circ f \approx \text{id}$. Then the induced chain maps are also chain homotopic to $\text{id}$, $f_\# \circ g_\# \approx \text{id}$ and $g_\# \circ f_\# \approx \text{id}$. Further by theorem 0.4.2, we have that the homology spaces $H_k(C_*(K_X))$ and $H_k(C_*(K_Y))$ are isomorphic.

The most important result to make note of is the implication of the above theorem which is enunciated as **Homotopy equivalent spaces have isomorphic homology groups**.

We noted in Section 0.2 referring to Figure 1, that homotopy equivalent spaces are similar in some sense (having a loop) even though they had different geometrical properties. The above theorem provides an alternative way of establishing such an equivalence by way of their homology groups (which are computable) being isomorphic.

In addition to providing tools to compare topological spaces, Homology groups also help us infer topological properties of these spaces. This is described in the next section.

0.5.2 Inferring Topological Properties from Homology groups

In order to understand what homology groups tell us about the topological space, we need to carefully look at the action of the boundary operators. Let us look at the null space (kernel) of $\partial_1$. Consider a cycle $c = e_4 + e_5 - e_1$ as shown in Figure 4 which is homeomorphic to a loop. The action of $\partial_1$ is given as:

$$
\partial_1(c) = \partial(e_4 + e_5 - e_1)
= (v_4 - v_1) + (v_2 - v_4) + (v_1 - v_2) = 0
$$

This implies that the null space of $\partial_1$ consists of all closed cycles (chains with zero boundaries). And as we saw in Equation 4, the boundaries of $k + 1$-simplices are closed cycles in $C_k$, and they belong to $\ker(\partial_k)$. This means that $\ker(\partial_1)$ also consists of closed cycles which are boundaries of 2-simplices. But we know that 2-simplices are homeomorphic
to disks. Therefore, if we remove all the cycles which are boundaries of 2-simplices, the cycles that remain are those circling a hole. From the definition of the homology group $H_1(C_\ast(K_X) = \ker(\partial_1)/\text{Img}(\partial_2)$, it is clear that $H_1$ counts the number of holes in our topological space. A similar explanation can be given to higher order homology groups; $H_2$ counts the number of 3-dimensional voids and so on, albeit it is difficult to visualize what the homology groups of order 3 and higher count. We now present an example to illustrate the basic mechanism of this procedure.

**Example**

We will now show how to compute the first homology space for the simplicial complex shown in Figure 3, repeated in Figure 4. We start by constructing the chain spaces $C_0, C_1,$ and $C_2$. Since there are no simplices of order greater than 2, all the chain spaces $C_k, k > 2$ are trivial. $C_0, C_1,$ and $C_2$ are vector spaces with the following basis sets:

- $C_0 : \{v_1, v_2, v_3, v_4\}$
- $C_1 : \{e_1, e_2, e_3, e_4, e_5\}$
- $C_2 : \{\sigma_1\}$

The matrix representation of the boundary operator $\partial_1 : C_1 \to C_0$ is given as:

$$
\partial_1 = \begin{pmatrix}
1 & 0 & -1 & 1 & 0 \\
-1 & 1 & 0 & 0 & -1 \\
0 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 & 1
\end{pmatrix}
$$

after column reduction, we have

$$
\partial_1 = \begin{pmatrix}
e_1 & e_2 & e_3 & e_4 & e_5 \\
v_1 & 1 & 0 & -1 & 1 & 0 \\
v_2 & -1 & 1 & 0 & 0 & -1 \\
v_3 & 0 & -1 & 1 & 0 & 0 \\
v_4 & 0 & 0 & 0 & -1 & 1
\end{pmatrix}
$$

where $z_1 = e_1 + e_2 + e_3,$ and $z_2 = e_4 + e_5 - e_1.$ Therefore, $\ker(\partial_1)$ is spanned by $\{z_1, z_2\},$ and $\text{Img}(\partial_2)$ is spanned by $\{e_1+e_2+e_3\}$. The first homology space is then $H_1 = \ker(\partial_1)/\text{Img}(\partial_2) \cong \text{span}\{e_4 + e_5 - e_1\}.$ The first homology has rank 1 which is equal to the number of holes in our topological space.

Let us look more closely at the quotient space $H_1.$ Consider the cycles $c_1$ and $c_2$ in Figure 4. It is clear from the figure, that they surround the same hole. The quotient space $H_1$ makes
Figure 4: Figure showing two homologous cycles $c_1$ and $c_2$ (in red). The difference between the cycles lies in $Img(\partial_2)$.

this similarity concrete. The difference of the cycles $c_2 - c_1 = e_1 + e_2 + e_3 \in Img(\partial_2)$. Two cycles in $ker(\partial_1)$ are homologous (equivalent) in $H_1$ if their difference lies in $Img(\partial_2)$, and hence, $c_1$ and $c_2$ are homologous.
Bibliography


