REGULAR CELL COMPLEXES IN TOTAL POSITIVITY

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Abstract. Fomin and Shapiro conjectured that the neighborhood of the origin in the Bruhat stratification of the totally nonnegative part of the unipotent radical of a Borel subgroup in a semisimple, simply connected algebraic group over $\mathbb{C}$ split over $\mathbb{R}$ is a regular CW complex homeomorphic to a ball. The main result of this paper is a proof of this conjecture. This completes the solution of the question of Bernstein of finding regular CW complexes arising naturally from representation theory having the lower intervals of Bruhat order as their closure posets. A key ingredient is a new criterion for determining whether a finite CW complex is regular with respect to a choice of characteristic maps; this is based on an interplay of combinatorics with codimension one topology.

1. Introduction

In this paper, the following conjecture of Sergey Fomin and Michael Shapiro from [12] is proven.

Conjecture 1.1. *Let Y be the neighborhood of the origin in the totally nonnegative part of the unipotent radical of a Borel subgroup B in a semisimple, simply connected algebraic group over $\mathbb{C}$ split over $\mathbb{R}$ and let $B_u = B^-uB^-$ for u in the Weyl group W. Then the stratification of Y into Bruhat cells $Y \cap B_u$ is a regular CW complex homeomorphic to a ball. Moreover for each $w \in W$, $Y_w = \cup_{u \le w} Y \cap B_u$ is a regular CW complex homeomorphic to a ball, as is the link of each of its cells.*

This result appears as Theorem 3.1. As a special case, we deduce that the neighborhood of the origin in the Bruhat stratification of the space of upper triangular matrices with 1's on the diagonal, all of whose minors are nonnegative, is a regular CW complex homeomorphic to a ball; more specifically, this is the collection of upper triangular, totally nonnegative matrices with 1's on the diagonal and entries immediately

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above the diagonal summing to a constant. The cells are specified by which minors are strictly positive and which are 0. The poset (partially ordered set) of closure relations is Bruhat order.

This positively answers the following question regarding naturally arising regular CW complexes with the intervals of Bruhat order as their closure posets. This question appears in a paper of Björner (see [4]), but was actually posed by Joseph Bernstein (personal communication, Anders Björner):

Question 1.2. It would be of considerable interest to know which (CW) posets can be reasonably interpreted as face posets of cellular decompositions of complex algebraic varieties, and whether there is a synthetic construction for doing so. In particular, can 'synthetic Schubert varieties' be naturally associated with the (lower) Bruhat intervals of any Coxeter group.

Results of Björner [4] combine with results of Björner and Wachs [6] to imply that each interval of Bruhat order is the closure poset of a regular CW complex. This is what led to Question 1.2. Fomin and Shapiro prove in [12] that the closure relations for $Y_w = \cup_{u \le w} Y \cap B_u$ are exactly those of the lower interval $(1, w)$ in Bruhat order, obtain substantial results regarding homological structure (especially in type A), and formulate Conjecture 1.1.

Motivation for studying these totally nonnegative varieties comes from a connection observed by Lusztig to his theory of canonical bases. The change of coordinate maps resulting from applying a braid move to the reduced expression specifying a canonical basis is a tropicalized version of the corresponding change of coordinates for a totally nonnegative variety (cf. [22], [23], [24]). Trying to understand such changes of coordinates was an inspiration also for the theory of cluster algebras (see [2], [14]). Our topological approach seems to give a somewhat new perspective on these changes of coordinates. The rather explicit collapses we develop later will also give quite explicit information about fibers of a map whose inverse has been the subject of considerable study (see e.g. [14] and the preprint: A. Postnikov, Total positivity, Grassmannians, and networks, arXiv:math.CO/0609184). For instance, we deduce connectedness of fibers indirectly, using our Theorem 1.3 below.

Our starting point was the following new criterion for deciding which finite CW complexes are regular with respect to a choice of characteristic maps. This provided a way of approaching Conjecture 1.1.

Theorem 1.3. *Let K be a finite CW complex with characteristic maps $f_\alpha : B^{\dim e_\alpha} \to \overline{e_\alpha}$. Then K is regular with respect to the characteristic maps $\{f_\alpha\}$ if and only if the following conditions hold:*

1. *For each α, $f_\alpha(B^{\dim e_\alpha})$ is a union of open cells*
2. *For each f_α, the preimage of open cells of dimension $\dim e_\alpha - 1$ form a dense subset of the boundary of $B^{\dim e_\alpha}$.*
3. *The closure poset of K is thin. Additionally, each open interval (u, v) with $rk(v) - rk(u) > 2$ is connected.*
4. *For each α, the restriction of f_α to the preimages of the open cells of dimension exactly one less than e_α is an injective map.*
5. *For each $e_\sigma \subseteq \overline{e_\alpha}$, f_σ factors as an embedding $\iota : B^{\dim e_\sigma} \longrightarrow B^{\dim e_\alpha}$ followed by f_α.*

Theorem 1.3 is proven in Section 2. Examples are also given in Section 2 demonstrating that each of Conditions 2, 3, 4, and 5 is not redundant. Conditions 1 and 2 imply that the closure poset is graded by cell dimension. Condition 4, i.e. codimension one injectivity, turns out to be much easier to check than injectivity in general in the setting of Conjecture 1.1 because we are able to use the exchange axiom for Coxeter groups in a way that does not generalize to higher codimensions. Theorem 1.3 seems to capture how combinatorics (encoded in condition 3) substantially reduces what one must check topologically, spreading the injectivity requirement for attaching maps around in a way that appears to be more readily checkable.

The proof of Conjecture 1.1 also involved developing a combinatorial topological toolkit for performing a series of collapses on a convex polytope in a manner that preserves regularity and homeomorphism type, reducing the number of cells at each step by eliminating some cells and identifying with each other some other pairs of cells. This is done in Section 5. We realize the stratified spaces Y_w from Conjecture 1.1 as images of continuous maps $f_{(i_1,\dots,i_d)}$ from polytopes to spaces of matrices with $f_{(i_1,\dots,i_d)}$ acting homeomorphically on the interior of its associated polytope; the collapses only identify points having the same image under $f_{(i_1,\dots,i_d)}$. Each collapse results from covering a face F of the polytope (which in our main application is a simplex) with a collection of curves homeomorphic to parallel line segments and mapping these to their endpoints in a boundary face $G \subseteq \overline{F}$ while stretching a collar for the closure of the complement of F within the boundary of a cell of dimension one higher than F to cover F. After collapsing in this manner all the faces upon which $f_{(i_1,\dots,i_d)}$ acts non-homeomorphically, namely ones indexed by non-reduced subexpressions of a reduced expression $s_{i_1} \cdots s_{i_d}$, we use Theorem 1.3 (the new regularity criterion) to prove that the induced map $\overline{f_{(i_1,\dots,i_d)}}$ on the resulting quotient space is a homeomorphism, yielding Conjecture 1.1. In particular, this implies that once we eliminate all regions indexed by non-reduced subexpressions of a reduced expression $s_{i_1}\cdots s_{i_d}$, in the process we will have identified all regions indexed by different subexpressions that are reduced expressions for the same Coxeter group element.

While the collapses developed in Section 5 are topological in nature, defined as explicit maps on spaces, we have gone to considerable effort to make the criteria one must check in order to use them as combinatorial as possible. This is done not only to make the proof of Conjecture 1.1 more combinatorial, but also to facilitate possible future applications to other stratified spaces of interest such as the double Bruhat decomposition for the totally nonnegative part of the Grassmannian or the totally nonnegative part of the flag variety. Throughout the paper, we deliberately include a high level of detail to help readers whose main expertise is in only one of the areas of the paper.

Another important ingredient in the proof of Conjecture 1.1 is the development of properties of reduced and non-reduced expressions of a 0-Hecke algebra, appearing in Section 4. The relations of a 0-Hecke algebra capture in a completely natural way which faces of a polytope should be identified with each other via the aforementioned collapses, and in fact which faces of the polytope have the same image under $f_{(i_1,\dots,i_d)}$. Thus, the 0-Hecke algebra gives a dictionary from the topology of point identifications to the combinatorics of cell identifications. The lack of a cancellation law introduces a fair bit of subtlety in how properties of Coxeter groups translate to statements we will need in the associated 0-Hecke algebra.

Remark 1.4. Lusztig and Rietsch have studied a combinatorial decomposition for the totally nonnegative part of a flag variety, namely the decomposition into double Bruhat cells (cf. [22] and [32]). Lusztig proved contractibility of the entire space in [22] while Rietsch and Williams proved contractibility of each cell closure in [33]. Williams conjectured in [38] that this is a regular CW complex homeomorphic to a ball. It seems quite plausible that Theorem 1.3 together with tools from Section 5 could also be used to prove that conjecture, though we believe that significant further new ideas would also be needed.

Rietsch determined the closure poset of this decomposition in [32]. Williams proved in [38] that this poset is shellable and thin, so in particular meets condition 3 of Theorem 1.3. Postnikov, Speyer and Williams proved in [28] for the special case of the Grassmannian that its decomposition is a CW decomposition; Rietsch and Williams subsequently generalized this to all flag varieties in [33]. In each case, it remains open whether these CW complexes are regular. Also still open is the question of whether the spaces themselves are homeomorphic to balls. These papers show that the Euler chararacteristic is what one would expect in order for these CW complexes to be regular, providing further evidence for Williams' conjecture.

Remark 1.5. This conjecture for the flag variety is related to Conjecture 1.1 in that the spaces we are considering arise as links of cells in the double Bruhat stratification of the flag variety. However, Williams' conjecture does not imply Conjecture 1.1 since links of cells in regular CW complexes are not always themselves regular. For example, consider the double suspension of a Poincare homology 3-sphere with a big cell glued in (personal communication, Anders Björner).

1.1. Background and Terminology.

Now we review basic terminology and facts from topology and combinatorics that will be essential to this paper. See e.g. [7], [27] or [36] for further background.

Definition 1.6. A *CW complex* is a space X and a collection of disjoint open cells e_α whose union is X such that:

(1) X is Hausdorff.
(2) For each open m-cell e_α of the collection, there exists a continuous map $f_\alpha : B^m \to X$ that maps the interior of B^m homeomorphically onto e_α and carries the boundary of B^m into a finite union of open cells, each of dimension less than m.
(3) A set A is closed in X if $A \cap \overline{e}_\alpha$ is closed in $\overline{e}_\alpha$ for each α.

An *open m-cell* is any topological space which is homeomorphic to the interior of an m-ball B^m. Our conventions are that the boundary of B^0 is empty, so that an open 0-cell is a point. We refer to the restriction of a characteristic map f_α to the boundary of B^m as an attaching map. Denote the closure of a cell α by $\overline{\alpha}$. A *finite CW complex* is a CW complex with finitely many open cells.

Definition 1.7. A CW complex is *regular* if additionally each of the maps f_α restricts to a homeomorphism from the boundary of B^m onto a finite union of lower dimensional open cells.

Definition 1.8. The *closure poset* of a finite CW complex is the partially ordered set (or poset) of open cells with $\sigma \le \tau$ iff $\sigma \subseteq \overline{\tau}$. By convention, we adjoin a unique minimal element $\hat{0}$ which is covered by all the 0-cells, which may be regarded as representing the empty set.

Definition 1.9. The *order complex* of a finite partially set is the simplicial complex whose i-dimensional faces are the chains $u_0 < \cdots < u_i$ of $i+1$ comparable poset elements.

Remark 1.10. The order complex of the closure poset of a finite regular CW complex K (with $\hat{0}$ removed) is the first barycentric subdivision of K, hence is homeomorphic to K. In particular, this implies that the order complex for any open interval $(\hat{0}, v)$ in the closure poset of K will be homeomorphic to a sphere $S^{rk(v)-2}$.

A poset is graded if for each $u \le v$, all saturated chains $u = u_0 \prec u_1 \prec \cdots \prec u_k = v$ involve the same number k of covering relations $u_i \prec u_{i+1}$ (i.e. $u_i < u_{i+1}$ such that $u_i \le v \le u_{i+1}$ implies $v = u_i$ or $v = u_{i+1}$). Recall that a finite, graded poset with unique minimal and maximal elements is *Eulerian* if each interval $[u, v]$ has equal numbers of elements at even and odd ranks. This is equivalent to its Möbius function satisfying $\mu(u, v) = (-1)^{rk(v)-rk(u)}$ for each pair $u < v$, or in other words the order complex of each open interval (u, v) having the same Euler characteristic as that of a sphere $S^{rk(v)-rk(u)-2}$. A finite, graded poset is *thin* if each rank two closed interval $[u, v]$ has exactly four elements, in other words if each such interval is Eulerian.

In [4], Björner characterized which finite, graded posets are closure posets of regular CW complexes, calling such posets *CW posets*. In particular, he proved that these are the finite posets having a unique minimal element $\hat{0}$, at least one additional interval, and each open interval $(\hat{0}, u)$ having order complex homeomorphic to a sphere $S^{rk(u)-2}$. Results of Danaraj and Klee in [9] give a convenient way to prove that a poset is a CW poset , namely by proving it is thin and shellable.

Two finite CW complexes may have the same closure poset in spite of having very different topological structure, so proving that the closure poset of a stratified space is a CW poset gives evidences that the stratified space is a regular CW complex, but is not enough to determine topological structure of the stratified space itself.

Definition 1.11. A *convex polytope* is the convex hull of a finite collection of points in $\mathbb{R}^n$, or equivalently it is an intersection of closed half spaces that is bounded.

For simplicial complexes and polytopes, the closure poset is often called the face poset. Let $[\sigma, \tau]$ denote the subposet consisting of elements z such that $\sigma \le z \le \tau$, called the *closed interval* from σ to τ. Likewise, the *open interval* from σ to τ, denoted (σ, τ), is the subposet of elements z with $\sigma < z < \tau$. A cell σ *covers* a cell ρ, denoted $\rho \prec \sigma$, if $\rho < \sigma$ and each z with $\rho \le z \le \sigma$ must satisfy $z = \rho$ or $z = \sigma$.

For a regular cell complex in which the link of any cell is also regular, $\Delta(u, v)$ is homeomorphic to the link of u within the boundary of v, hence is homeomorphic to $S^{\dim(v)-\dim(u)-2}$.

Remark 1.12. If each closed interval $[u, v]$ of a finite poset P is Eulerian and shellable, then each open interval has order complex homeomorphic to a sphere $S^{rk(v)-rk(u)-2}$, implying condition 3 of Theorem 1.3.

The stratified spaces we consider for our main result have closure posets that are the intervals of Bruhat order, well known to be thin and proven shellable by Björner and Wachs in [6].

2. A NEW REGULARITY CRITERION FOR CW COMPLEXES

Before proving Theorem 1.3, the main result of this section, we first give a few examples demonstrating the need for its various hypotheses. The CW complex consisting of an open 2-cell with its entire boundary attached to a 0-cell does not have closure poset graded by dimension; it violates condition 2 of Theorem 1.3. Condition 2 is designed also to preclude examples such as a CW complex whose 1-skeleton is the simplicial complex comprised of the faces $\{v_1, v_2, v_3, e_{1,2}, e_{1,3}, e_{2,3}\}$, also having a two cell with a closed interval of its boundary not equalling a point mapped to v_2 and the remainder of its boundary mapped homeomorphically to the rest of the 1-skeleton.

Remark 2.1. In this case, one may choose a different characteristic map which is a homeomorphism even at the boundary. Whether or not this can always be done for finite CW complexes with characteristic maps satisfying conditions 1, 3, 4, and 5 seems subtle at best, in light of examples such as the Alexander horned ball: a 3-ball which cannot be contracted to a point without changing the homeomorphism type of the complement.

The next example is a non-regular CW complex satisfying conditions 1, 2, 4, and 5 of Theorem 1.3, but violating condition 3.

Example 2.2. Let K be a 2-dimensional CW complex whose 1-skeleton is the simplicial complex with maximal faces $\{e_{1,2}, e_{2,3}, e_{1,3}, e_{3,4}, e_{4,5}, e_{3,5}\}$ and which has a unique 2-cell σ. The boundary of σ is mapped by f_σ to the 1-cycle $(e_{3,1}, e_{1,2}, e_{2,3}, e_{3,4}, e_{4,5}, e_{5,3})$. The point v_3 is hit twice, while the rest of the 1-skeleton is hit once, so the attaching map f_σ is not a homeomorphism.

One might ask if the connectedness part of requirement 3 is redundant, at least if one requires the closure poset be Eulerian. Closure posets do have the property that open intervals $(\hat{0}, u)$ with $rk(u) > 2$ are connected, by virtue of the fact that the image of a continuous map from a sphere S^d with $d > 0$ is connected. However, there are closure posets of CW complexes which are Eulerian and have disconnected intervals (u,v) with $rk(v) - rk(u) > 2$ (personal communication, Hugh Thomas). Nonetheless, it is still plausible that condition 3 in Theorem 1.3 could be replaced by the requirement that the closure poset be Eulerian.

Next is a non-regular CW decomposition of $\mathbb{R}P_2$ satisfying conditions 1, 2, 3, and 5 of Theorem 1.3, but failing condition 4.

Example 2.3. Let K be the CW complex having as its 1-skeleton the simplicial complex with maximal faces $e_{1,2}, e_{2,3}, e_{1,3}$. Additionally, K has a single 2-cell whose boundary is mapped to the 1-cycle which goes twice around the 1-cycle (v_1, v_2, v_3). Notice that this CW decomposition of $\mathbb{R}P_2$ has the same closure poset as a 2-simplex, but the attaching map for the 2-cell is a 2 to 1 map onto the lower dimensional cells.

Finally, we give an example (due to David Speyer) of a CW complex with characteristic maps meeting conditions 1, 2, 3 and 4, but failing condition 5, though this CW complex is regular with respect to a different choice of characteristic maps. David Speyer also helped with the formulation of condition 5.

Example 2.4. Let the 2-skeleton be the boundary of a pyramid. Now attach a 3-cell which is a triangular prism by sending an entire edge of one of the rectangular faces to the unique vertex of degree 4 in the pyramid, otherwise mapping the boundary of the prism homeomorphically to the boundary of the pyramid.

Proposition 2.5. *Conditions 1 and 2 of Theorem 1.3 imply that the closure poset is graded by cell dimension.*

Proof. Consider any $e_\rho \subseteq \overline{e_\sigma}$ with $\dim(e_\sigma) - \dim(e_\rho) > 1$. Choose a point p in e_ρ expressible as $f_\sigma(x)$ for some $x \in S^{\dim(e_\sigma)-1}$. If we take an infinite series of smaller and smaller open sets about x, by condition 2 each must include a point sent by f_σ to an open cell of higher dimension than e_ρ; finiteness of the CW complex then implies some such open cell e_τ is mapped into infinitely often, implying $p \in \overline{e_\tau}$. Thus, $e_\rho < e_\sigma$ for $\dim(e_\sigma) - \dim(e_\rho) > 1$ implies there exists e_τ with $e_\rho < e_\tau < e_\sigma$. □

This motivates us to say that a finite CW complex is *dimension-graded* whenever it meets conditions 1 and 2 of Theorem 1.3. Now to the proof of Theorem 1.3.

Proof. It is clear that conditions 1, 2, and 4 are each necessary. The necessity of 3 follows easily from the fact that a regular CW complex is homeomorphic to the order complex of its closure poset. To see that 5 is also necessary, note that if K is regular with respect to the characteristic maps $\{f_\alpha\}$, then $e_\sigma \subseteq \overline{e_\tau}$ implies that f_σ factors as $f_\tau \circ f_\tau^{-1}|_\sigma \circ f_\sigma$ where $f_\tau^{-1}|_\sigma \circ f_\sigma$ is the desired embedding.

Now to the sufficiency of these five conditions. We must prove that each attaching map f_σ is a homeomorphism from $\partial(B^{\dim \sigma})$ to the set of open cells comprising $\overline{e_\sigma} \setminus e_\sigma$. Since K is a CW complex in which the closure of each cell is a union of cells, f_σ must be continuous and surjective onto a union of lower dimensional cells, leaving us to prove injectivity of f_σ and continuity of f_σ^{-1}. However, once we prove injectivity, we may use the fact that any bijective, continuous map from a compact set to a Hausdorff space is a homeomorphism to conclude continuity of the inverse, so it suffices to prove injectivity.

If the attaching maps for K were not all injective, then we could choose open cells e_ρ, e_σ with $\dim(e_\sigma) - \dim(e_\rho)$ as small as possible such that $e_\rho \in \overline{e_\sigma}$ and f_σ restricted to the preimage of e_ρ is not 1-1. Then we could choose a point $z \in e_\rho$ with $|f_\sigma^{-1}(z)| = k$ for some $k > 1$. By condition 4, $\dim(e_\sigma) - \dim(e_\rho)$ must be at least 2. We will now show that the open interval (e_ρ, e_σ) in the closure poset has at least k connected components, which by condition 3 forces $[e_\rho, e_\sigma]$ to have rank exactly two. The point is to show for each $\overline{\text{point } p_i \in f_\sigma^{-1}(z)}$ that there is an open cell $e_{\tau_i} \subseteq \overline{e_\sigma}$ such that $p_i \in \overline{\iota(B^{\dim e_{\tau_i}})}$, and then to show for distinct points $p_i, p_j \in f_\sigma^{-1}(z)$ that the open cells e_{τ_i}, e_{τ_j} are incomparable in the closure poset. To prove the first part, take an infinite sequence of smaller and smaller balls about p_i, which by condition 2 must each intersect $f_\sigma^{-1}(e_\tau)$ for some $e_\tau < e_\sigma$ with $\dim e_\sigma - \dim e_\tau = 1$; by finiteness of K, the preimage of some such e_{τ_i} is hit infinitely often, implying $p_i \in \overline{f_\sigma^{-1}(e_{\tau_i})}$, hence $e_\rho \subseteq \overline{e_{\tau_i}}$. We prove next that the collections of cells whose closures contain the various points in $f_\sigma^{-1}(z)$ must belong to distinct components of (e_ρ, e_σ), yielding the desired k components in the open poset interval. Consider $p_1 \neq p_2$ with $p_i \in \overline{f_\sigma^{-1}(e_{\tau_i})}$ for $i = 1, 2$. If $e_{\tau_i} < e_{\tau_j}$ in the closure poset for $\{i, j\} = \{1, 2\}$, then condition 5 would imply $\overline{f_\sigma^{-1}(e_{\tau_i})} \subseteq \overline{f_\sigma^{-1}(e_{\tau_j})}$, and hence $p_1, p_2 \in \overline{f_\sigma^{-1}(e_{\tau_j})}$, contradicting the fact that f_{τ_j} restricted to the preimage of ρ is a homeomorphism. Thus, (e_ρ, e_σ) has no comparabilities between cells whose preimages under f_σ have closures containing distinct points of $f^{-1}(z)$; in particular, (e_ρ, e_σ) has at least k connected components, hence must be rank two.

Finally, we show that (e_ρ, e_τ) has at least $2k$ elements, forcing k to be 1, by the thinness requirement in condition 3. This will contradict our assumption that k was strictly larger than 1. Lemma 2.6 provides the desired $2k$ elements by showing that for each of the k preimages of

z, there are at least two open cells e_τ in (e_ρ, e_σ) with $\overline{f_\sigma^{-1}(e_\tau)}$ containing that particular preimage of z. ☐

Lemma 2.6. *If a CW complex K meets the conditions of Theorem 1.3, then it also satisfies the following condition: for each open cell e_τ and each $x \in \overline{e_\tau} \backslash e_\tau$ with $f_\tau(x)$ in an open cell $e_\rho \subseteq \overline{e_\tau}$ with $\dim e_\tau - \dim e_\rho = 2$, there exist distinct open cells $e_{\sigma_1}, e_{\sigma_2}$ with $\dim e_{\sigma_i} = 1 + \dim e_\rho$ and $x \in \overline{f_\tau^{-1}(e_{\sigma_i})}$ for $i = 1, 2$.*

Proof. Condition 2 ensures that the boundary of $B^{\dim e_\tau}$ does not include any open $(\dim e_\tau - 1)$-ball, all of whose points map are mapped by f_τ into e_ρ. In particular, each such ball containing x includes points not sent by f_τ to e_ρ. Since K is finite, there must be some particular cell e_{σ_1} such that points arbitarily close to x within the boundary of $B^{\dim \tau}$ map into e_{σ_1}, implying $x \in \overline{e_{\sigma_1}}$, with $\dim e_\rho < \dim e_{\sigma_1} < \dim e_\tau$. Thus, $e_\rho \subseteq \overline{e_{\sigma_1}}$ and $\dim e_{\sigma_1} = \dim e_\rho + 1$, just as needed.

Now let us find a suitable e_{σ_2}. Here we use the fact that removing the boundary of e_{σ_1} from a sufficiently small ball $B^{\dim e_\tau - 1}$ about x yields a disconnected region, only one of whose components may include points from e_{σ_1}. This forces the existence of the requisite open cell e_{σ_2} which includes points of the other component and has x in its closure. ☐

The following (which appears as Theorem 38.2 in [27]) will enable us to build CW complexes inductively.

Theorem 2.7. *Let Y be a CW complex of dimension at most $p-1$, let $\sum B_\alpha$ be a topological sum of closed p-balls, and let $g : \sum Bd(B_\alpha) \to Y$ be a continuous map. Then the adjunction space X formed from Y and $\sum B_\alpha$ by means of g is a CW complex, and Y is its $(p-1)$-skeleton.*

The next result, which is easy to prove, explains the general manner in which Theorem 1.3 will be used later in the paper. In particular, it singles out conditions 3 and 4 to be checked separately in order to prove that $f(K)$ is a regular CW complex, given a suitable regular CW complex K and continuous function f.

Corollary 2.8. *Let K be a finite, regular CW complex of dimension p and let f be a continuous function from K to a Hausdorff space L. Suppose that f is a homeomorphism on the interior of each open cell and on the closure of each cell of the $(p-1)$-skeleton of K. Then $f(K)$ is a finite CW complex satisfying conditions 1, 2, and 5 of Theorem 1.3, with the restrictions of f to various closed cells in K serving as the characteristic maps.*

Proof. The restrictions of f to a collection of closures of cells of the $(p-1)$-skeleton give the characteristic maps needed to prove that the $(p-1)$-skeleton of $f(K)$ is a finite CW complex. Now we use Theorem 2.7 to attach the p-cells and deduce that $f(K)$ is a finite CW complex with characteristic maps given by the various restrictions of f.

Conditions 1 and 2 are immediate from our assumptions on f. If there are two open cells σ_1, σ_2 in K (of dimension at most $p-1$) with identical image under f, then the fact that $\overline{\sigma}_1$ and $\overline{\sigma}_2$ are both regular with isomorphic closure posets gives a homeomorphism from σ_1 to σ_2 preserving cell structure, namely the map sending each x to the unique y with $f(y) = f(x)$. This allows us to use the embedding of either $\overline{\sigma}_1$ or $\overline{\sigma}_2$ in the closure of any higher cell of K to deduce condition 5. Finally, we use the fact that L is Hausdorff and that the number of cells is finite to deduce requirements (1) and (3) in our definition of CW complex. $\square$

Although the requirements of Corollary 2.8 may seem quite demanding, Corollary 2.8 is well-suited to proving for a family of regular CW complexes that their images under f are also regular CW complexes by an induction on dimension. In our application, each of the complexes in the family will have a unique maximal cell, and for any member of the family each of its closed cells will be another member of the family. The place where we will use Corollary 2.8 is in the proof of Theorem 3.13, an inductive result that is key to our main application.

3. Proof of the Fomin-Shapiro Conjecture

In this section, we apply Theorem 1.3 to the stratified spaces Y_w introduced by Fomin and Shapiro in [12] to prove the following.

Theorem 3.1. *The Bruhat decomposition Y_w of the neighborhood of the origin in the totally nonnegative part of the unipotent radical of a Borel in a semisimple, simply connected group over $\mathbb{C}$ split over $\mathbb{R}$ is a regular CW complex homeomorphic to a ball. Moreover the link of each cell, as defined in* [12]*, is as well.*

Theorem 1.3 will enable us to prove the conjecture of Fomin and Shapiro that these are regular CW complexes homeomorphic to balls which have the intervals of Bruhat order as their closure posets. We will work in terms of the description due to Lusztig of each such space as the image within a space of matrices of a continuous function $f_{(i_1,\dots,i_d)}$ from the slice of $\mathbb{R}^d_{\geq 0}$ given by intersecting with a hyperplane $\sum x_i = 1$. We sometimes will refer to $f_{(i_1,\dots,i_d)}$ as Lusztig's map. Our strategy will be to construct a regular CW complex K upon which $f_{(i_1,\dots,i_d)}$ will act in a manner that meets all the requirements of Corollary 2.8, and then to show that condition 4 of Theorem 1.3 follows easily from the exchange axiom for Coxeter groups. K will be a quotient space of a simplex, and its construction will rely heavily on the collapsing lemmas of Section 5 as well as 0-Hecke algebra lemmas of Section 4. Readers may find it useful to look ahead to those sections. Condition 3 of Theorem 1.3 is immediate from the result of Björner and Wachs in [6] that Bruhat order is thin and shellable. Thus, we may conclude that $f_{(i_1,\dots,i_d)}(K)$ is a regular CW complex homeomorphic to a ball, with the restrictions of $f_{(i_1,\dots,i_d)}$ to the closed cells of K giving the characteristic maps.

3.1. Background on totally nonnegative spaces.

Recall that a real matrix is totally nonnegative (resp. totally positive) if each minor is nonnegative (resp. positive). The totally nonnegative part of $SL_n(\mathbb{R})$ consists of the matrices in $SL_n(\mathbb{R})$ whose minors are all nonnegative. Motivated by connections to canonical bases, Lusztig generalized this dramatically in [22] as follows. The totally nonnegative part of a reductive algebraic group G over $\mathbb{C}$ which is split over $\mathbb{R}$ is the semigroup generated by the sets $\{x_i(t)|t \in \mathbb{R}_{>0}, i \in I\}, \{y_i(t)|t \in \mathbb{R}_{>0}, i \in I\}$, and $\{t \in T|\chi(t) > 0 \text{ for all } \chi \in X^*(T)\}$, for I indexing the simple roots. More generally, we have $x_i(t) = I_n + tE_{i,i+1}$, namely the n by n identity matrix modified to have the value t in position $(i, i+1)$, and $y_i(t) = I_n + tE_{i+1,i}$. In any type, $x_i(t) = exp(te_i)$ and $y_i(t) = exp(tf_i)$ for $\{e_i, f_i|i \in I\}$ the Chevallay generators. In other words, if we let ϕ_i be the homomorphism of SL_2 into G associated to the i-th simple root, then

$$x_i(t) = \phi_i \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \text{ and } y_i(t) = \phi_i \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}.$$

Let B^+, B^- be opposite Borels with N^+ (or simply N) and N^- denoting their unipotent radicals. In type A, we may choose B^+, B^- to consist of the upper and lower triangular matrices in $GL(n)$, respectively. In this case, N^+, N^- are the matrices in B^+, B^- with diagonal entries all equalling one. The totally nonnegative part of N^+, denoted Y, is the submonoid generated by $\{x_i(t_i)|i \in I, t_i \in \mathbb{R}_{>0}\}$. Let W be the Weyl group of G. One obtains a combinatorial decomposition of Y by taking the usual Bruhat decomposition of G and intersecting each open Bruhat cell B_w for $w \in W$ with Y to obtain an open cell $Y_w^o := Y \cap B_w$ in Y for $B_w = B^-wB^-$. We follow [22] in using the standard topology on $\mathbb{R}$ throughout this paper. See e.g. [17] for further background on algebraic groups.

Remark 3.2. Lusztig proved for $(i_1, \dots, i_d)$ any reduced word for w that the map $f_{(i_1,\dots,i_d)}$ sending $(t_1, \dots, t_d)$ to $x_{i_1}(t_1)\cdots x_{i_d}(t_d)$ is a homeomorphism from $\mathbb{R}^d_{>0}$ to Y^o_w (see Proposition 2.7 in [22]).

The closure of Y^o_w, denoted Y_w, is the image of this same map applied to $\mathbb{R}^d_{\geq 0}$. Since $x_i(0)$ is the identity matrix, the cells in the closure of Y^o_w are obtained by choosing subwords of $(i_1, \dots, i_d)$, i.e. $Y_w = \cup_{u\leq w} Y^o_u$ for $u \leq w$ in Bruhat order on W. Fomin and Shapiro suggested for each $u < w$ in Bruhat order that the link of the open cell Y^o_u within Y_w should serve as a good geometric model for the Bruhat interval $(u, w]$, namely as a naturally arising regular CW complex with $(u, w]$ as its closure poset. This required introducing a suitable notion of link, i.e. of $lk(u, w)$.

To this end, Fomin and Shapiro introduced a projection map $\pi_u : Y_{\geq u} \to Y^o_u$ which may be defined as follows. Letting $N(u) = u^{-1}Bu \cap N$ and $N^u = B^- u B^- \cap N$, Fomin and Shapiro proved that each $x \in Y_{\geq u}$ has a unique expression as $x = x_u x^u$ with $x_u \in N^u$ and $x^u \in N(u)$. In light of results in [FS], $\pi_u(x)$ may be defined as equalling $x_u \in N^u$. Now let $lk(u, w) = (\pi_u^{-1}(x_u)) \cap Y_{[u,w]} \cap S_\epsilon(x_u)$ for x_u an arbitrary point in Y^o_u and $S_\epsilon(x_u)$ a sufficiently small sphere about x_u (cf. p. 11 in [12]). Thus, points of $lk(u, w)$ belong to cells $Y_{u'}$ for $u < u' \leq w$, and closure relations are inherited from Y. Fomin and Shapiro proved that each open cell in $lk(u, w)$ is indeed homeomorphic to $\mathbb{R}^n$ for some n, i.e. is a cell.

Later in Section 3, we will produce a regular CW complex that will map homeomorphically to $lk(1, w)$. To this end, we develop an identification map on $\mathbb{R}^d_{\geq 0}$ described by a series of face collapses whose restrictions to $\mathbb{R}^d_{\geq 0} \cap S^{d-1}_1$, for S^{d-1}_1 the plane with coordinates summing to 1, each preserving homeomorphism type as well as regularity. Denote by $\sim$ the equivalence relation describing the point identifications resulting from the collapses. We will eventually prove that $\sim$ identifies exactly those points having the same image under Lusztig's map $f_{(i_1,\dots,i_d)} : (t_1, \dots, t_d) \to x_{i_1}(t_1)\cdots x_{i_d}(t_d)$ given by any reduced word $(i_1, \dots, i_d)$ for $w \in W$. Letting $\overline{f_{(i_1,\dots,i_d)}}$ denote the induced map from the quotient space $(\mathbb{R}^d_{\geq 0} \cap S^{d-1}_1)/\sim$ to $Y_{s_{i_1}\cdots s_{i_d}}$, we will prove in Theorem 3.13 that $\overline{f_{(i_1,\dots,i_d)}}$ is a homeomorphism from $(\mathbb{R}^d_{\geq 0} \cap S^{d-1}_1)/\sim$ to $Y_{s_{i_1}\cdots s_{i_d}}$ with $\overline{f_{(i_1,\dots,i_d)}}$ sending the open cells of $(\mathbb{R}^d_{\geq 0} \cap S^{d-1}_1)/\sim$ to the cells Y^o_u with $u \leq w = s_{i_1}\cdots s_{i_d}$.

We will eventually show that the identifications under $\sim$ all follow by transitivity from relations $x_i(t_1)x_i(t_2) = x_i(t_1 + t_2)$ along with the braid relations discussed next. These make the 0-Hecke algebra exactly the right algebraic gadget for determining which open cells in $\Delta_{d-1} = \mathbb{R}^d_{\geq 0} \cap S_1^{d-1}$ are sent to the same open cell by $f_{(i_1,\dots,i_d)}$. Section 4 develops combinatorial properties of the 0-Hecke algebra that we will need.

Recall from [22], [14] the relations (1) $x_i(t_1)x_j(t_2) = x_j(t_2)x_i(t_1)$ for any s_i, s_j which commute, and (2) $x_i(t_1)x_j(t_2)x_i(t_3) = x_j(\frac{t_2t_3}{t_1+t_3})x_i(t_1 + t_3)x_j(\frac{t_1t_2}{t_1+t_3})$ for any s_i, s_j with $(s_is_j)^3 = e$ and any $t_1+t_3 \neq 0$. These are not difficult to verify directly. In [22], it is proven that there are more general relations of a similar nature for each braid relation $(s_is_j)^{m(i,j)} = e$ in W, i.e., relations $x_i(t_1)x_j(t_2)\cdots = x_j(t_1')x_i(t_2')\dots$ of degree $m(i,j)$ for $t_1', \dots t_{m(i,j)}'$ rational functions of $t_1, \dots t_s$ each mapping $\mathbb{R}^d_{>0}$ to $\mathbb{R}_{>0}$.

Sections 3.4 and 3.5 provide the construction of $(\mathbb{R}^d_{\geq 0} \cap S_1^{d-1})/\sim$ and the proof that it is indeed a regular CW complex homeomorphic to a ball. The homeomorphism $\overline{f_{(i_1,\dots,i_d)}}$ will allow us to regard Y_w as the regular CW complex having Bruhat interval $(1,w)$ as its closure poset, i.e. to regard Y_w as $lk(1,w)$. This in turn enables us to define $lk(u,w)$ as the link of the cell indexed by u within $lk(1,w)$. Proposition 5.23 then yields regularity of $lk(u,w)$ from regularity of $lk(1,w)$.

3.2. **Background on Coxeter groups and a map w from associated 0-Hecke algebras.**

Recall e.g. from [5] the following terminology and basic properties of Coxeter groups. An *expression* for a Coxeter group element w is a way of writing it as a product of simple reflections $s_{i_1}\cdots s_{i_r}$. An expression is *reduced* when it minimizes r among all expressions for w, in which case r is called the *length* of w. Breaking now from standard terminology, we also speak of the *wordlength* of a (not necessarily reduced) expression $s_{i_1}\cdots s_{i_r}$, by which we again mean r. Given simple reflections s_i, s_j, define $m(i,j)$ to be the least positive integer such that $(s_is_j)^{m(i,j)} = 1$. We will only consider Coxeter groups in which each $m(i,j)$ is finite, in particular enabling us to regard them as reflection groups.

The following basic lemma will be key to our proof that the complexes Y_w satisfy Condition 4 in our CW complex regularity criterion:

Lemma 3.3. *Given a reduced word $s_{i_1}s_{i_2}\cdots s_{i_r}$ for a Coxeter group element w, any two distinct subwords of length $r-1$ which are both themselves reduced must give rise to distinct Coxeter group elements.*

We include a short proof of this vital fact for completeness sake.

Proof. Suppose deleting s_{i_j} yields the same Coxeter group element which we get by deleting s_{i_k} for some pair $1 \leq j < k \leq r$. This implies $s_{i_j}s_{i_{j+1}}\cdots s_{i_{k-1}} = s_{i_{j+1}}\cdots s_{i_{k-1}}s_{i_k}$. Multiplying on the right by

s_{i_k} yields

$$s_{i_j}s_{i_{j+1}}\cdots s_{i_{k-1}}s_{i_k} = s_{i_{j+1}}\cdots s_{i_{k-1}}(s_{i_k})^2 = s_{i_{j+1}}\cdots s_{i_{k-1}},$$

contradicting the fact that the original expression was reduced. $\square$

Notice that the statement of the above lemma no longer holds if we replace $r-1$ by $r-i$ for $i>1$, as indicated by the example of the reduced word $s_1s_2s_1$ in the symmetric group on 3 letters, where s_i denotes the adjacent transposition $(i,i+1)$ swapping the letters i and $i+1$ (or more generally the i-th simple reflection of a Coxeter group W). For this reason, it really seems to be quite essential to our proof of the conjecture of Fomin and Shapiro that Theorem 1.3 will enable us to focus on codimension one cell incidences.

The *Bruhat order* is the partial order on the elements of a Coxeter group W having $u \le v$ iff there are reduced expressions $r(u), r(v)$ for u, v with $r(u)$ a subexpression of $r(v)$. Bruhat order is also the closure order on the cells $B_w = B^-wB^-$ of the Bruhat stratification of the reductive algebraic group having W as its Weyl group.

Given a (not necessarily reduced) expression $s_{i_1}\cdots s_{i_d}$ for a Coxeter group element w, define a *braid-move* to be the replacement of a string of consecutive simple reflections $s_is_j\cdots$ by $s_js_i\cdots$ yielding a new expression for w by virtue of a braid relation $(s_is_j)^{m(i,j)}=1$ with $i\neq j$. Define a *nil-move* to be the replacement of a substring s_is_i appearing in consecutive positions by 1. We call braid moves with $m(i,j)=2$ *commutation moves* and those with $m(i,j)>2$ *long braid moves.* The following may be found e.g. as Theorem 3.3.1 in [5].

Theorem 3.4. *Let (W,S) be a Coxeter system consisting of Coxeter group W and minimal generating set of simple reflections S. Consider $w\in W$.*

1. *Any expression $s_{i_1}s_{i_2}\cdots s_{i_d}$ for w can be transformed into a reduced expression for w by a sequence of nil-moves and braid-moves.*
2. *Every two reduced expressions for w can be connected via a sequence of braid-moves.*

An expression $s_{i_1}\cdots s_{i_d}$ may be represented more compactly by its *word*, namely by $(i_1,\dots,i_d)$. Let us say that an expression is *stuttering* if it admits a nil-move and call it *non-stuttering* otherwise. An expression is *commutation equivalent to a stuttering expression* if a series of commutation relations may be applied to it to obtain a stuttering expression. See [16] or [5] for further background on Coxeter groups.

Associated to any Coxeter system (W,S) is a 0-Hecke algebra, with generators $\{x_i | i \in S\}$ and the following relations: for each braid relation $s_i s_j \cdots = s_j s_i \cdots$ in W, there is an analogous relation $x_i x_j \cdots = x_j x_i \cdots$, again of degree $m(i,j)$; there are also relations $x_i^2 = -x_i$ for each $i \in S$. In our set-up, we will need relations $x_i^2 = x_i$, but this sign change is inconsequential to all of our proofs, and we abuse language and call the algebra with relations $x_i^2 = x_i$ the 0-Hecke algebra of W. This variation on the usual 0-Hecke algebra has previously arisen in work on Schubert polynomials (see e.g. [13] or [25]). We refer to $x_i^2 \to x_i$ as a *modified nil-move*. It still makes sense to speak of reduced and non-reduced expressions, and many properties (including Lemma 3.3 and Theorem 3.4) carry over to the 0-Hecke algebra by virtue of having the same braid moves; there are important differences though too, resulting e.g. from the fact that cancellation is no longer available.

It is natural (and will be helpful) to associate a Coxeter group element $w(x_{i_1} \cdots x_{i_d})$ to any 0-Hecke algebra expression $x_{i_1} \cdots x_{i_d}$. This is done by applying braid moves and modified nil-moves to $x_{i_1} \cdots x_{i_d}$ to obtain a new expression $x_{j_1} \cdots x_{j_s}$ such that $s_{j_1} \cdots s_{j_s}$ is reduced, then letting $w(x_{i_1} \cdots x_{i_d}) = s_{j_1} \cdots s_{j_s}$. The fact that this does not depend on the choice of braid moves and modified nil-moves will follow from the geometric description for $w(x_{i_1} \cdots x_{i_d})$ given next in Proposition 3.5.

Define *regions* or *faces* in $X = \mathbb{R}^d_{\ge 0}$ or in $X = \mathbb{R}^d_{\ge 0} \cap S_1^{d-1}$ as the sets $R_S = \{(t_1, \dots, t_d) \in \mathbb{R}^d_{\ge 0} \cap S_1^{d-1} | t_i > 0 \text{ iff } i \in S\}$. We will study $f_{(i_1,\dots,i_d)}(\mathbb{R}^d_{\ge 0} \cap S_1^{d-1})$, using the nonstandard convention of S_1^{d-1} equalling the hyperplane with coordinates summing to 1. We will then make extensive use of the fact that $\mathbb{R}^d_{\ge 0} \cap S_1^{d-1}$ is a simplex, hence is piecewise linear. Call a region R_S *non-reduced* if the word $(i_{j_1}, \dots, i_{j_k})$ for $S = \{j_1, \dots, j_k\}$ is not reduced.

In upcoming combinatorial arguments, all that will matter is which parameters are nonzero, so we often suppress parameters and associate the 0-Hecke algebra expression $x_{i_{j_1}} \cdots x_{i_{j_k}}$ to the region $R_{\{j_1,\dots,j_k\}}$. We call this the *x-expression* of $R_{\{j_1,\dots,j_k\}}$, denoted $x(R_{\{j_1,\dots,j_k\}})$. To keep track of the positions of the nonzero parameters, we sometimes also include 1's as placeholders, e.g. describing the region $R_{\{1,3\}} = \{x_1(t_1)x_2(0)x_1(t_3) | t_1, t_3 > 0\}$ by the expression $x_1 \cdot 1 \cdot x_1$. The relations $x_i(t_1)x_i(t_2) = x_i(t_1 + t_2)$ which hold for all $t_1, t_2 \in \mathbb{R}_{\ge 0}$ and any x_i yield the modified nil-moves $x_i x_i \to x_i$. The braid moves of the Coxeter group W also hold in a suitable sense, as explained in Section 3.1, which is why it makes sense to regard these x-expressions as expressions in the 0-Hecke algebra associated to W.

Proposition 3.5. *Lusztig's map* $f_{(i_1,\dots,i_d)}$ *sends the region* R_S *of* $\mathbb{R}^d_{\geq 0}$ *with* $S = \{j_1, \dots, j_k\}$ *to the open cell* Y_u^o *for* $u = w(x_{i_{j_1}} \cdots x_{i_{j_k}})$.

Proof. This follows from Theorem 3.4, which ensures the existence of a series of braid moves and modified nil-moves which may be applied to $x_{i_{j_1}} \cdots x_{i_{j_k}}$ mapping the points of R_S onto the points of some cell R_T indexed by a reduced expression, sending each $x \in R_S$ to some $y \in R_T$ with the property that $f_{(i_1,\dots,i_d)}(x) = f_{(i_1,\dots,i_d)}(y)$. ☐

Corollary 3.6. *The Coxeter group element* $w(x_{i_{j_1}} \cdots x_{i_{j_k}})$ *does not depend on the series of braid moves and modified nil-moves used to convert* $x_{i_{j_1}} \cdots x_{i_{j_k}}$ *into a reduced expression.*

Proof. Any series of braid moves and modified nil-moves leading to a reduced expression may be used since the end result is determined solely by the image of the cell under the map $f_{(i_1,\dots,i_d)}$. ☐

Corollary 3.7. *If* $A = x_{j_1} \cdots x_{j_r}$ *and* $B = x_{k_1} \cdots x_{k_s}$ *satisfy* $\{j_1, \dots, j_r\} \subseteq \{k_1, \dots, k_s\}$, *then* $w(A) \leq_{Bruhat} w(B)$. *Thus, w is a poset map from a Boolean algebra to Bruhat order.*

Proof. A is obtained from B by setting some parameters to 0, hence the open cell to which A maps is in the closure of the open cell to which B maps. But Bruhat order is the closure order on cells of Y_w. ☐

See [1] for additional properties of this poset map from a Boolean algebra, i.e. the closure poset of a simplex, to Bruhat order, i.e. the closure poset for Y_w.

The relations $x_i x_i = x_i$ yield the following 0-Hecke algebra variant on the deletion exchange property:

Lemma 3.8. *If* $w(x_{i_1} \cdots \hat{x}_{i_r}) = w(\hat{x}_{i_1} \cdots x_{i_r})$, *then*

$$w(x_{i_1} \cdots x_{i_r}) = w(\hat{x}_{i_1} \cdots x_{i_r}) = w(x_{i_1} \cdots \hat{x}_{i_r}).$$

Proof. Since $w(x_{i_1} \cdots \hat{x}_{i_r}) = w(\hat{x}_{i_1} \cdots x_{i_r})$, we right multiply both expressions by x_{i_r} to obtain $w(x_{i_1} \cdots \hat{x}_{i_r} x_{i_r}) = w(\hat{x}_{i_1} \cdots x_{i_r} x_{i_r})$. Thus,

$$w(x_{i_1} \cdots x_{i_r}) = w(x_{i_1} \cdots \hat{x}_{i_r} x_{i_r}) = w(\hat{x}_{i_1} \cdots x_{i_r} x_{i_r}) = w(\hat{x}_{i_1} \cdots x_{i_r}).$$

☐

This leads us to the following 0-Hecke algebra notion which will play a key role in various lemmas in Section 3.5 , i.e. in the combinatorial justification that cell collapses preserve regularity.

Definition 3.9. Define a *deletion pair* in a 0-Hecke algebra expression $x_{i_1} \cdots x_{i_d}$ to be a pair $\{x_{i_r}, x_{i_s}\}$ such that the subexpression $x_{i_r} \cdots x_{i_s}$ is not reduced but $\hat{x}_{i_r} \cdots x_{i_s}$ and $x_{i_r} \cdots \hat{x}_{i_s}$ are each reduced.

For example, in type A the expression $x_1x_2x_1x_2$ has deletion pair $\{x_{i_1}, x_{i_4}\}$.

3.3. **Proof framework and deducing regularity of Y_w from regularity of $(\mathbb{R}^d_{\geq 0} \cap S_1^{d-1})/\sim$.** Fix a reduced word $(i_1, \dots, i_d)$ for w. We will use the map $\overline{f_{(i_1,\dots,i_d)}} : (\mathbb{R}^d_{\geq 0} \cap S_1^{d-1})/\sim \rightarrow Y_w$ and its restriction to various closed cells to provide the characteristic maps with respect to which Y_w will turn out to be a regular CW complex. One of the biggest challenges will be proving that $(\mathbb{R}^d_{\geq 0} \cap S_1^{d-1})/\sim$ is a regular CW complex homeomorphic to a ball. This is done in Section 3.5, where $\sim$ is defined carefully. Now we sketch the main idea of how $\sim$ will be constructed, before proving that a suitable such $\sim$ would indeed imply that $\overline{f_{(i_1,\dots,i_d)}}$ is a homeomorphism and hence that Y_w is a regular CW complex homeomorphic to a ball.

Lemma 3.10. *The new parameters after applying a braid relation will have the same sum as the old ones; moreover, this preservation of sum refines to the subset of parameters for any fixed x_i.*

Proof. This follows from the description of $x_i(t)$ as $exp(te_i)$, simply by comparing the linear terms in the expressions $x_i(t_1)x_j(t_2)\cdots = x_j(t_1')x_i(t_2')\dots$ appearing in a braid relation. □

Lemma 3.10 justifies that our description of $\mathbb{R}^d_{\geq 0} \cap S_1^{d-1}$ may be used even after a change of coordinates (as in Lemma 3.40) resulting from a braid relation, in that braid relations map $\mathbb{R}^d_{\geq 0} \cap S_1^{d-1}$ to itself.

Associate to each point $(t_1, \dots, t_d) \in \mathbb{R}^d_{\geq 0}$ the x-expression $x_{i_{j_1}} \cdots x_{i_{j_k}}$ given by the word $(i_{j_1}, \dots, i_{j_k})$ where $j_1, \dots, j_k$ are the indices of the nonzero entries in $(t_1, \dots, t_d)$. In Sections 3.4 and 3.5, we will define an equivalence relation $\sim$ on $\mathbb{R}^d_{\geq 0} \cap S_1^{d-1}$ using the following idea: if the x-expression $x_{i_{j_1}} \cdots x_{i_{j_k}}$ associated to a point $(t_1, \dots, t_d) \in \mathbb{R}^d_{\geq 0}$ is not reduced, then we may apply commutation moves and braid moves to it, causing a coordinate change to new coordinates $(u_1, \dots, u_d)$ in which we may apply a substitution $x_i(u_r)x_i(u_{r+1}) = x_i(u_r + u_{r+1})$ called a modified nil-move. If we choose to use this particular modified nil-move to collapse the region $R_{\{j_1,\dots,j_k\}}$, then we will say $(u_1, \dots, u_d) \sim (u_1', \dots, u_d')$ for those points $(u_1', \dots, u_d') \in \mathbb{R}^d_{\geq 0} \cap S_1^{d-1}$ such that $u_r' + u_{r+1}' = u_r + u_{r+1}$ and $u_i' = u_i$ for $i \neq r, r+1$. However, for each non-reduced subword $(i_{j_1}, \dots, i_{j_s})$ of $(i_1, \dots, i_d)$, we only choose one such way of identifying points of the open cell $R_{\{j_1,\dots,j_s\}}$ with points having strictly fewer nonzero parameters, namely the identifications dictated by the collapse we apply to the region $R_{\{j_1,\dots,j_s\}}$.

Remark 3.11. Additional equivalences will hold by transitivity of $\sim$, but it will be quite important to our collapsing argument that we only specify those identifications that are caused by the particular series of collapses that we perform.

When we collapse a region $R_{\{j_1,\dots,j_s\}}$ given by a nonreduced expression that requires long braid moves in order to apply a modified nil-move, each of the long braid moves will necessitate a change of coordinates within $\overline{R_{\{j_1,\dots,j_s\}}}$. We apply the same series of braid moves to all points in $\overline{R_{\{j_1,\dots,j_s\}}}$, enabling a modified nil-move specifying entire curves, each to be collapsed onto a boundary point. In this manner, $\sim$ will be defined by a series of region collapses on a regular CW complex homeomorphic to a ball identifying collections of points. Each collapse will be shown to preserve homeomorphism type and regularity using Lemma 5.21. In particular, we will need to check that the various (mainly combinatorial) hypotheses of Lemma 5.21 hold just prior to any particular collapse, assuming we successfully carried out all earlier collapses. Thus, the end result of the collapsing process, namely $(\mathbb{R}^d_{\geq 0} \cap S_1^{d-1})/\sim$, will also be a regular CW complex homeomorphic to a ball.

We use the regularity of the quotient complex $(\mathbb{R}^d_{\geq 0} \cap S_1^{d-1})/\sim$ to prove that Y_w is also a regular CW complex homeomorphic to a ball. To this end, let us now verify condition 4 of Theorem 1.3 for the characteristic maps given by $\overline{f_{(i_1,\dots,i_d)}} : (\mathbb{R}^d_{\geq 0} \cap S_1^{d-1})/\sim \longrightarrow Y_w$ and their restrictions to the closed cells of $(\mathbb{R}^d_{\geq 0} \cap S_1^{d-1})/\sim$. The proof of Lemma 3.12 below will only require the following properties of $\sim$, which will be immediate from how $\sim$ is defined later:

(1) Each $p \in \partial(\mathbb{R}^d_{\geq 0} \cap S_1^{d-1})$ whose x-expression is not reduced is identified by $\sim$ with a point having more parameters set to 0.
(2) $p \sim q$ implies $w(f_{(i_1,\dots,i_d)}(p)) = w(f_{(i_1,\dots,i_d)}(q))$.

The points in a cell boundary, i.e. the preimage of one of the attaching maps, are obtained by letting parameters go to 0.

Lemma 3.12. *Given a reduced word* $(j_1,\dots,j_s)$ *which is a subword of reduced word* $(i_1,\dots,i_d)$*, then* $\overline{f_{(i_1,\dots,i_d)}}$ *restricted to the codimension one faces of* $F = \overline{R_{\{j_1,\dots,j_s\}}}/\sim$ *is an injection into* $Y_{s_{j_1}\cdots s_{j_s}}$.

Proof. Notice first that $\overline{f_{(i_1,\dots,i_d)}}|_F = \overline{f_{(j_1,\dots,j_s)}}$. By Lemma 3.3, this means that $w(x_{j_1}\cdots \hat{x}_{j_r}\cdots x_{j_s}) \neq w(x_{j_1}\cdots \hat{x}_{j_{r'}}\cdots x_{j_s})$ for $r \neq r'$, provided both expressions are reduced, since being reduced that the map w just replaces each x_i by the corresponding simple reflection s_i. Consequently, boundary points obtained by sending distinct single parameters to 0 to obtain reduced expressions of length one shorter must be sent to distinct cells, so in particular have distinct images under $\overline{f_{(j_1,\dots,j_s)}}$. On the other hand, changing values of the nonzero parameters when a fixed parameter is set to 0 and the resulting expression is reduced must also yield points with distinct images under $\overline{f_{(j_1,\dots,j_s)}}$, since $f_{(j_1,\dots,j_s)}$ acts homeomorphically on $\mathbb{R}^s_{>0}$ for $(j_1,\dots,j_s)$ reduced. Combining yields that $\overline{f_{(j_1,\dots,j_s)}}$ is injective on the codimension one cells, as desired. □

The next theorem is phrased so as to enable a proof by induction on the length d of a Coxeter group element, without assuming that $\overline{f_{(i_1,\dots,i_d)}}((\mathbb{R}^d_{\ge 0}\cap S_1^{d-1})/\sim)$ is even a CW complex. To use Theorem 1.3, we will need the preimages of the various characteristic maps to be closed cells in $(\mathbb{R}^d_{\ge 0}\cap S_1^{d-1})/\sim$, since this will give condition 5 of our regularity criterion. It is not clear that taking the closure in $(\mathbb{R}^d_{\ge 0}\cap S_1^{d-1})/\sim$ of an open cell which is sent by $\overline{f_{(i_1,\dots,i_d)}}$ to Y_σ for a Coxeter group element σ of length $d' < d$ is the same as constructing a complex $(\mathbb{R}^{d'}_{\ge 0}\cap S_1^{d'-1})/\sim$ for σ itself directly; this issue is handled by allowing flexibility in the choice of $\sim$ in the statement of the next theorem, in particular allowing the collapsing maps for the closure of a cell which is not top-dimensional to be induced from the collapsing maps on the entire complex.

Theorem 3.13. *Let $\sim$ be the identifications given by any series of face collapses (cf. Definition 5.5) on $\mathbb{R}^d_{\ge 0}\cap S_1^{d-1}$ such that (1) $x\sim y$ implies $f_{(i_1,\dots,i_d)}(x) = f_{(i_1,\dots,i_d)}(y)$, and (2) the series of collapses eliminates all regions whose words are not reduced. Then $\overline{f_{(i_1,\dots,i_d)}}$ acts homeomorphically on $(\mathbb{R}^d_{\ge 0}\cap S_1^{d-1})/\sim$.*

Proof. The proof is by induction on d, with the case $d = 1$ being trivial. Therefore, we may assume the result for all finite Coxeter group elements of length strictly less than d. Remark 5.4 enables us to deduce continuity of $\overline{f_{(i_1,\dots,i_d)}}$ from continuity of $f_{(i_1,\dots,i_d)}$. Notice that $f_{(i_1,\dots,i_d)}$ restricts to any region obtained by setting some t_i's to 0 since $x_i(0)$ is the identity matrix. Whenever the resulting subword is reduced, results in [22] guarantee that $f_{(i_1,\dots,i_d)}$ acts homeomorphically on this open cell. The requirements of Corollary 2.8 regarding closures of cells in the $(d-1)$-skeleton follow from our inductive hypothesis, along with the fact that any series of face collapses will restrict to one on the closure of any cell. Thus, we may apply Corollary 2.8 to deduce that $\overline{f_{(i_1,\dots,i_d)}}((\mathbb{R}^d_{\ge 0}\cap S_1^{d-1})/\sim)$ is a finite CW complex with the restrictions of $\overline{f_{(i_1,\dots,i_d)}}$ to the various cell closures in $(\mathbb{R}^d_{\ge 0}\cap S_1^{d-1})/\sim$ giving the characteristic maps, and that this CW complex structure satisfies conditions 1,2 and 5 of Theorem 1.3. Lemma 3.12 confirmed condition 4 of Theorem 1.3, while results of [6] that Bruhat order is thin and shellable give condition 3. Thus, by Theorem 1.3, $\overline{f_{(i_1,\dots,i_d)}}((\mathbb{R}^d_{\geq 0} \cap S_1^{d-1})/\sim)$ is a regular CW complex with characteristic maps given by the restrictions of $\overline{f_{(i_1,\dots,i_d)}}$ to the various cell closures, which is exactly what is needed. ☐

In Section 3.5, we will construct such an identification map $\sim$ given by a composition of collapsing maps, enabling us therefore to deduce:

Corollary 3.14. *If $(i_1,\dots,i_d)$ is a reduced word for w, then $f_{(i_1,\dots,i_d)}$ induces a homeomorphism $\overline{f_{(i_1,\dots,i_d)}} : (\mathbb{R}^d_{\geq 0} \cap S_1^{d-1})/\sim \rightarrow Y_w$ which preserves the cell structure. Therefore, Y_w is a regular CW complex homeomorphic to a ball, providing the conjectured regular CW complex structure for $lk(1,w)$.*

Proof. By Theorem 3.46, $K = (\mathbb{R}^d_{\geq 0} \cap S_1^{d-1})/\sim$ is a regular CW complex homeomorphic to a ball. The map $f_{(i_1,\dots,i_d)}$ is well-defined on K because $x \sim y$ implies $f_{(i_1,\dots,i_d)}(x) = f_{(i_1,\dots,i_d)}(y)$. Using this, together with Lusztig's result that $f_{(i_1,\dots,i_d)}$ is continuous on $\mathbb{R}^d_{\geq 0} \cap S_1^{d-1}$ and is a homeomorphism on $\mathbb{R}^d_{>0} \cap S_1^{d-1}$ (see [Lu, Section 4]), and the fact that our collapsing process is given by an identification map, Proposition 13.5 of Ch. 1 of [7] gives that $\overline{f_{(i_1,\dots,i_d)}}$ is also continuous on K. Therefore, K meets all the requirements of Theorem 3.13, yielding the result. ☐

Corollary 3.15. *For $(t_1,\dots,t_d),(t'_1,\dots,t'_d) \in \mathbb{R}^d_{\geq 0}$ and $(i_1,\dots,i_d)$ any reduced word,*

$$x_{i_1}(t_1)\cdots x_{i_d}(t_d) = x_{i_1}(t'_1)\cdots x_{i_d}(t'_d) \text{ if and only if } (t_1,\dots,t_d) \sim (t'_1,\dots,t'_d).$$

We define $lk(u,w)$ as the link of u within $lk(1,w)$ for any $u \leq w$. Proposition 5.23 ensures for $(\mathbb{R}^d_{\geq 0} \cap S_1^{d-1})/\sim$ that this is a regular CW complex. This property will carry over to Y_w via our homeomorphism $\overline{f_{(i_1,\dots,i_d)}}$. Now let us check that this coincides, up to homeomorphism, with the notion of Fomin and Shapiro. To this end, we use that $f_{(i_1,\dots,i_d)}$ acts homeomorphically on $(\mathbb{R}^d \cap S_1^{d-1})/\sim$ together with Theorem 4.2 of [FS] which asserts that $\pi_u^{-1}(x_u)$ within a neighborhood of x_u is homeomorphic to an affine space of the desired dimension. Thus, a tiny sphere about x_u restricted to this subspace intersected with $\mathbb{R}^d_{\geq 0}$ gives a region homeomorphic to $lk(u,w)$.

Corollary 3.16. *The stratified space $lk(u,w)$ defined in [FS] is equivalent, up to homeomorphism, to the link of u in $lk(1,w) = (\mathbb{R}^{l(w)}_{\geq 0} \cap S_1^{l(w)-1})/\sim$. Thus, $lk(u,w)$ as defined in* [12] *is a regular CW complex homeomorphic to a ball, as conjectured in* [12].

It would be interesting to better understand how $lk(u,w)$ relates both to subword complexes (cf. [19], [20], [21], [18]) and also to the synthetic CW complexes for Bruhat intervals studied in [30].

3.4. Homeomorphism type and regularity of the quotient space $(\mathbb{R}^d_{\ge 0} \cap S^{d-1}_1)/\sim_C$.

Let us begin by collapsing the faces of $\mathbb{R}^d_{\ge 0} \cap S^{d-1}_1$ whose words are commutation equivalent to stuttering words by identifying collections of points. Denote the resulting identifications by $\sim_C$. We prove that $(\mathbb{R}^d_{\ge 0} \cap S^{d-1}_1)/\sim_C$ is a regular CW complex homeomorphic to a ball by proving that these properties persist after each collapsing step. We will start afresh in Section 3.5 for the much more difficult task of constructing the complex $(\mathbb{R}^d_{\ge 0} \cap S^{d-1}_1)/\sim$ obtained by collapsing all faces whose words are nonreduced and proving that regularity and homeomorphism type are again preserved under each collapse. A separate proof for $\sim_C$ is given first for two reasons: (1) it illustrates the strategy of the general case in a much simpler setting, and (2) it will be used in the proofs of Lemmas 3.36 and 3.40, two key ingredients to the general case.

Definition 3.17. The *omittable pairs* of an x-expression $x(F)$ are those pairs $\{x_{i_l}, x_{i_r}\} \subseteq x(F)$ such that $i_l = i_r$ and there exists a series of commutation moves applicable to $x(F)$ placing x_{i_l} and x_{i_r} into neighboring positions.

Example 3.18. The x-expression $x_1x_3x_4x_3x_1$ in type A has omittable pair $\{1,5\}$, often denoted $\{x_{i_1}, x_{i_5}\}$.

Order as follows all possible triples $(x(F), \{x_{i_l}, x_{i_r}\})$, where F is a face in $\mathbb{R}^d_{\ge 0} \cap S^{d-1}_1$, $x(F)$ is the x-expression for F, and $\{x_{i_l}, x_{i_r}\}$ is an omittable pair in $x(F)$ with $l < r$. Use linear order on the index r, then break ties with linear order on $r - l$, breaking further ties by reverse linear order on $\dim F$, and breaking any remaining ties arbitrarily.

Example 3.19. $(x_1x_3x_4x_3x_1, \{1,5\})$ precedes $(x_1x_3 \cdot x_3x_1, \{1,5\})$ because the former involves a higher dimensional face. However, $(x_1x_3 \cdot x_3x_1, \{2,4\})$ comes earlier than both due to the right endpoint in its deletion pair being farther to the left.

Using this order, we specify collapses as follows. Repeatedly choose the earliest triple $(x(F), \{x_{i_l}, x_{i_r}\})$ whose face F has not yet been collapsed. Denote by $(x(F_m), \{x_{i_{l_m}}, x_{i_{r_m}}\})$ the triple chosen for the m-th collapse. Collapse F_m by a collapsing map g_m which identifies all points in $\overline{F_m}$ satisfying $t_{i_l} + t_{i_r} = k$ and $t_{i_j} = k_j$ for each $j \ne l, r$ and any choice of constants $\{k, k_1, \dots, \hat{k}_l, \dots, \hat{k}_r, \dots, k_d\}$. We will prove that each such collapse fits the set-up needed for Lemma 5.21, hence may be accomplished by a map that preserves regularity and homeomorphism type.

Remark 3.20. It often will happen that the step collapsing (and thereby eliminating) a cell F_m will also collapse some other cells that are in its closure. However, each collapsing step will have one maximal cell among the cells being collapsed at that step, with all others in its closure.

Definition 3.21. Call the collections of points

$$\{(t_1,\dots,t_d)|t_{l_i}+t_{r_i}=k \text{ and } t_j=k_j \text{ for } j\notin\{l_i,r_i\}\}$$

in $\overline{F_i}$ for the various choices $k,k_1,\dots,\hat{k}_{l_i},\dots\hat{k}_{r_i},\dots k_d$ of constants the *level curves* of $\overline{F_i}$. These will turn out to be parallel-like (in the sense of Definition 5.18), allowing us to collapse F_i via Lemma 5.21 by mapping each curve to one of its endpoints and stretching a collar to cover what was previously $\overline{F_i}$.

Notation 3.22. If two cells G and G' are identified during one of the first $j-1$ collapsing steps, denote this by $G\sim_j G'$.

Remark 3.23. The collapse given by $(x(F_m),\{x_{i_{l_m}},x_{i_{r_m}}\})$ will also collapse those cells in $\partial(\overline{F_m})$ given by subexpressions of $x(F_m)$ having both $t_{i_{l_m}}>0$ and $t_{i_{r_m}}>0$, since deleting letters other than $x_{i_{l_m}}$ and $x_{i_{r_m}}$ preserves the property of $\{x_{i_{l_m}},x_{i_{r_m}}\}$ being an omittable pair. Thus, the collapse will identify faces having $t_{i_{l_m}}=0$ with ones instead having $t_{i_{r_m}}=0$.

To keep track combinatorially of which faces are identified by the collapses giving rise to $\sim_C$, define a *slide-move*, or simply a *slide*, to be the replacement of $S=\{j_1,\dots,j_s\}$ by $S'=\{k_1,\dots,k_s\}$ for $j_1<\cdots<j_s$ and $k_1<\cdots<k_s$ with $j_i=k_i$ for $i\neq r$ for some fixed r and $i_{j_r}=i_{k_r}$. An example in type A for $(i_1,\dots,i_d)=(1,2,3,1,2)$ is $S=\{1,5\}$ and $S'=\{4,5\}$. An *exchange* is the replacement of one letter by another letter than can be accomplished by a series of slide-moves and commutation moves. Now we use combinatorics to verify that the hypotheses needed for topological collapses in the sense of Section 5 are indeed met.

Lemma 3.24. *The collapses inducing $\sim_C$ satisfy Condition 5.15, i.e. the least upper bound condition. That is, if G_1 and G_2 are identified during the collapse of F via deletion pair $\{x_{i_{l_j}},x_{i_{r_j}}\}$, for $x_{i_{l_j}}\in x(G_1)$ and $x_{i_{r_j}}\in x(G_2)$, and F' is any least upper bound for G_1 and G_2 in the closure poset just prior to the collapse of F, then F' is also collapsed during the step collapsing F.*

Proof. By virtue of our set-up, $x(\overline{F'})$ must have subexpressions $x(G_1')$ and $x(G_2')$ with $G_1' \sim_j G_1$ and $G_2' \sim_j G_2$. Consider $x(F')$ and its earliest subexpressions (in our collapsing ordering on triples) which are x-expressions for some such $G_1' \sim_j G_1$ and $G_2' \sim_j G_2$. We first consider the case that $x_{i_{r_j}} \not\in x(F')$, which means that $x_{i_{r_j}}$ must have been exchanged with a letter x_{i_r} to its left during an earlier identification step involved in transforming G_2 to G_2'. Then $x(F')$ will have an omittable pair $\{x_{i_l}, x_{i_r}\}$ for some $r < r_j$, causing F' to have been collapsed strictly before the collapse of F, just as needed.

On the other hand, suppose $x_{i_{r_j}} \in x(F')$. Then F' will again be collapsed during or prior to the collapse of F unless $x_{i_{l_j}}$ has been exchanged for a letter $x_{i_l} \in x(F')$ to its left appearing instead in F'. But then by the fact that our collapsing order maximizes dimension among faces with the same omittable pair, this exchange $x_{i_{l_j}} \rightarrow x_{i_l}$ (or each such exchange in a series) would extend to a face including $x_{i_{r_j}}$, thereby identifying faces including x_{i_l} and $x_{i_{r_j}}$ with ones instead including $x_{i_{l_j}}$ and $x_{i_{r_j}}$. In particular, F would have been identified with F' prior to the collapse of F, implying F' is collapsed when F is. ☐

Example 3.25. The cell F with $x(F) = x_1x_1x_1$ is collapsed based on the deletion pair comprised of its leftmost two x_1's, identifying $x_1 \cdot 1 \cdot x_1$ with $1 \cdot x_1 \cdot x_1$ and in the process also identifying $x_1 \cdot 1 \cdot 1$ with $1 \cdot x_1 \cdot 1$. The region with expression $1 \cdot x_1 \cdot x_1$ is collapsed based on its pair of x_1's, identifying $1 \cdot x_1 \cdot 1$ with $1 \cdot 1 \cdot x_1$. Composing face identifications based on these two steps causes $x_1 \cdot 1 \cdot 1$ to be identified with $1 \cdot 1 \cdot x_1$, potentially causing the attaching map for the face given by $x_1 \cdot 1 \cdot x_1$ no longer to be injective; however, this face will itself have been collapsed by this time, by virtue of having already been identified with the face $1 \cdot x_i \cdot x_i$ which was already collapsed.

Condition 1 of Definition 5.18 follows immediately from our set-up. The next lemma ensures that Condition 2 of Definition 5.18 is also met by our level curves that we wish to use to perform collapses.

Lemma 3.26. *The collapses inducing $\sim_C$ satisfy Condition 5.20. That is, the two endpoints of any nontrivial level curve across which a cell F_i is collapsed live in distinct cells just prior to the collapse.*

Proof. Suppose $G_1 \subseteq \overline{F_i}$ with $t_{l_i} > 0$ and $t_{r_i} = 0$ had been identified already with $G_2 \subseteq \overline{F_i}$ instead having $t_{r_i} > 0$ and $t_{l_i} = 0$. This would have required a series of earlier slides, including one of the form $r_i \rightarrow r$ for some $r < r_i$. Our collapsing order requires $r_i - r < r_i - l_i$. By our maximality assumption in choosing which faces to collapse, the last of these slide moves shifting the right element of our deletion pair to the left would have been in a step which would have also collapsed F_i, by virtue of identifying it with a face already collapsed, a contradiction. □

Next we check Condition 3 of Definition 5.18.

Lemma 3.27. *Suppose a face $H_1 \subseteq G_1$ is collapsed prior to the collapse of F_j where H_1 is identified with H_2 in the collapsing step given by $(F_j, \{x_{i_r}, x_{i_s}\})$ by an exchange of $x_{i_r} \in H_1 \subseteq G_1$ for $x_{i_s} \in H_2 \subseteq G_2$ with $r < s$. Then the least upper bound $H_1 \vee H_2$ will have also been collapsed prior to the collapse of F_j in such a way that any two level curves in $H_1 \vee H_2$ having the same endpoint in H_1 are identified prior to the collapse $(F_j, \{x_{i_r}, x_{i_s}\})$ in such a way that their parametrizations coincide (cf. Section 5 for terminology).*

Proof. The fact that H_1 is collapsed before F_j means that $x(H_1)|_{x_{i_1}\cdots x_{i_s}}$ contains an omittable pair. However, $x(H_1)|_{x_{i_1}\cdots x_{i_s}} = x(H_1)|_{x_{i_1}\cdots x_{i_{s-1}}}$, implying $x(H_1)|_{x_{i_1}\cdots x_{i_{s-1}}}$ contains an omittable pair based upon which H_1 is collapsed. By our prioritization of higher dimensional faces in our collapsing order, the face $H_1 \vee \{x_{i_s}\} = H_1 \vee H_2$ will have been collapsed in the same way, yielding the desired result. □

Next we show that the requirements of Proposition 5.23 are met, enabling us to use it to show that links also remain regular under our collapses.

Lemma 3.28. *For H_1 and H_2 that are identified under a collapse of $H_1 \vee H_2$ via deletion pair $\{x_{i_r}, x_{i_s}\}$ with $x_{i_s} \notin H_1$ and $x_{i_r} \notin H_2$, we have* $\dim H_1 = \dim H_2 = \dim H_1 \vee H_2 - 1$.

Proof. For $H_1 \vee H_2$ not to have been collapsed earlier, it must be the case that $x(H_1)|_{x_{i_1}\cdots x_{i_s}}$ and $x(H_2)|_{x_{i_1}\cdots x_{i_s}}$ are both reduced, implying H_1 and H_2 also could not have been collapsed earlier. This includes that they cannot have been identified earlier with lower dimensional cells, since any such cell G would have $w(x(G)|_{x_{i_1}\cdots x_{i_s}}) = w(x(H_i)|_{x_{i_1}\cdots x_{i_s}})$ for $i = 1, 2$, with the Coxeter group length of the former strictly shorter than that of the latter, a contradiction. □

Theorem 3.29. $(\mathbb{R}^d_{\geq 0} \cap S^{d-1}_1)/\sim_C$ *is a regular CW complex homeomorphic to a ball.*

Proof. Lemma 5.21 is used to prove that each collapse on $\mathbb{R}^d_{\geq 0} \cap S^{d-1}_1$ used to induce $\sim_C$ may be accomplished by a map that preserves homeomorphism type and regularity.

Let us check that the hypotheses of Lemma 5.21 are indeed met at each collapsing step: the level curves for the $(i+1)$-st collapsing step are the images under $g_i \circ \cdots g_1$ of parallel line segments covering a closed cell of $\mathbb{R}^d_{\geq 0} \cap S_1^{d-1}$, since they are specified by the constraint $t_{i_l} + t_{i_r} = k$ for omittable pair $\{i_l, i_r\}$ together with the constraints $t_{i_j} = k_j$ for all $j \neq l, r$, and any set of constants $\{k\} \cup \{k_j | j \notin \{l, r\}\}$ summing to 1. Lemma 3.27 gives the distinctness requirement for the endpoints of the parallel-like curves within $\overline{G_1}$. To see that $g_i \circ \cdots \circ g_1$ acts on each level curve either homeomorphically or by sending it to a point, notice that by definition the interior of any nontrivial level curve lives entirely in some open cell $F \subseteq \overline{F_i}$, hence a cell upon which all earlier collapses act homeomorphically. Lemma 3.26 ensures that the two endpoints of any curve are distinct, so Condition 5.20 is met. Lemma 3.24 confirms Condition 5.15, i.e. the least upper bound condition. $\square$

Proposition 3.30. *Suppose $x(R_S)$ and $x(R_T)$ are not commutation equivalent to stuttering expressions. Then $R_S \sim_C R_T$ iff S and T differ from each other by a series of commutation moves and slide moves.*

Proof. Let $S = \{j_1, \ldots, j_s\}$ and $T = \{k_1, \ldots k_s\}$. We begin with pairs of words $x(R_S), x(R_T)$ differing by a single slide, so $S \cap T = S \setminus \{j_r\} = T \setminus \{k_r\}$ for some r with $i_{j_r} = i_{k_r}$. But then $x(R_{S\cup T})$ is stuttering, implying $R_{S\cup T}$ was collapsed by $\sim_C$. The fact that $x(R_S), x(R_T)$ are not commutation equivalent to stuttering expressions makes it impossible to apply commutation relations to $x(R_{S\cup T})$ to obtain any other stuttering pair. Thus, $R_{S\cup T}$ could have only been collapsed by identifying R_S with R_T. By transitivity of $\sim_C$, S and T differing by a series of slide moves give rise to R_S, R_T with $R_S \sim_C R_T$. Applying commutation relations to $x(R_S)$ to produce $\sigma(x(R_{\sigma(S)})$ which is slide equivalent to $x(R_T)$ again ensures $x(R_{S\cup T})$ also admits the same commutation moves leading to a stuttering word, and again $x(R_{S\cup T})$ does not admit any other stuttering pairs, so again $R_S \sim_C R_T$. Transitivity of $\sim_C$ yields the result. $\square$

Example 3.31. For $(i_1, \ldots, i_d) = (1, 2, 1)$ in type A, $R_{\{1\}}$ is identified with $R_{\{3\}}$ during the collapse of $R_{\{1,3\}}$. If $(i_1, \ldots, i_d) = (1, 3, 1)$, then $R_{\{1,2\}}$ is identified with $R_{\{2,3\}}$ during the collapse of $R_{\{1,2,3\}}$.

3.5. Homeomorphism type and regularity of the quotient space $(\mathbb{R}^d_{\geq 0} \cap S_1^{d-1})/\sim$.

Now we turn to the identifications $\sim$ induced by collapsing all faces whose words are nonreduced, i.e. collapsing those requiring long braid moves in addition to those faces whose words are commutation equivalent to stuttering words.

For any deletion pair $\{x_{i_r}, x_{i_s}\}$ with $r < s$ in any expression $x(F)$, Lemma 4.1 together with Theorem 3.4 guarantees that it is possible to apply braid and commutation moves to the subexpression $x_{i_r} \cdots x_{i_{s-1}}$, yielding an expression whose rightmost letter forms a stuttering pair together with x_{i_s}. For example, $(x_1x_2x_1)x_3x_2x_3 \to x_2x_1(x_2x_3x_2)x_3 \to x_2x_1x_3x_2x_3x_3$ gives stuttering pair x_3x_3 after two long braid moves. The existence of such a series of moves allows us to define the quantity $c(\{x_{i_r}, x_{i_s}\}; x(F))$ as the smallest number of long braid moves needed in such a series of moves.

Example 3.32. In type A, we have $c(\{x_{i_1}, x_{i_4}\}; x_1x_2x_1x_2) = 1$, because we may apply the relation $x_1x_2x_1 \to x_2x_1x_2$ to obtain $x_2x_1x_2x_2$.

Now let us order triples $(x(F), \{x_{i_l}, x_{i_r}\})$ where $\{x_{i_l}, x_{i_r}\}$ is a deletion pair of $x(F)$ so as to specify the series of cell collapses that will induce $\sim$. By convention, say $l < r$. Letting the statistics listed earliest take highest priority, and using later statistics to break ties, order the triples $(x(F), \{x_{i_l}, x_{i_r}\})$ by: (1) linear order on r, (2) linear order on $r - l$, (3) linear order on $c(\{x_{i_l}, x_{i_r}\}; x(F))$, and (4) reverse linear order on $\dim F$. We may break any remaining ties arbitrarily.

We repeatedly choose the earliest triple $(x(F), \{x_{i_l}, x_{i_r}\})$ among those for faces F not yet collapsed. Denote the triple chosen in this manner for the j-th collapse by $(x(F_j), \{x_{i_{l_j}}, x_{i_{r_j}}\})$. We will use Lemma 5.21 to perform the collapse of F_j by a collapsing map g_j. Our task now is to show that the hypotheses of Lemma 5.21 are indeed met. The level curves (cf. Definition 3.33) resulting from the deletion pair $\{x_{i_{l_j}}, x_{i_{r_j}}\}$ in $x(F_j)$ will serve as our so-called parallel-like curves. To obtain these curves, we will use a series of commutation moves and long braid moves involving exactly $c(\{x_{i_{l_j}}, x_{i_{r_j}}\}; x(F_j))$ long braid moves to create a stuttering pair. This will require a change of coordinates, but Lemma 3.40 gives the requisite change of coordinate map and proves it is a homeomorphism on the closed cell where it is to be applied. The proof that a stuttering pair gives rise to a family of parallel-like curves will be similar to the $\sim_C$ case once we develop suitable change-of-coordinates homeomorphisms; recall that in the $\sim_C$ case the curves resulted from the stuttering relation $x_i(t_1)x_i(t_2) = x_i(t_1 + t_2)$.

Definition 3.33. Generalizing the $\sim_C$ situation, define the *level curves* of F_j with respect to deletion pair $\{x_{i_{l_j}}, x_{i_{r_j}}\}$ as follows. Suppose $(i_1, \dots, i_d)$ is transformed to $(j_1, \dots, j_d)$ by a series of braid moves on $(i_{l_j}, \dots, i_{r_j - 1})$ using exactly $c(x(F), \{x_{i_{l_j}}, x_{i_{r_j}}\})$ long braid moves. Choose new coordinates $(u_1, \dots, u_d)$ for the points of F_j by requiring $f_{(i_1,\dots,i_d)}(t_1, \dots, t_d) = f_{(j_1,\dots,j_d)}(u_1, \dots, u_d)$, as justified by Lemma 3.40.

Then the level curves of F_j are the collections of points $(u_1, \dots, u_d)$ satisfying $u_{r_j-1} + u_{r_j} = k$ and $u_m = k_m$ for all $m \; /\!\int \{r_j - 1, r_j\}$ for the various choices of constants $\{k\} \cup \{k_m | m \notin \{r_j - 1, r_j\}\}$ for $m \not\int \{r_j - 1, r_j\}$.

Example 3.34. Applying braid moves to $x_1x_2x_1x_3x_2x_3$ yields $x_2x_1x_3x_2x_3x_3$. Collapsing based on the resulting stuttering pair will cause the proper face $1 \cdot x_2x_1x_3x_2x_3$ to be identified with the face $x_1x_2x_1x_3x_2 \cdot 1$. The proper face $x_1x_2 \cdot 1 \cdot 1 \cdot 1 \cdot x_3$ is neither collapsed nor identified with another face in the process since not all requisite braid moves apply to it. On the other hand, the face $x_1x_2 \cdot 1 \cdot x_3x_2x_3$ would have already been collapsed at an earlier step, hence need not be considered in the next lemma as part of the boundary of the cell indexed by $x_1x_2x_1x_3x_2x_3$.

Next we show in Lemma 3.40 that the braid moves needed to create stuttering pairs may be accomplished by change of coordinate maps that are homeomorphisms; first we need a few ingredients.

Remark 3.35. For any braid relation $(s_is_j)^{m(i,j)} = e$ and any $t_2, \dots, t_{m(i,j)} > 0$, there is a unique $(t_1', \dots, t_{m(i,j)}') \in \mathbb{R}_{\geq 0}^{m(i,j)}$ satisfying

$$x_i(0)x_j(t_2)x_i(t_3)\cdots = x_j(t_1')x_i(t_2')x_j(t_3')\cdots ,$$

namely $t_1' = t_2, t_2' = t_3, \dots, t_{m(i,j)-1}' = t_{m(i,j)}$, and $t_{m(i,j)}' = 0$.

Lemma 3.36. *Consider the reduced expression $s_is_j\dots$ of length $m(i,j)$ comprised of alternating s_i's and s_j's. Then the resulting regular CW complex $\Delta = (\mathbb{R}_{\geq 0}^{m(i,j)} \cap S_1^{m(i,j)-1})/\sim_C$ given by $(i,j,\dots)$ is homeomorphic via the map $f_{(j,i,\dots)}^{-1} \circ f_{(i,j,\dots)}$ to the regular CW complex $\Delta' = (\mathbb{R}_{\geq 0}^{m(i,j)} \cap S_1^{m(i,j)-1})/\sim_{C'}$ given by $(j,i,\dots)$.*

Proof. We may use the fact that $f_{(i,j,\dots)}$ and $f_{(j,i,\dots)}$ act homeomorphically on the interior of the big cell. Each point $x \in \Delta$ not in the interior of the big cell must instead belong to a region $R_{\{i_{j_1},\dots,i_{j_k}\}}$ whose associated Coxeter group element $w(x_{i_{j_1}}\cdots x_{i_{j_k}})$ has a unique reduced expression, namely one with the appropriate alternation of s_i's and s_j's. Thus, x must be sent to a point in Δ' having this same reduced expression, so that by Proposition 3.30 the only choices to be made are equivalent to each other under $\sim_{C'}$. This map from Δ to Δ' is therefore a composition of two homeomorphisms, namely $f_{(i,j,\dots)}$ and $f_{(j,i,\dots)}^{-1}$. ☐

Example 3.37. The type A relation $s_is_{i+1}s_i = s_{i+1}s_is_{i+1}$ gives rise to the map $(t_1, t_2, t_3) \to (t_1', t_2', t_3')$ for $(t_1', t_2', t_3') = (\frac{t_2t_3}{t_1+t_3}, t_1 + t_3, \frac{t_1t_2}{t_1+t_3})$ on the interior of $\{f_{(1,2,1)}(t_1, t_2, t_3) | t_1, t_2, t_3 \geq 0\}$. The above proposition shows that by virtue of $\sim_C$, this map on $(\mathbb{R}_{\geq 0}^d \cap S_1^{d-1})/\sim_C$ extends to the boundary, e.g. sending $(t_1, t_2, 0)$ to $(0, t_1, t_2)$ and sending $(0, t_2, 0)$ to the $\sim_{C'}$-equivalence class $\{(t_1', 0, t_3')|t_1' + t_3' = t_2\}$.

Lemma 3.38. *Given a reduced word* $(i_1, \dots, i_d)$ *and some* $d' < d$ *such that we may assume by induction that our series of collapses for* $(i_1, \dots, i_{d'})$ *successfully applies to* $\mathbb{R}^{d'}_{\geq 0} \cap S_1^{d'-1}$*, with each collapse preserving regularity and homeomorphism type, then this same series of collapses may be successfully performed on* $\mathbb{R}^{d}_{\geq 0} \cap S_1^{d-1}$*.*

Proof. First apply each collapse to the subcomplex of $\mathbb{R}^{d}_{\geq 0} \cap S_1^{d-1}$ in which $t_{d'+1} = \cdots = t_d = 0$, since this is exactly $\mathbb{R}^{d'}_{\geq 0} \cap S_1^{d'-1}$. Then extend continuously to the family of subspaces with $t_{d'+1} = \cdots = t_d = k$ for $0 \leq k \leq \frac{1}{d-d'+1}$, using that each of these is also isomorphic to $\mathbb{R}^{d'}_{\geq 0} \cap S_1^{d'-1}$. Geometrically, we are adding a cone point and extending the collapse to the coned complex. Continuity follows from the fact that the level curves we collapse across hold all coordinates constant except the two involved in the deletion pair, after suitable change of coordinates which in particular holds $t_{d'+1}, \dots, t_d$ constant. $\square$

Remark 3.39. Suppose that the k-th collapsing step uses deletion pair $\{x_{i_r}, x_{i_s}\}$ for $r < s$. Then all possible identifications based on deletion pairs $\{x_{i_u}, x_{i_v}\}$ with $\max(u, v) < s$ will have already been accomplished prior to step k, as proven next.

Proof. Consider the map $f_{(i_1,\dots,i_{s-1})}$ on just the first $s-1$ positions in our reduced word. By induction on wordlength, we may use Theorem 3.13 to deduce that $\overline{f_{(i_1,\dots,i_{s-1})}}$ is a homeomorphism from $(\mathbb{R}^{s-1}_{\geq 0} \cap S_1^{s-2})/\sim$ to $Y_{s_{i_1} \cdots s_{i_{s-1}}}$. By Lemma 3.38, this means in particular that the collapses based on deletion pairs using only positions $1, \dots, s-1$ are enough to accomplish all the desired identifications. $\square$

The next lemma relies heavily on our collapsing order. Denote by $\sim_k$ the equivalence relation comprised of the identifications from the first $k-1$ collapsing steps, i.e. those done prior to the collapse given by $(x(F_k), \{x_{i_r}, x_{i_s}\})$. Denote by $\sim^p$ the set of identifications under $\sim$ resulting from collapses based on deletion pairs only involving the leftmost p positions.

Lemma 3.40. *Each long braid move used in the series of braid moves enabling the collapsing step given by* $(x(F_k); \{x_{i_r}, x_{i_s}\})$ *gives rise to a change of coordinates map ch which is a homeomorphism from the region* $\overline{F_k}/\sim_k$ *being collapsed within* $(\mathbb{R}^{d}_{\geq 0} \cap S_1^{d-1})/\sim_k$ *to itself. Moreover,* $ch(t_1, \dots, t_d) = (t_1', \dots, t_d')$ *implies* $x_{i_1}(t_1) \cdots x_{i_d}(t_2) = x_{j_1}(t_1') \cdots x_{j_d}(t_d')$ *where* $(j_1, \dots, j_d)$ *is the word obtained from* $(i_1, \dots, i_d)$ *by applying the braid move.*

Proof. We first define the map ch, prove it is well-defined and bijective, then finally prove continuity of ch and its inverse. We begin with the case where ch is the first long braid move in the chosen series of braid moves on $x_{i_r} \cdots x_{i_{s-1}}$ using exactly $c(x(F_k), \{x_{i_r}, x_{i_s}\})$ long braid moves to create a stutter with x_{i_s}. The case of subsequent long braid moves will follow easily from our treatment of this case.

We may assume that the long braid move under consideration involves the indexing positions r' through s' for some $r' < s' < s$, and that it takes the form $x_i x_j \cdots \rightarrow x_j x_i \cdots$. Lemma 3.36 allows us to define ch on the closed region $G \subseteq \overline{F_k}/\sim_k$ where all parameters are zero except those involved in this braid move, i.e. $t_1 = \cdots = t_{r'-1} = 0 = t_{s'+1} = \cdots = t_d$, letting $ch = f^{-1}_{(j,i,\dots)} \circ f_{(i,j,\dots)}$ with maps composed right to left, and $f^{-1}_{(j,i,\dots)}$ sending $f_{(i,j,\dots)}(t_1, \dots, t_d)$ to the equivalence class under $\sim_k$ of points $(t'_1, \dots, t'_d)$ satisfying $f_{(j,i,\dots)}(t'_1, \dots, t'_d) = f_{(i,j,\dots)}(t_1, \dots, t_d)$. This is well-defined because our collapsing order ensures by Remark 3.39 that the identifications due to slide-moves on the subword to which the braid is being performed will have already been achieved by $\sim_k$, by virtue of their deletion pairs having right endpoint strictly to the left of x_{i_s}.

We now extend this definition for ch from G to $\overline{F_k}/\sim_k$. Our first step is to define ch on $\overline{F_k}/\sim_C^{s'-r'+1}$ where $\sim_C^{s'-r'+1}$ has just the identifications resulting from slide moves on positions r' through s'. Let ch act as the identity on all coordinates to the left of $x_{i_{r'}}$ as well as those to the right of $x_{i_{s'}}$, i.e., letting $ch = (id_{r'-1} \otimes f^{-1}_{(j,i,\dots)} \otimes id_{d-s'}) \circ (id_{r'-1} \otimes f_{(i,j,\dots)} \otimes id_{d-s'})$. To define ch on $\overline{F_k}/\sim_k$, we impose the additional identifications $ch(x) \sim_k ch(y)$ for each pair x, y having $x \sim_k y$ prior to the application of ch. We will show that this map is indeed a homeomorphism, enabling us to regard its image also as $\overline{F_k}/\sim_k$.

Let us first check that ch is indeed bijective on $\overline{F}_k/\sim_C^{s'-r'+1}$. By induction on length (cf. Remark 3.39), we may assume for $d' < d$ that $\overline{f_{(i_1,\dots,i_{d'})}}$ acts homeomorphically on $(\mathbb{R}^{d'}_{\geq 0} \cap S_1^{d'-1})/\sim$, ensuring all possible identifications based on deletion pairs $\{x_{i_{r''}}, x_{i_{s''}}\}$ with $r'' < s'' < s$ will have been accomplished earlier. From this, we deduce injectivity of $id \otimes f_{(i,j,\dots)} \otimes id$ (resp. $id \otimes f_{(j,i,\dots)} \otimes id$) on $(\mathbb{R}^d_{\geq 0} \cap S_1^{d-1})/\sim_C$ (resp. $(\mathbb{R}^d_{\geq 0} \cap S_1^{d-1})/\sim_{C'}$), since $f_{(i,j,\dots)}(x) = f_{(i,j,\dots)}(y)$ implies $x \sim_C y$, and likewise for $f_{(j,i,\dots)}$; here, $\sim_{C'}$ is as defined in Lemma 3.36. This along with the fact that $im(f_{(i,j,\dots)}) = im(f_{(j,i,\dots)})$ yields bijectivity of $(id \otimes f^{-1}_{(j,i,\dots)} \otimes id) \circ (id \otimes f_{(i,j,\dots)} \otimes id)$.

Next observe that ch acts homeomorphically on $H_k = \overline{F_k}/(\sim^{r'-1} \otimes \sim_C^{s'-r'+1})$ where H_k has exactly the identifications based on deletion pairs with right endpoint to the left of $x_{i_{r'}}$ along with the slide-move identifications on positions r' through s'; the point is that ch acts as the identity on the coordinates outside the braided segment and acts as the homeomorphism $f^{-1}_{(j,i,\dots)} \circ f_{(i,j,\dots)}$ on the braided coordinates, hence acts homeomorphically on H_k. Remark 3.39 also implies that $\overline{F_k}/\sim_k$ is a quotient space of H_k, i.e. all identifications present in H_k also occur in $\overline{F_k}/\sim_k$. We will use Lemma 5.8 applied repeatedly, once after each in a series of collapses, to deduce that the homeomorphism ch on $\overline{F_k}/(\sim^{r'-1} \otimes \sim_C^{s'-r'+1})$ induces a homeomorphism on $\overline{F_k}/\sim_k$ which we also call ch.

To this end, we now show that $\overline{F_k}/\sim_k$ may be obtained from H_k by a series of face collapses. Apply our collapsing order to H_k, i.e. at each step choosing a deletion pair $\{x_{i_l}, x_{i_r}\}$ for $l < r$ with r as small as possible, breaking ties by making $r - l$ as small as possible, then breaking further ties by choosing face $F \subseteq H_k$ to be collapsed with $c(x(F), \{x_{i_l}, x_{i_r}\})$ as small as possible, and finally making the dimension of F as large as possible among such faces F, breaking any remaining ties arbitrarily. We may use Lemma 5.21 to accomplish each such collapse, for the following reasons: we may assume by induction that all steps prior to the k-th collapsing step were performed successfully; this also implies the analogous steps on H_k may be performed successfully, since the only change to the situation is that our quotienting now identifies faces that coincide except for slide moves on positions $r', \dots, s'$ and modified nil-moves. In theory, these extra identifications could cause a face to be collapsed earlier by virtue of being identified with a face collapsed earlier, but this in fact cannot happen: the slide moves and modified nil moves merely identify leftmost representatives with expressions obtained by sliding letters to the right, but our collapsing order ensures that the faces with letters slid to the right cannot be collapsed earlier than their leftmost counterparts since they cannot have earlier deletion pairs in our collapsing order.

Each of these collapses performed on H_k (or the quotient space derived from it by earlier collapses) to obtain $\overline{F_k}/\sim_k$ induces a collapsing map on $ch(H_k)$ (or the appropriate quotient space of it), by virtue of ch carrying any family of parallel-like curves for H_k (or a quotient space of it) to a family of parallel-like curves for $ch(H_k)$ (or the appropriate quotient space of it); thus, our collapses on H_k induce a series of face collapses on $ch(H_k)$ yielding $ch(\overline{F_k}/\sim_k)$. This implies that the map sending $ch(H_k)$ onto its quotient space $ch(\overline{F_k}/\sim_k)$ is a composition of identification maps given by collapses. After each of these collapses, we use Lemma 5.8 to deduce that ch still acts homeomorphically on the derived quotient space. In particular, this yields that ch acts homeomorphically on $\overline{F_k}/\sim_k$, as desired.

When more than one long braid move is needed, for each such move we define its change of coordinates map ch in terms of the coordinates just prior to applying ch. The above arguments then immediately apply. $\square$

Remark 3.41. A face σ of $\overline{F_i}$ having $t_r = 0$ is identified with the face σ' instead having $t_s = 0$ iff the long braid moves apply to the subexpression of $x(F_i)$ involving exactly the letters in σ in such a way so as to move into neighboring positions the two letters of the deletion pair.

Now that changes of coordinates are homeomorphisms, as proven in Lemma 3.40, this enables collapses to be defined. Next we use combinatorics to verify that our collapsing steps indeed meet the conditions needed for our topological collapsing lemmas, namely the requirements of Lemmas 5.21 and 5.23.

When a collapse identifies cells A and A' via a deletion pair $\{x_{i_u}, x_{i_v}\}$, we say that x_{i_u} is *exchanged* for x_{i_v}, denoted $x_{i_u} \to x_{i_v}$. For subexpressions $x(A)$ and $x(B)$ of $x_{i_1} \cdots x_{i_d}$, let $x(A) \vee x(B)$ be the expression made of the union of the indexing positions from $x(A)$ and $x(B)$.

Next we show that our level curves meet the third requirement in our definition of parallel-like curves, Definition 5.18.

Lemma 3.42. *If a face $H_2 \subseteq G_2$ is collapsed prior to the collapse of F_j where H_2 is to be identified with H_1 in the collapsing step given by $(F_j, \{x_{i_r}, x_{i_s}\})$ by an exchange of $x_{i_s} \in H_2$ for $x_{i_r} \in H_1$ for $r < s$, then $H_1 \vee H_2$ will also be collapsed prior to the collapse of F_j in such a way that any two level curves with the same endpoint in H_1 will have already been identified with each other in a manner that preserves the parametrization.*

Proof. Suppose $x(H_2)|_{x_{i_1}\cdots x_{i_s}} = x_{i_{j_1}} \cdots \hat{x}_{i_r} \cdots x_{i_s}$ and $x(H_1)|_{x_{i_1}\cdots x_{i_s}} = x_{i_{j_1}} \cdots x_{i_r} \cdots \hat{x}_{i_s}$. The fact that H_2 has already been collapsed means $x_{i_{j_1}} \cdots \hat{x}_{i_r} \cdots x_{i_s}$ is not reduced. But this equals $x_{i_{j_1}} \cdots x_{i_r} \cdots \hat{x}_{i_s}$ which has the same wordlength and therefore is also not reduced. By our collapsing order, which uses right endpoint in the optimal deletion pair to order collapses, only turning to other data to break ties, H_1 will also have been collapsed already. But the fact that we maximize dimension in choosing collapses with a given deletion pair and number of long braid moves and that we use linear order on the number of long braid moves for faces with a fixed optimal deletion pair means that $H_1 \vee H_2$ will then also have already been collapsed using the same deletion pair as in H_1, hence the same parametrization. $\square$

Lemma 3.43. *If the collapse of F is across level curves each having one endpoint in $\overline{H}_1$ and the other in $\overline{H}_2$, then* $\dim H_1 = \dim H_2 = \dim F - 1$.

Proof. Let $\{x_{i_r}, x_{i_2}\}$ be the deletion pair inducing the collapse of F. Then $x(H_1)|_{x_{i_1}\cdots x_{i_s}}$ and $x(H_2)|_{x_{i_1}\cdots x_{i_s}}$ must both be reduced, since otherwise F would have been collapsed earlier. Thus, neither H_1 nor H_2 will have been collapsed earlier, from which the result follows by comparing wordlengths. $\square$

Lemma 3.44. *The collapses inducing $\sim$ satisfy Condition 5.20, i.e. the distinct endpoints condition.*

Proof. Suppose F is collapsed via deletion pair $\{x_{l_i}, x_{r_i}\}$. Thus, t_{l_i}, t_{r_i} are the parameters of the deletion pair. Let u_r, u_s be the new parameters after suitable coordinate changes as in Lemma 3.40 for the stuttering pair obtained from our deletion pair by a series of braid moves. Let G_1 and G_2 be the faces containing opposite endpoints of the level curves across which F is collapsed, so G_1 has $t_{l_i} = 0$ and G_2 has $t_{r_i} = 0$, with G_1 and G_2 otherwise both coinciding with F in terms of which parameters are zero and which are nonzero. (This implies that one of these faces will have $u_r = 0$ while the other instead has $u_s = 0$.) Our goal is to prove that G_1 and G_2 are not identified in an earlier collapse.

Suppose they are identified earlier. We may assume by induction that the complex is regular at each step prior to collapsing F, precluding G_1 and G_2 from being incomparable in the closure poset just prior to their identification, because such an incomparability would make $\overline{F}$ non-regular just after the identification of G_1 and G_2, a contradiction. Suppose on the other hand we have $G_1 \subseteq \overline{G_2}$ or $G_2 \subseteq \overline{G_1}$ just prior to their identification. But $x(G_1)$ and $x(G_2)$ differ only in the deletion pair $\{x_{i_{l_i}}, x_{i_{r_i}}\}$, which means one must have collapsed onto a lower dimensional cell to be in the boundary of the other, implying both would be collapsed prior to our current collapsing step by Lemma 3.42. But $x(G_2)|_{x_{i_1}\cdots x_{i_{r_i}}}$ has no deletion pairs, precluding G_2 from having been collapsed earlier, a contradiction. $\square$

Lemma 3.45. *The collapses inducing $\sim$ satisfy Condition 5.15, i.e. the least upper bound condition.*

Proof. Suppose that the k-th collapsing step collapses a face F via deletion pair $\{x_{i_r}, x_{i_s}\}$ across parallel-like curves (cf. Definition 5.18) each having one endpoint in $\overline{A}$ and the other endpoint in $\overline{B}$, where A is obtained by omitting x_{i_r} from $x(F)$, and B is obtained from F by omitting x_{i_s}. Remark 3.39 allows us to assume that all possible identifications based on deletion pairs strictly to the left of x_{i_s} have already been done before the k-th collapse. Let $x(F')$ be an x-expression for any other least upper bound for A and B just prior to step k. $A, B \in \overline{F'}$ implies that there are subexpressions $x(A_u)$ and $x(B_v)$ of $x(F')$ such that $A_u \sim_k A, B_v \sim_k B$, with $x(F') = x(A_u) \vee x(B_v)$.

Suppose $x_{i_s} \notin x(F')$. We will show in this case that A_u must have already been identified with B_v in an earlier collapsing step, yielding a contradiction to Lemma 3.44 holding. Because we have only done identifications based on deletion pairs whose right endpoint is at position x_{i_s} or to its left, this means that A, A_u, B, and B_v all must coincide with each other to the right of x_{i_s}. We also know that $w(A_u|_{x_{i_1}\cdots x_{i_s}}) = w(A|_{x_{i_1}\cdots x_{i_s}})$ and $w(B_v|_{x_{i_1}\cdots x_{i_s}}) = w(B|_{x_{i_1}\cdots x_{i_s}})$, since A was identified with A_u and B was identified with B_v prior to step k by a series of exchanges not involving any letters to the right of x_{i_s}. Hence, $w(A_u|_{x_{i_1}\cdots x_{i_s}}) = w(B_v|_{x_{i_1}\cdots x_{i_s}})$, since by definition $w(A|_{x_{i_1}\cdots x_{i_s}}) = w(B|_{x_{i_1}\cdots x_{i_s}})$. Since $x_{i_s} \notin A_u \vee B_v$, we must also have $w(A_u|_{x_{i_1}\cdots x_{i_{s-1}}}) = w(B_v|_{x_{i_1}\cdots x_{i_{s-1}}})$. But now we may use Remark 3.39 to deduce that $A_u|_{x_{i_1}\cdots x_{i_{s-1}}}$ must get identified with $B_v|_{x_{i_1}\cdots x_{i_{s-1}}}$ in the collapsing process for the smaller complex given by $(i_1, \dots, i_{s-1})$. But using Lemma 3.38, this implies $A_u \sim_k B_v$, completing this case.

Now to the case $x_{i_s} \in F'$. Begin reading A_u and B_v from left to right, and consider the first letter that is present in one but not the other. Without loss of generality, suppose this letter x_{i_m} appears in A_u. Since $w(A|_{x_{i_1}\cdots x_{i_s}}) = w(B|_{x_{i_1}\cdots x_{i_s}})$, there exists $x_{i_n} \in B_v$ with $n \le s$ and $R(x_{i_n})$ in B_v equalling $R(x_{i_m})$ in A_u.

Suppose $m < r$. Then it follows easily from Lemma 4.7, $\{x_{i_m}, x_{i_n}\}$ comprise a deletion pair in $B_v \cup \{x_{i_m}\}$; this is due to the fact that as long as words are reduced, we may instead regard them in the associated Coxeter group, enabling cancellation of $s_{i_1} \cdots s_{i_{\min(m,n)-1}}$ from both expressions, putting us exactly in the situation to which Lemma 4.7 applies. For $n < s$, the exchange $x_{i_m} \to x_{i_n}$ on B_v must occur prior to step k, by Remark 3.39, implying B_v is equivalent to some $B_{v'}$ in which x_{i_n} is replaced by x_{i_m}. Note that either $A_u \vee B_v = A_u \vee B_{v'}$ or $A_u \vee B_{v'} \subsetneq A_u \vee B_v$. The latter case contradicts our assumption that $A_u \vee B_v$ is a least upper bound just prior to step k. In the former case, we repeat the argument, replacing B_v by $B_{v'}$. Doing this repeatedly as long as we have $n < s$ yields new A_u and B_v which agree from their left endpoints up until some position m' where (without loss of generality) B_v has a letter accomplishing the same reflection that x_{i_s} gives in A_u.

When we get to a step having $n = s$ and $m = m'$, we skip this step. Now proceed to steps with $m > m'$, restricting attention now to $A_u|_{x_{i_{m'+1}}\cdots x_{i_{s-1}}}$ and $B_v|_{x_{i_{m'+1}}\cdots x_{i_{s-1}}}$. We can do this restriction because $\{x_{i_{m'}}, x_{i_s}\}$ must comprise a deletion pair for $B_v \vee \{x_{i_s}\} = A_u \vee \{x_{i_{m'}}\}$, implying $w(A_u|_{x_{i_{m'+1}}\cdots x_{i_{s-1}}}) = w(B_v|_{x_{i_{m'+1}}\cdots x_{i_{s-1}}})$. This $m' < m < s$ case is handled similarly to our earlier $m < m'$ and $n < s$ case, i.e. proceeding left to right on $x_{i_{m'+1}} \cdots x_{i_{s-1}}$ eliminating discrepancies. In this manner, we obtain $A_{u'}$ and $B_{v'}$ which coincide except in positions m' and s, with $A_{u'} \sim_k A_u$, $B_{v'} \sim_k B_v$ and $A_{u'} \vee B_{v'} = A_u \vee B_v$. The fact that the wordlength of $A_{u'} \vee B_{v'}|_{x_{i_1}\cdots x_{i_s}}$ is exactly one more that the Coxeter-theoretic length with x_{i_s} redundant implies that $(A_{u'} \vee B_{v'})|_{x_{i_1}\cdots x_{i_{s-1}}}$ is reduced with $w((A \vee B)|_{x_{i_1}\cdots x_{i_{s-1}}}) = w((A_{u'} \vee B_{v'})|_{x_{i_1}\cdots x_{i_{s-1}}})$. This ensures that $(A_{u'} \vee B_{v'})|_{x_{i_1}\cdots x_{i_{s-1}}}$ is identified with $(A \vee B)|_{x_{i_1}\cdots x_{i_{s-1}}}$ before step k, by Remark 3.39. The fact that collapses are chosen to maximize dimension among those with the same deletion pair and number of long braid moves implies then that $A \vee B$ will have likewise been identified with $A_{u'} \vee B_{v'}$ before the k-th collapse, giving the desired result. $\square$

We have now assembled the ingredients to prove:

Theorem 3.46. $(\mathbb{R}^d_{\geq 0} \cap S_1^{d-1})/\sim$ *is a regular CW complex homeomorphic to a ball. Moreover, the link of any cell is itself a regular CW complex homeomorphic to a ball.*

Proof. This proof is structured similarly to that of Theorem 3.29, with added care needed in handling changes of coordinates. We begin with $\mathbb{R}^d_{\geq 0} \cap S_1^{d-1}$ which is a simplex and serves as K_0. We use Lemma 5.21 to perform collapses across level curves, thereby eliminating all regions given by non-reduced subwords of $(i_1, \dots, i_d)$, using the collapsing order given just after Example 3.32. Each collapsing step is specified by a deletion pair $\{x_{i_r}, x_{i_s}\}$ in an x-expression $x(F_i)$, with $\overline{F_i}$ collapsed across level curves each having one endpoint in the closed face $\overline{G_1}$ which consists of the points of $\overline{F_i}$ with $t_{i_r} = 0$ and the other endpoint in the closed face $\overline{G_2}$ instead having $t_{i_s} = 0$.

Let us check that the hypotheses of Lemma 5.21 are met, implying that each collapse preserves regularity and homeomorphism type. Lemmas 3.45 and 3.44 confirm Conditions 5.15 and 5.20. By Lemma 3.40, each long braid move to be used is given by a change of coordinates homeomorphism ch on $\overline{F_i}$ sending our level curves to ones which are the image under $g_{j-1} \circ \cdots g_1$, composing right to left, of a family of parallel line segments, meaning that they are the image under $ch^{-1} \circ g_{j-1} \cdots \circ g_1$ of a family of parallel line segments. Lemma 3.27 gives that distinct level curves have distinct endpoints in G_1. Thus, the level curves meet all the requirements to be a parallel-like family of curves.

While it is important for each point $x \in \overline{F_i}$ that $f_{(i_1,\dots,i_d)}(x) = f_{(j_1,\dots,j_d)}(ch(x))$, where $(j_1,\dots,j_d)$ is the word obtained from $(i_1,\dots,i_d)$ by the braid relation causing the change of coordinates ch, this property is not needed for points in $N \setminus \overline{F_i}$, since the collapse acts injectively outside of $\overline{F_i}$; thus, we may extend ch to a neighborhood N of $\overline{F_i}$ by thickening the boundary of $\overline{F_i}$ to $\partial(\overline{F_i}) \times [0,\epsilon)$ and letting $ch(x,t) = (ch(x),t)$ for each $t \in [0,\epsilon)$.

The last claim follows from Lemmas 5.23 and 3.43. ☐

The fact that collapsing maps are identification maps which only identify points belonging to the same fiber of $f_{(i_1,\dots,i_d)}$ yields:

Corollary 3.47. *The map* $f_{(i_1,\dots,i_d)}$ *induces a continuous, bijective function* $\overline{f_{(i_1,\dots,i_d)}}$ *from* $(\mathbb{R}^d_{\geq 0} \cap S_1^{d-1})/\sim$ *to* $Y_{s_{i_1}\cdots s_{i_d}}$.

4. 0-Hecke algebra lemmas

In Section 3.2, we introduced the notion of a deletion pair for a 0-Hecke algebra as Definition 3.9.

Lemma 4.1. *If* $\{x_{i_r}, x_{i_s}\}$ *comprise a deletion pair, then* $w(x_{i_r}\cdots x_{i_s}) = w(\hat{x}_{i_r}\cdots x_{i_s}) = w(x_{i_r}\cdots \hat{x}_{i_s})$*, but these do not equal* $w(\hat{x}_{i_r}\cdots \hat{x}_{i_s})$.

Proof. $w(x_{i_r}\cdots \hat{x}_{i_s}) \leq w(x_{i_r}\cdots x_{i_s})$ and $w(\hat{x}_{i_r}\cdots x_{i_s}) \leq w(x_{i_r}\cdots x_{i_s})$ in Bruhat order, while all three of these Coxeter group elements have the same length, so the equalities follow. The inequality is immediate from the fact that $x_{i_r}\cdots \hat{x}_{i_s}$ is reduced. ☐

Lemma 4.2. *If an expression* $x_{i_1}\cdots x_{i_d}$ *has deletion pairs* $\{x_{i_r}, x_{i_s}\}$ *and* $\{x_{i_s}, x_{i_t}\}$ *for* $r < s < t$*, then* $x_{i_r}\cdots \hat{x}_{i_s}\cdots x_{i_t}$ *is not reduced.*

Proof. Since $x_{i_r}\cdots \hat{x}_{i_s}$ and $\hat{x}_{i_r}\cdots x_{i_s}$ are reduced expressions for the same Coxeter group element, we may apply braid relations to the former to obtain the latter. Likewise, we may apply braid relations to $\hat{x}_{i_s}\cdots x_{i_t}$ to obtain $x_{i_s}\cdots \hat{x}_{i_t}$. Applying these same braid relations to the first and second parts of $x_{i_r}\cdots \hat{x}_{i_s}\cdots x_{i_t}$ yields $\hat{x}_{i_r}\cdots x_{i_s}x_{i_s}\cdots \hat{x}_{i_t}$, implying $x_{i_r}\cdots \hat{x}_{i_s}\cdots x_{i_t}$ is not reduced. ☐

See [11] for a faithful representation of the 0-Hecke algebra in which the simple reflections which do not increase length act by doing nothing. In some sense, this idea translates to our set-up as follows.

Lemma 4.3. *If* $x_{i_r}\cdots \hat{x}_{i_u}\cdots x_{i_s}$ *is reduced but* $x_{i_r}\cdots x_{i_s}$ *is not, then* x_{i_u} *belongs to a deletion pair within* $x_{i_r}\cdots x_{i_s}$.

Proof. Without loss of generality, assume $\{x_{i_r}, x_{i_s}\}$ is a deletion pair in $x_{i_r} \cdots x_{i_s}$; we are assured there is no deletion pair strictly to the right or strictly to the left of x_{i_u}, since $x_{i_r} \cdots \hat{x}_{i_u} \cdots x_{i_s}$ is reduced. There must be a series of braid moves transforming $x_{i_r} \cdots \hat{x}_{i_s}$ into $\hat{x}_{i_r} \cdots x_{i_s}$. We may apply this same series of moves to $x_{i_r} \cdots \hat{x}_{i_u} \cdots \hat{x}_{i_s}$, provided that x_{i_u} never appears in an interior position of a braid move, by saying that the braid move sends a word beginning with $\hat{x}_{i_u}$ to one instead ending with some $\hat{x}_j$ or vice versa. The fact that $x_{i_r} \cdots \hat{x}_{i_u} \cdots x_{i_s}$ is reduced ensures that $\hat{x}_{i_u}$ will never appear at an interior position of a braid move, because the first instance of this would imply the existence of a deletion pair in $x_{i_r} \cdots \hat{x}_{i_u} \cdots x_{i_s}$. At the end of the series of braid moves, we will have a stuttering pair; the only way for $x_{i_r} \cdots \hat{x}_{i_u} \cdots x_{i_s}$ not to lead to the same stuttering pair is for the stuttering pair to involve x_{i_u}, hence for x_{i_u} to be part of a deletion pair. $\square$

Corollary 4.4. *If $\{x_{i_r}, x_{i_s}\}$ is a deletion pair in $x_{i_1} \cdots x_{i_r} \cdots x_{i_s} \cdots x_{i_d}$, then any $x_{i_r} \cdots \hat{x}_{i_u} \cdots x_{i_s}$ obtained by deleting one letter from $x_{i_r} \cdots x_{i_s}$ for $u \neq r, s$ is not reduced.*

Remark 4.5. If $\{x_{i_u}, x_{i_v}\}$ comprise a deletion pair in a word $x_{i_1} \cdots x_{i_d}$ and we apply a braid relation in which x_{i_u} is the farthest letter from x_{i_v} in the segment being braided, then the resulting expression will have as a deletion pair x_{i_v} together with the nearest letter to it in the segment that was braided. We regard this as a braided version of the same deletion pair. For example, applying a braid relation to $x_1x_2x_1x_2$ yields $x_2x_1x_2x_2$; we regard the third and fourth letter in the new expression as a braided version of the deletion pair comprised of the first and fourth letters in the original expression.

Remark 4.6. One might ask what happens to a deletion pair upon application of a braid move having a letter of the deletion pair at a non-extreme position. We set up our collapsing order so that we never use any such braid relations, hence need not consider this question.

Given a reduced expression $x_{i_1} \cdots x_{i_d}$, associate a Coxeter generator, denoted $R(x_{i_j})$ to each x_{i_j} by letting $R(x_{i_j}) = s_{i_1} \cdots s_{i_{j-1}} s_{i_j} s_{i_{j-1}} \cdots s_{i_1}$. In a nonreduced expression, if $w(x_{i_1} \cdots x_{i_j}) = w(x_{i_1} \cdots x_{i_{j-1}})$, then we find the largest $j' < j$ such that $s_{i_{j'}} \cdots s_{i_{j-1}} = s_{i_{j'+1}} \cdots s_{i_j}$ and let $R(x_{i_j}) = R(x_{i_{j'}})$.

Sergey Fomin helped us generalize our proof of the following lemma from reflection groups to Coxeter groups, enabling us to avoid assuming that our Coxeter groups are finite.

Lemma 4.7. *Given a reduced expression* $x_{i_1} \cdots x_{i_m}$ *in the 0-Hecke algebra of a Coxeter group in which* $R(x_{i_m}) = s_{i_0}$*, then* $x_{i_0} x_{i_1} \cdots x_{i_m}$ *has* $\{x_{i_0}, x_{i_m}\}$ *as a deletion pair.*

Proof. Let us show that $x_{i_0} x_{i_1} \cdots x_{i_{m-1}}$ is reduced whereas $x_{i_0} x_{i_1} \cdots x_{i_m}$ is not. By exercise 8 in [BB, Chapter 1], it suffices to prove $R(x_{i_j}) \neq R(x_{i_k})$ for all $0 \leq j < k \leq m-1$ along with proving $R(x_{i_0}) = R(x_{i_m})$. But $x_{i_1} \cdots x_{i_{m-1}}$ is reduced, which implies $s_{i_1} \cdots s_{i_{j-1}} s_{i_j} s_{i_{j-1}} \cdots s_{i_1} \neq s_{i_1} \cdots s_{i_{k-1}} s_{i_k} s_{i_{k-1}} \cdots s_{i_1}$ for $1 \leq j < k \leq m-1$. This in turn implies $R(x_{i_j}) \neq R(x_{i_k})$ for $1 \leq j < k \leq m-1$ with respect to the expression $x_{i_0} x_{i_1} \cdots x_{i_m}$, since we simply conjugate the previous inequalities by s_{i_0} to obtain the desired inequalities. On the other hand, $R(x_{i_0}) = s_{i_0} = s_{i_0}^3 = s_{i_0}(s_{i_1} \cdots s_{i_m} \cdots s_{i_1}) s_{i_0}$, completing the proof. □

5. Combinatorial-topological collapsing lemmas

In this section, we prove that certain types of collapses performed sequentially on a convex polytope preserve homeomorphism type as well as the property of being a finite regular CW complex (though do not preserve polytopality). To this end, we define these collapsing maps rather explicitly topologically in Lemma 5.21. In preparation for this, we first introduce some helpful properties a topological space or a map may have, with the common feature that these properties may be checked using primarily combinatorics. We adopt conventions and terminology from [Br]. Let us begin with a simple example that conveys something of the flavor of our collapses.

Example 5.1. Let Δ_2 be the convex hull of $(0,0), (1,0), (0,1/2)$ in $\mathbb{R}^2$, and let Δ_1 be the convex hull of $(0,0)$ and $(1,0)$ in $\mathbb{R}^2$. We will construct a surjective, continuous function $h : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ that acts homeomorphically on $\mathbb{R}^2 - \Delta_2$ sending it to $\mathbb{R}^2 - \Delta_1$. The idea is to map parallel line segments covering Δ_2 to individual points covering Δ_1, then take a neighborhood N of Δ_2 and have h stretch it to cover Δ_2. For $0 \leq x \leq 1$ and $0 \leq y \leq -x/2 + 1/2$, let $h(x,y) = (x,0)$; this maps Δ_2 onto Δ_1. For $0 \leq x \leq 1$ and $-x/2 + 1/2 \leq y \leq 1$, let $h(x,y) = (x, \frac{x - 1/2 + x/2}{1 - 1/2 + x/2} + (1/2 - x/2)$. For $-1 \leq x \leq 0$ and $0 \leq y \leq x/2 + 1$, let $h(x,y) = (x, y\frac{-x/2}{x/2+1})$. For $-1 \leq x \leq 0$ and $x/2 \leq y \leq 1$, let $h(x,y) = (x, (y - (x/2+1))\frac{1 + x/2}{1 - (x/2+1)} + x/2 + 1)$. Let h act as the identity outside $R = \{(x,y) : -1 \leq x \leq 1, 0 \leq y \leq 1\}$.

Remark 5.2. See p. 42-43 in [35] for a closely related, though fundamentally different homeomorphism given by explicit maps.

In what follows, we will typically have a topological space X endowed with the structure of a finite, regular CW complex K. We denote this by X_K and call X_K a CW space. We will speak of a cell σ being contained in $\overline{\tau}$ (resp. $\partial\tau$) in K, by which we mean $\sigma \leq \tau$ (resp. $\sigma < \tau$) in the closure poset for K. Let $\partial\tau$ denote $\overline{\tau} \setminus \tau$, i.e. the boundary of $\overline{\tau}$.

Definition 5.3. Given a continuous, surjective function $g : X \to Y$, define the *quotient topology* on Y as the topology whose open sets are the sets whose inverse images are open in X. Say that g is an *identification map* if the topology on Y is the quotient topology induced by g; this is automatic for X, Y compact and Hausdorff.

Thus, for a pair of spaces X, Y endowed with topologies, to prove that a map $g : X \to Y$ is an identification map, we must prove that g is continuous and surjective, and also that all open sets of Y arise as those given by the quotient topology.

Remark 5.4. Given an identification map $g : X \to Y$ and a function $f : X \to Z$ such that $g(x) = g(y)$ implies $f(x) = f(y)$, then Proposition 13.5 of [Br] implies that f is continuous iff the induced function $\overline{f} : Y \to Z$ satisfying $f = \overline{f} \circ g$ is continuous. In our application, this allows us to deduce from continuity of $f_{(i_1,\dots,i_d)}$ continuity of the induced map $\overline{f_{(i_1,\dots,i_d)}}$ on the quotient space after a series of collapses each of which is given by an identification map.

Denote by $K/(\ker g)$ the quotient space obtained from an identification map g on a topological space K by setting $x \sim y$ iff $g(x) = g(y)$. Let us also establish the following convenient general notion.

Definition 5.5. Given a finite regular CW complex K on a set X and an open cell L in K, define a *face collapse* of $\overline{L}$ onto $\overline{\tau}$ for τ an open cell contained in ∂L to be an identification map $g : X \to X$ such that:

1. Each open cell of $\overline{L}$ is mapped surjectively onto an open cell of $\overline{\tau}$ with L mapped onto τ
2. g restricts to a homeomorphism from $K \setminus \overline{L}$ to $K \setminus \overline{\tau}$ and acts homeomorphically on $\overline{\tau}$.
3. The images under g of the cells of K form a regular CW complex with new characteristic maps obtained by composing the original characteristic maps of K with g^{-1} for those cells of K contained in $\overline{\tau}$ or $(K \setminus \overline{L})$.

We call such a map g a *collapsing map.*

Remark 5.6. The map $\overline{g} : X_K/(\ker g) \to X$ induced by a collapsing map g is continuous (cf. Remark 5.4) as well as obviously bijective.

Since K is Hausdorff and $K/(\ker g)$ is compact, we may conclude that $\overline{g}$ is a homeomorphism. Thus, face collapses preserve homeomorphism type.

Remark 5.7. If g is a collapsing map on X_K and h is a homeomorphism from the underlying space X to itself, then $g \circ h$ is also a collapsing map. This is important in our application when we do changes of coordinates to accomplish long braid moves.

Lemma 5.8. *Suppose K_1 and K_2 are topological spaces, f is a homeomorphism from K_1 to K_2, $\pi_1 : K_1 \to K_1$ and $\pi_2 : K_2 \to K_2$ are identification maps giving rise to quotient spaces $K_1/\sim$ and $K_2/\sim'$. If we also have $x \sim y$ in K_1 iff $f(x) \sim' f(y)$ in K_2, then $K_1/\sim$ is homeomorphic to $K_2/\sim'$ under the induced map $\overline{f}$.*

Proof. This follows easily from Proposition 13.5 in Ch. 1 of [7] by constructing a suitable commutative diagram. Simply note that $\overline{f}$ is continuous if and only if $\pi_1 \circ \overline{f}$ is, but $\pi_1 \circ \overline{f} = f \circ \pi_2$ yields continuity of the latter. A similar argument shows $\overline{f}^{-1}$ is continuous. Bijectivity is immediate from our set-up. ☐

Next we review the notion of a collaring. This will be crucial to defining our collapses in Lemma 5.21.

Definition 5.9. A *topological n-manifold* is a Hausdorff space M having a countable basis of open sets, with the property that every point of M has a neighborhood homeomorphic to an open subset of $\mathbb{H}^n$, where $\mathbb{H}^n$ is the half-space of points $(x_1, \dots, x_n)$ in $\mathbb{R}^n$ with $x_n \geq 0$. The *boundary* of M, denoted ∂M, is the set of points $x \in M$ for which there exists a homeomorphism of some neighborhood of x onto an open set in $\mathbb{H}^n$ taking x into $\{(x_1, \dots, x_n) | x_n = 0\} = \partial \mathbb{H}^n$.

Definition 5.10. Given a topological manifold M with boundary, a *collar* or *collaring* for M is a closed neighborhood N of ∂M contained in M that is homeomorphic to $\partial M \times [0, 1]$.

A proof of the following may be found in Appendix II of [37].

Theorem 5.11. *If M is a compact, topological manifold with boundary ∂M, then there exists a neighborhood W of ∂M in M such that W is homeomorphic to $\partial M \times [0, 1]$ in such a way that ∂M corresponds naturally to $\partial M \times 0$.*

Definition 5.12. Given a continuous, surjective function $g_{i+1} : X \to X$, define an *interpolating family* of maps $\{g_{i+1,t} | t \in [0, 1]\}$ from X to X as a collection of maps with $g_{i+1,0} = g_{i+1}$ and $g_{i+1,1} = id$, requiring for each $t \in (0,1]$ that $g_{i+1,t}$ be a homeomorphism from X to X and that the map $h_{i+1} : X \times [0,1] \to X$ defined by $h_{i+1}(x,t) = g_{i+1,t}(x)$ be continuous.

Lemma 5.13. *If a collapsing map g_{i+1} on a regular CW space X_K homeomorphic to a sphere gives rise to an interpolating family of maps $\{g_{i+1,t} | t \in [0,1]\}$ on the topological space X_K, then g_{i+1} extends to a collapsing map on any ball B which is a compact manifold with boundary having X_K as its boundary. Thus, for such a B which is also a regular CW complex, $g_{i+1}(B)$ is a regular CW complex homeomorphic to a ball.*

Proof. Choose a collar $K \times [0,1]$ for B. This exists by Theorem 5.11. For each $(x,t) \in K \times [0,1]$, let $g_{i+1}(x,t) = g_{i+1,t}(x)$. Let g_{i+1} act as the identity map on $B \setminus (K \times [0,1])$. □

The next condition is designed to enable extension via collars of a collapsing map from a low dimensional subcomplex within which a low-dimensional cell is more easily collapsed to a full complex. See Lemma 5.25 for this extension process.

Condition 5.14. A finite regular CW complex K has the *inductive manifold condition* if for each open d-cell τ and each $(d+1)$-cell σ such that $\tau \subseteq \partial\sigma$, $\overline{\partial\sigma \setminus \tau}$ is a compact manifold with boundary.

Now to a purely combinatorial condition which will be quite helpful for proving that certain types of identification maps preserve regularity.

Condition 5.15. Given an identification map g on a regular CW space X_K, suppose that g maps cells onto cells, maps an open cell F onto one of its boundary cells, and acts homeomorphically on $X_K \setminus \overline{F}$. Then we say that g satisfies the *least upper bound condition* if for any pair of open cells $A, B \subseteq \overline{F}$ such that $g(A) = g(B)$ and any face F' that is a least upper bound for A and B in the closure poset just prior to the application of g, we have that F' is also mapped onto one of its boundary cells by g.

Remark 5.16. In our collapsing construction, there will always be one cell being collapsed such that all others being collapsed are in its closure. In this setting, the least upper bound condition will imply for any pair of cells F, F' which are both least upper bounds for the same pair of cells that if F is collapsed in a particular step, then so is F' as well as a larger cell having both F and F' in its closure.

Remark 5.17. Given a collection of parallel line segments $\mathcal{C}$ covering a face F of a polytope P, define a length function $len : F \to \mathbb{R}$ by letting *len*(x) be the length of the element of $\mathcal{C}$ containing x. Convexity of P implies *len* is continuous. Now define a parametrization function $p : F \to [0,1]$ by letting the restriction of p to any $c \in \mathcal{C}$ be the linear function from c to $[0,1]$. Continuity of *len* implies that p is also continuous. Moreover, if h is any homeomorphism from $int(F)$ to another topological space, then $p \circ h^{-1}$ is also continuous.

This will allow us to perform collapses across so-called parallel-like curves, as defined next, according to a continuous function that stretches a collar to cover a face being collapsed.

Definition 5.18. Let K_0 be a convex polytope, and let $\mathcal{C}_i^0$ be a family of parallel line segments covering a closed face L_i^0 in ∂K_0 with the elements of $\mathcal{C}_i^0$ given by linear functions $c : [0,1] \to L_i^0$. Suppose that there is a pair of closed faces G_1, G_2 in ∂L_i^0 with $c(0) \in G_1$ and $c(1) \in G_2$ for each $c \in \mathcal{C}_i^0$ and there is a composition $g_i \circ \cdots \circ g_1$ of face collapses on K_0 such that:

1. $g_i \circ \cdots \circ g_1$ acts homeomorphically on $int(L_i^0)$.
2. For each $c \in \mathcal{C}_i^0$, $g_i \circ \cdots \circ g_1$ either sends c to a single point or acts homeomorphically on c.
3. Suppose $g_i \circ \cdots \circ g_1(c(t)) = g_i \circ \cdots \circ g_1(c'(t'))$ for $c \neq c' \in \mathcal{C}_i^0$ and $t \neq 1$ or $t' \neq 1$. Then $t = t'$ and $g_i \circ \cdots \circ g_1(c) = g_i \circ \cdots \circ g_1(c')$, where c, c' are parametrized by a parametrization function p as defined in Remark 5.17.

Then call $\mathcal{C}_i = \{g_i \circ \cdots \circ g_1(c) | c \in \mathcal{C}_i^0\}$ a family of *parallel-like* curves on the closed cell $L_i = g_i \circ \cdots \circ g_1(L_i^0)$ of the finite regular CW complex $K_i = g_i \circ \cdots \circ g_1(K_0)$.

Notice that (3) in Definition 5.18 implies the curves are nonoverlapping except perhaps in $g_i \circ \cdots \circ g_1(G_2)$.

Remark 5.19. In our main application, i.e. the application given in Section 3, (3) follows easily from the manner in which we use deletion pairs. In practice, we will verify (2) by verifying Condition 5.20 below; this suffices because by definition $g_i \circ \cdots \circ g_1$ must act homeomorphically on each open cell not collapsed by any g_j for $j \leq i$.

We say that a curve is *nontrivial* if it includes more than one point.

Condition 5.20. A collection $\mathcal{C}$ of curves covering the closure of a cell F satisfies the *distinct endpoints condition* if for any nontrivial curve $c \in \mathcal{C}$, the two endpoints of c live in distinct cells in $\overline{F}$.

Now we are ready to prove our main topological collapsing lemma. The point of its inductive statement is to allow a series of collapses by showing that after each collapse the properties are preserved that are needed to apply the lemma again, i.e. to perform another collapse. The proof is largely devoted to defining explicitly a suitable continuous, surjective map based on a collection of parallel-like curves covering a cell to be collapsed.

Lemma 5.21. *Let K_0 be a convex polytope. Let $g_1, \dots, g_i$ be collapsing maps with $g_j : X_{K_{j-1}} \to X_{K_j}$ for regular CW complexes $K_0, \dots, K_i$ all having underlying space X. Suppose K_i satisfies Condition 5.14, the inductive manifold condition, and that there is an open cell L_i^0 in ∂K_0 upon which $g_i \circ \cdots \circ g_1$ acts homeomorphically and a collection $\mathcal{C} = \{g_i \circ \cdots \circ g_1(c) | c \in \mathcal{C}_i^0\}$ of parallel-like curves covering $\overline{L_i}$ for $L_i = g_i \circ \cdots \circ g_1(L_i^0) \in K_i$. Then there is an identification map $g_{i+1} : X_{K_i} \to X_{K_{i+1}}$ specified by $\mathcal{C}$; moreover, if this map g_{i+1} satisfies Condition 5.15, i.e. the least upper bound condition, then g_{i+1} is a collapsing map and K_{i+1} is a regular CW complex also satisfying Condition 5.14.*

Proof. We may assume L_i is top dimensional in ∂K_i in our argument, because otherwise Lemma 5.25 will enable us to extend g_{i+1} from a subpolytope having this property to all of K_0. More specifically, choose a cell L' in K_i with $\dim L' = \dim L_i + 1$ and $L_i \subseteq \partial L'$. We will define the collapsing map on $\partial L'$, then use Lemma 5.25 to extend g_{i+1} to the entire complex. In fact, we focus on defining g_{i+1} on $\partial L'$ in such a way that Lemma 5.13 enables its extension to $\overline{L'}$.

The idea will be to construct a continuous function g_{i+1} that sends entire curves in $\mathcal{C}$ to points in G_2 (cf. Definition 5.18), thereby collapsing $\overline{L_i}$ onto G_2, and in the process identifying each point of G_1 with a point of G_2. In preparation to do this, we first extend each nontrivial curve in $\mathcal{C}$ not contained in ∂L_i to obtain a curve in a new (larger) collection $\mathcal{C}'$ of curves covering a closed neighborhood N_i of $\overline{L_i}$; N_i is obtained by taking the union of L_i with a collar for $\overline{\partial L' \setminus L_i}$. Such a collar exists since K_i meets Condition 5.14, enabling the use of Theorem 5.11. We will stretch the portion of each extended curve living outside $\overline{L_i}$, i.e. living in the collar, via g_{i+1} to cover the portion of that curve contained in $\overline{L_i} \setminus G_2$. For each curve $c \in \mathcal{C}$ having $c \subseteq \partial L_i$, we will create a family of curves in $\mathcal{C}'$ to cover the portion of a collar for $\overline{\partial L' \setminus L_i}$ given by $c \times [0, 1]$. The action of g_{i+1} on this family of curves is a bit more involved, designed to interpolate from a collapse at the inside of the collar to the identity function at the outside of the collar. Thus, we will map $\overline{L_i}$ onto G_2 and stretch the collar for $\overline{\partial L' \setminus L_i}$ to $\overline{L_i} \setminus G_2$, doing this in such a way that g_{i+1} acts as the identity at the boundary of N_i as well as outside N_i.

Let us now describe precisely how we obtain $\mathcal{C}'$ from $\mathcal{C}$. First consider any $c \in \mathcal{C}$ with $c \cap \partial(L_i) = \{x, y\}$ for points $x \in \overline{G_1}$ and $y \in \overline{G_2}$. Extend c to include all points (y, t) and (x, t) for $t \in [0, 1]$. Since $c_1 \neq c_2$ for $c_1, c_2 \in \mathcal{C}$ implies that c_1 and c_2 have distinct endpoints in G_1 by part 4 of Definition 5.18, the curve extensions $\{(x, t)|t \in [0, 1]\}$ given by the various points $x \in G_1$ are nonoverlapping. It will not matter if distinct $c_1, c_2 \in \mathcal{C}$ have the same endpoint $y \in G_2$, because g_{i+1} will act as the identity map on $G_2 \times [0, 1]$. In this situation, let $y \times [0, 1]$ belong to the extensions of both c_1 and c_2. Definition 5.18, part 2, guarantees $x \neq y$ for each nontrivial $c \in \mathcal{C}$. For each $c \in \mathcal{C}$ where c is a single point in $\partial(L_i)$, extend to $c' = \{(c, t)|t \in [0, 1]\}$, and let g_{i+1} act as the identity on c'. For each nontrivial curve c with $c \subseteq \partial(L_i)$, we create a family F_c of curves in N_i so that F_c covers exactly $\{c\} \times [0, 1] = \{(x, t)|x \in c; t \in [0, 1]\}$. We make one such curve $c_t \in F_c$ for each $t \in [0, 1]$, doing this in such a way that we have $c \subseteq c_0$. To this end, parametrize c as $\{c(t)|t \in [0, 1]\}$ with $c(0) \in G_1$ and $c(1) \in G_2$. Now for each $t \in [0, 1]$, let $c_t = \{(c(t/2), t')|t' \geq t\} \cup \{(c(t''), t)|t'' \geq t/2\}$. $\mathcal{C}'$ is comprised exactly of the union of these families of curves, with one such family F_c for each nontrivial curve $c \subseteq \partial L_i$, along with the extended curves resulting from the elements of $\mathcal{C}$ which are trivial or only intersect $\partial(L_i)$ in their endpoints.

Now we define $g_{i+1} : K_i \to K_i$ by specifying how it maps each $c' \in \mathcal{C}'$ surjectively onto itself. First consider any $c' \in \mathcal{C}'$ obtained by extending some nontrivial $c \in \mathcal{C}$. Represent the points of c' as $\{c(t)|t \in [-1, 2]\}$, where $[-1, 0]$ gives $c' \cap (G_1 \times [0, 1])$, i.e. the part of the collar sitting over the endpoint of c' in G_1, whereas $[0, 1]$ specifies the points in $c' \cap L_i$ by our usual parametrization, and $[1, 2]$ gives the points in $c' \cap (G_2 \times [0, 1])$. Now let $g_{i+1}(c(t)) = c(1) \in \overline{G_2}$ for $t \in [0, 1]$, let $g_{i+1}(c(t)) = c(2t + 1)$ for $t \in [-1, 0]$, and let $g_{i+1}(c(t)) = c(t)$ for $t \in [1, 2]$.

Next consider any family F_c of elements of $\mathcal{C}'$ covering $c \times [0, 1]$ for some $c \in \mathcal{C}$ with $c \subseteq \partial(L_i)$. Then g_{i+1} sends $\{(c(t'), t)|t/2 \leq t' \leq 1\}$ to $\{(c(t'), t)|1 - t/2 \leq t' \leq 1\}$, and g_{i+1} stretches the remainder of c_t by reparametrization by a suitable linear scaling factor so as to cover both itself and also the segment $\{(c(t'), t)|t/2 \leq t' \leq 1 - t/2\}$. More specifically, this stretching is done by sending $\{(c(t/2), t')|\frac{1+t}{2} \leq t' \leq 1\}$ to $\{(c(t/2), t')|t \leq t' \leq 1\}$ and sending $\{(c(t/2), t')|t \leq t' \leq \frac{1+t}{2}\}$ to $\{(c(t'), t)|t/2 \leq t' \leq 1 - t/2\}$.

Note that g_{i+1} acts as the identity on $\partial(N_i)$ and acts injectively on $N_i \setminus \overline{L_i}$. We choose the parametrizations of various curves in $\mathcal{C}'$ in a way that will make g_{i+1} continuous by construction; specifically, we use that the relative interiors of the curves in $\mathcal{C}$ have preimages that are a family of parallel line segments covering a convex region (a face of a polytope), enabling us to choose parametrizations there which induce suitable ones for $\mathcal{C}$; the point is that the collection of parametrizations on a family of parallel line segments covering a simplex L_0^i in our original PL ball may collectively be regarded as a function from L_0^i to $[0,1]$, so all we need to do is to make the parametrizations compatible in the sense of requiring this function to by continuous. This can be done by Remark 5.17. From such parametrizations, we obtain suitable parametrizations for $\mathcal{C}'$ using that the collar we use is homeomorphic to $\partial(L_i) \times [0,1]$.

Surjectivity and continuity of g_{i+1} imply it induces a continuous, bijective function $\overline{g_{i+1}}$ from $K_i/(\ker g_{i+1})$ to K_{i+1}. Continuity of $(\overline{g_{i+1}})^{-1}$ is then immediate, because any bijective, continuous function from a compact set to a Hausdorff space has continuous inverse. Thus, $\overline{g_{i+1}}$ is a homeomorphism, implying $g_{i+1}(K_i)$ is homeomorphic to K_i.

Now let us check that regularity is preserved under g_{i+1}. By Condition 5.15, notice for any face G not collapsed by $g_{i+1} \circ \cdots \circ g_1$ that any faces $\sigma_1, \sigma_2 \subseteq \partial(G)$ identified by g_{i+1} must have some least upper bound $A \subseteq \partial(G)$ which is also collapsed by g_{i+1}. Thus, our homeomorphism $\overline{g_{i+1}}$ will restrict to $\overline{G}/(\ker g_{i+1})$, enabling us to define the attaching map for G by composing the one for $K_i|_{\overline{G}}$ with $\overline{g_{i+1}}$ with $g_{i+1}^{-1}|_{\overline{G}_{final}}$ where $\overline{G}_{final}$ is the set of cells mapped homeomorphically to themselves by g_{i+1} or in other words the cells the various fibers of g_{i+1} are mapped onto. Lemmas 5.13 and 5.24 show that this may be extended to yield a characteristic map for all of $\overline{G}$, making it a regular CW complex homeomorphic to a ball.

Lemma 5.22 will verify Condition 5.14, i.e., the inductive manifold condition, for K_{i+1}. ☐

Lemma 5.22. *Collapses as in Lemma 5.21 preserve Condition 5.14, i.e. the inductive manifold condition.*

Proof. Consider any pair of cells $\tau \subseteq \overline{\sigma}$ in K_{i+1} with $\dim \sigma = \dim \tau + 1$. We must show that $\overline{\partial\sigma \setminus \tau}$ is a compact manifold with boundary. By our definition of collapsing map, there must be cells τ_i, σ_i in K_i with g_{i+1} mapping τ_i homeomorphically to τ and σ_i homeomorphically to σ with $\tau_i \subseteq \overline{\sigma_i}$. The proof of Lemma 5.21 shows that $\overline{g_{i+1}}$ is a homeomorphism from $\overline{\sigma_i}/(\ker g_{i+1})$ to $\overline{\sigma}$. In particular, this implies that $\overline{g_{i+1}}$ gives a bijection from $(\overline{\partial\sigma_i \setminus \tau_i})/(\ker g_{i+1})$ to $\overline{\partial\sigma \setminus \tau}$. By definition, $\overline{g_{i+1}}$ is continuous and $\ker(\overline{g_{i+1}}) = \{(x,x') | \overline{g_{i+1}}(x) = \overline{g_{i+1}}(x')\} \subseteq K_i \times K_i$ is closed, implying that $\overline{g_{i+1}}((\overline{\partial\sigma \setminus \tau})/(\ker g_{i+1}))$ is compact and Hausdorff. Since $\overline{g_{i+1}}$ is a continuous, bijective map from a compact set to a

Hausdorff space, $\overline{g_{i+1}}$ also has continuous inverse. Hence, the property of being a compact manifold with boundary transfers as desired. $\square$

Our construction also yields:

Proposition 5.23. *Collapses as in Lemma 5.21 preserve the property that the link of any cell (other than the big cell) is itself a closed ball with induced cell decomposition making it a regular CW complex, provided that the cells* G_1 *and* G_2 *such that each of the parallel-like curves has one endpoint in* $\overline{G_1}$ *and the other in* $\overline{G_2}$ *satisfy* $\dim G_1 = \dim G_2 = \dim L_i - 1$.

Proof. For most cells this is immediate from the fact that g_{i+1} acts homeomorphically except on the closure of the maximal cell F_j it is collapsing. Let $\overline{G_2}$ be the closed cell onto which g_{i+1} maps $\overline{F_j}$.

Now let us check the result for the link of G_2. The point is to check that the preimage (under g_{i+1}) of the link of $int(G_2)$ is a ball and that g_{i+1} preserves its homeomorphism type. To this end, first note that the preimage may be regarded as a union of links of the points in the preimage of G_2, where we define the link of a point x to be those $(x, t_1, \dots, t_r)$ in a series of collars over G_2 where $t_i \in [0,1]$ for each i with some t_i equalling 1. This is equivalent to intersecting the set of points that project to x in a higher dimensional closed cell (by the forgetful map sending $(x, t_1, \dots, t_r)$ to x) with a sphere of suitable dimension about x. Our use of parallel-like curves allows us to decompose the preimage of the link into three pieces that are each balls, namely points in a collar sitting over $\overline{G_1}, \overline{G_2}$ and $\overline{F_j} \setminus (\overline{G_1} \cup \overline{G_2})$, with the parallel-like curves indicating how these three pieces are glued together in a natural way that makes the union also a ball. This would be much more difficult without our equidimensionality assumption for G_1 and G_2. Exactly the same approach applies for the link of any other cell also contained in $\overline{G_2}$ which likewise is not collapsed by g_{i+1} but has some cell of dimension higher than it collapsed onto it by g_{i+1}.

Finally, consider the link of any cell σ that is not collapsed by g_{i+1} but is in the closure of a cell that is collapsed by g_{i+1}. We have already handled all such σ except those covered by parallel-like curves of length 0. Then the point is that our homeomorphism from $X/(\ker g_{i+1})$ to X naturally restricts to a neighborhood of any point in σ by choosing the neighborhood to have boundary that is a union of points upon which g_{i+1} acts trivially together with entire curves from our family of parallel-like curves. $\square$

Lemma 5.24. *Suppose a collapsing map g collapses across a family of parallel-like curves $\mathcal{C}'$ where each $c \in \mathcal{C}'$ is sent to itself by a monotonically increasing, piecewise linear function $g : [0,1] \to [0,1]$. Then g gives rise to an interpolating family of maps.*

Proof. By definition, we must have $0 = a_1 < \cdots < a_k = 1$ and $0 = b_1 \leq \cdots \leq b_k = 1$ such that g maps $[a_i, a_{i+1}]$ to $[b_i, b_{i+1}]$ by a linear map for each $1 \leq i \leq k-1$. Then define g_t instead to map $[a_i, a_{i+1}]$ linearly to $[ta_i + (1-t)b_i, ta_{i+1} + (1-t)b_{i+1}]$. □

Lemma 5.25. *Let K be a regular CW complex having a unique maximal cell and satisfying Condition 5.14, i.e. the inductive manifold condition. Let τ be an i-cell in the boundary of an $(i+1)$-cell σ. Let g_{i+1} be a collapsing map on $\partial\sigma$ that collapses the cell τ. If g_{i+1} gives rise to an interpolating family of maps, then g_{i+1} may be extended to a collapsing map on K.*

Proof. First choose a series of cells $\tau = \sigma_1, \sigma = \sigma_2, \sigma_3, \ldots, \sigma_k$ such that $\sigma_j \subseteq \overline{\sigma_{j+1}}$ with $\dim \sigma_{j+1} = \dim \sigma_j + 1$ for each j, letting σ_k be the unique maximal cell of K. We are given g_{i+1} defined on $\partial\sigma = \partial\sigma_2$ and will now describe for each $2 \leq j \leq k-1$ how to extend g_{i+1} from $\partial\sigma_j$ to $\overline{\sigma_j}$ and then to $\partial\sigma_{j+1}$.

Theorem 5.11 ensures that $\overline{\sigma_j}$ has a collar. This enables us to extend g_{i+1} from $\partial\sigma_j$ to $\overline{\sigma_j}$ by Lemma 5.13. If σ_j is the big cell, we are done. Otherwise, choose σ_{j+1} with $\overline{\sigma_j} \subseteq \partial\sigma_{j+1}$ and take a collar for $\overline{\partial\sigma_{j+1} \setminus \overline{\sigma_j}}$ within $\partial\sigma_{j+1}$ (which exists by Condition 5.14 and Theorem 5.11). Use an interpolating family to extend g_{i+1} from $\overline{\sigma_j}$ to $\partial\sigma_{j+1}$, defining the interpolating family as follows.

We alternate between choosing collars for $\overline{\sigma_j}$ and collars for $\overline{\partial\sigma_{j+1} \setminus \sigma_j}$, with j increasing by one each time r increases by 2. Now for each r in turn, let g_{i+1} send $(x, t_1, t_2, \ldots, t_r)$ to $(g_{i+1,1-(1-t_1)\cdots(1-t_r)}(x), t_2, \ldots, t_r)$ for each $(t_1, \ldots, t_r) \in [0,1]^r$ and act as the identity on all points of $\overline{\sigma_r}$ (resp. $\partial\sigma_{r+1} \setminus \overline{\sigma_r}$) not in our collar for $\overline{\sigma_r}$ (resp. $\partial\sigma_{r+1} \setminus \overline{\sigma_r}$) as well as for points $(x,t) \in \partial\sigma_r \times [0,1]$ in either collar such that g_{i+1} acts as the identity on $x \in \partial\sigma_r$ (as determined at the previous step in our inductive construction by dimension). In this manner, we use a series of collars enabling us to extend g_{i+1} to higher and higher dimensional cells; the fact that $1 - (1-t_1)\cdots(1-t_r) = 1$ whenever any t_i is 1 ensures that we achieve the identity map once we reach the outer boundary of any one of these collars encountered in series. □

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