

RANDOM WALKS ON QUASISYMMETRIC FUNCTIONS

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ABSTRACT. Conditions are provided under which an endomorphism on quasisymmetric functions gives rise to a left random walk on the descent algebra which is also a lumping of a left random walk on permutations. Spectral results are also obtained. Several important random walks are now realized this way: Stanley's QS -distribution results from endomorphisms given by evaluation maps, a -shuffles result from the a -th convolution power of the universal character, and the Tchebyshev operator of the second kind introduced recently by Ehrenborg and Readdy yields traditional riffle shuffles. A conjecture of Ehrenborg regarding the spectra for a family of random walks on ab -words is proven. A theorem of Stembridge from the theory of enriched P -partitions is also recovered as a special case.

1. INTRODUCTION

Quasisymmetric functions have long been used for encoding and manipulating enumerative combinatorial data. They admit a natural graded Hopf algebra structure $\mathcal{Q} = \bigoplus_{n=0}^{\infty} \mathcal{Q}_n$ central to the study of combinatorial Hopf algebras [2]. Their dual relationship to Solomon's descent algebras [31, 20, 27], as well as to noncommutative symmetric functions [19], are an important part of the story and have inspired a broad literature.

A major goal of this paper is to identify and develop bridges between some of this literature and the body of work surrounding Bidigare, Hanlon, and Rockmore's far-reaching generalizations of Markov chains for various common shuffling and sorting schemes, the main references being [4, 5, 10, 11, 12]. Stanley first recognized and established a connection between quasisymmetric functions and this work in [34]. We deepen this connection and draw combinatorial Hopf algebras into the story by proving that endomorphisms on quasisymmetric functions which satisfy certain nonnegativity requirements always give rise to random walks both on permutations and also on the descent algebra. Stanley's QS -distribution may be regarded as the special case where the endomorphism corresponds to an evaluation map. Some consequences of this relationship are as follows: (1) whenever a random walk on permutations or the descent algebra arises this way, its transition matrix will be lower triangular with respect to the monomial basis of quasisymmetric functions; (2) these transition matrices often turn out to be diagonalizable with respect to bases that are quite natural from the viewpoint of quasisymmetric functions; (3) in some cases, one may directly transfer known spectral results from one setting to the other.

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Our original motivation was to understand the probabilistic behavior of an important endomorphism $\Theta : \mathcal{Q} \rightarrow \mathcal{Q}$ introduced by Stembridge [35] in his development of enriched P -partitions. Billera, Hsiao, and van Willigenburg observed [8] that Θ can be represented by a stochastic matrix (after normalization) with unique stationary distribution equal to the distribution of peak sets of random permutations. The associated random walk on peak sets was conjectured to specialize a family of random walks on permutations having uniform stationary distribution, a conjecture which turns out to be correct.

It was soon pointed out [2, 6] that the Θ -map is dual to a specialization at $q = -1$ of the $A \mapsto (1 - q)A$ transformation on noncommutative symmetric functions introduced by Kroh, Leclerc, and Thibon [24]. They develop a series of general results about these transformations that include a complete description of their spectral decompositions. They also describe these transformations as acting by multiplication on Solomon's descent algebra, and based on their description it is easy to resolve the conjecture made in [8]. The dual relationship between Θ and a multiplicative operator on the descent algebra turns out to be one instance of a much more general phenomenon, which we develop in Section 3.

Section 2 briefly gives background on quasisymmetric functions, the descent algebra, noncommutative symmetric functions and lumping of random walks. Section 3 draws together results about characters on combinatorial Hopf algebras and makes some new observations so as to characterize which endomorphisms on quasisymmetric functions give rise to Markov chains. In addition, a characterization of when the stationary distribution is unique (in which case it is uniform) is given. In Section 4, it is then shown how in this setting one may read off the eigenvalues, and in fact an eigenbasis is constructed by a recursive procedure. Section 5 describes the resulting transition matrices quite explicitly.

Turning now to applications, Section 6 expresses Stanley's QS -distribution as the special case of random walks driven by the endomorphism on \mathcal{Q} resulting from evaluating the quasisymmetric functions at a specified point (r_1, r_2, \dots) where each r_i is a scalar. Section 7 deals with the well-studied a -shuffles where a deck of cards is split into a (possibly empty) piles which are then shuffled; this random walk results from a very natural endomorphism on \mathcal{Q} , namely the a -th convolution power of the universal character. Connecting all this to the literature on enumeration in posets, we show that Ehrenborg and Readdy's Tchebyshev operator of the second kind on \mathcal{Q} encodes standard riffle shuffles [18], and in Section 8 we prove a conjecture of Ehrenborg regarding the spectra for a certain family of random walks on ab -words that arise from the r -Birkoff transform [17]. In fact, we explicitly describe the transition probabilities of these walks in a way that generalizes a theorem of Stembridge from [35].

For an overview of the literature on shuffling and related topics, the survey by Diaconis [14] is a very helpful resource. We also recommend [36] and [2, 3] for further background on Hopf algebras and on combinatorial Hopf algebras, respectively.

2. BACKGROUND

This section reviews background on quasisymmetric functions, noncommutative symmetric functions, and the descent algebra, including bases and products to be used in later sections,

as well as a key property of the resulting Markov chains called lumping. We work over the rational numbers \mathbb{Q} , although most of our results hold over any field of characteristic 0.

2.1. Quasisymmetric functions. Let x_1, x_2, \dots be an ordered list of variables. Let $n \geq 0$ and $\alpha = (a_1, \dots, a_k)$ be a *composition* of n , that is, a sequence of positive integers that sum to $|\alpha| = n$. The abbreviated notation $\alpha = a_1 \dots a_k$ will often be used. The *monomial quasisymmetric function* indexed by α is the formal power series

$$M_\alpha = \sum_{i_1 < \dots < i_k} x_{i_1}^{a_1} \cdots x_{i_k}^{a_k}.$$

Any linear combination of monomial quasisymmetric functions is called a *quasisymmetric function*. In order for a formal power series to be a quasisymmetric function, notice that for any composition α and any two monomials $x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_k}^{a_k}$ and $x_{j_1}^{a_1} x_{j_2}^{a_2} \cdots x_{j_k}^{a_k}$ with $i_1 < \dots < i_k$ and $j_1 < \dots < j_k$, these monomials must have the same coefficient in the quasisymmetric function; thus, quasisymmetric functions are indexed by the compositions of integers in exactly the way that symmetric functions are indexed by number partitions. They were introduced by Gessel as generating functions for weights of P -partitions [20].

Let $\text{Comp}(n)$ denote the set of compositions of n and \mathcal{Q}_n denote the linear span of $\{M_\alpha\}_{\alpha \in \text{Comp}(n)}$. The vector space $\mathcal{Q} = \bigoplus_{n \geq 0} \mathcal{Q}_n$ of quasisymmetric functions has the structure of a graded Hopf algebra: The product is ordinary multiplication of power series and the coproduct is defined on a monomial function by

$$\Delta_{\mathcal{Q}}(M_\alpha) = \sum_{\alpha = \beta \cdot \gamma} M_\beta \otimes M_\gamma,$$

the sum being over all ways of writing α as the concatenation of two (possibly empty) compositions β and γ . The set of (Hopf algebra) endomorphism on \mathcal{Q} is denoted $\text{End}(\mathcal{Q})$.

Define a partial order on $\text{Comp}(n)$ by setting $\alpha \leq \beta$ if β is a refinement of α ; that is, β is the concatenation of compositions $\beta = \beta_1 \cdots \beta_k$ such that $\alpha = (|\beta_1|, \dots, |\beta_k|)$. Given $\alpha = (a_1, \dots, a_k) \in \text{Comp}(n)$ let us define $S_\alpha \subseteq [n-1]$ by $S_\alpha = \{a_1, a_1 + a_2, \dots, a_1 + \dots + a_k\}$. The correspondence $\alpha \mapsto S_\alpha$ is an isomorphism of posets between $\text{Comp}(n)$ and the set of subsets of $[n-1]$ under inclusion. The composition corresponding to $S \subseteq [n-1]$ is denoted $\text{co}(S)$. Thus, $\text{co}(S_\alpha) = \alpha$.

For $\alpha \in \text{Comp}(n)$, define the *fundamental quasisymmetric function* F_α by

$$(2.1) \quad F_\alpha = \sum_{\substack{i_1 \leq i_2 \leq \dots \leq i_n \\ i_k \in S_\alpha \implies i_k < i_{k+1}}} x_{i_1} \cdots x_{i_n} = \sum_{\beta \in \text{Comp}(n): \beta \geq \alpha} M_\beta.$$

By inclusion-exclusion,

$$M_\alpha = \sum_{\beta \in \text{Comp}(n): \beta \geq \alpha} (-1)^{\ell(\beta) - \ell(\alpha)} F_\beta,$$

where $\ell(\alpha)$ denotes the number of parts, or *length*, of α . Thus $\{F_\alpha\}$ is a basis for \mathcal{Q} .

See [33, 27] for further background on quasisymmetric functions.

2.2. Descent algebra. For $n \geq 0$, let \mathfrak{S}_n denote the set of permutations of $[n]$. A permutation $\sigma \in \mathfrak{S}_n$ will be represented as a sequence $\sigma = (\sigma_1, \dots, \sigma_n)$, where $\sigma_i = \sigma(i)$. The descent set of σ is defined by $\text{Des}(\sigma) = \{i \in [n-1] : \sigma_i > \sigma_{i+1}\}$. The *descent composition* of σ is defined by $D(\sigma) = \text{co}(\text{Des}(\sigma))$. The set of permutations that have the same descent composition is called a *descent class*.

For $\alpha \in \text{Comp}(n)$, define $Y_\alpha \in \mathcal{D}_n$ and $X_\alpha \in \mathcal{D}_n$ by

$$(2.2) \quad Y_\alpha = \sum_{\sigma \in \mathfrak{S}_n : D(\sigma) = \alpha} \sigma$$

$$(2.3) \quad X_\alpha = \sum_{\beta \in \text{Comp}(n) : \beta \leq \alpha} Y_\beta = \sum_{\sigma \in \mathfrak{S}_n : D(\sigma) \leq \alpha} \sigma.$$

A well-known result due to Solomon [31] asserts that the vector space spanned by $\{X_\alpha\}_{\alpha \in \text{Comp}(n)}$ (or equivalently $\{Y_\alpha\}_{\alpha \in \text{Comp}(n)}$) is a subalgebra of the group algebra $\mathbb{Q}[\mathfrak{S}_n]$. This subalgebra is called the *descent algebra* and we denote it by \mathcal{D}_n .

Let $\mathcal{D} = \bigoplus_{n=0}^{\infty} \mathcal{D}_n$ and $\widehat{\mathcal{D}} = \prod_{i=0}^{\infty} \mathcal{D}_i$. An element $W \in \widehat{\mathcal{D}}$ can be represented uniquely as a formal series $W = \sum_{n \geq 0} W_n$, where $W_n \in \mathcal{D}_n$. From this viewpoint \mathcal{D} is the subspace of $\widehat{\mathcal{D}}$ consisting of those formal series having only finitely many nonzero terms. We will think of $\widehat{\mathcal{D}}$ as an algebra with component-wise multiplication: $(\sum W_n) \cdot (\sum V_n) = \sum W_n \cdot V_n$. Note that the series $X = \sum_{n \geq 0} X_n$ is the identity element.

Let $\langle \cdot | \cdot \rangle : \widehat{\mathcal{D}} \times \mathcal{Q} \rightarrow \mathbb{Q}$ be the bilinear form defined for any pair of compositions α, β by

$$\langle X_\alpha | M_\beta \rangle = \delta_{\alpha, \beta}, \quad \text{or equivalently} \quad \langle Y_\alpha | F_\beta \rangle = \delta_{\alpha, \beta}.$$

We hereby use the pairing $\langle \cdot | \cdot \rangle$ to identify $\widehat{\mathcal{D}}$ with the dual vector space of \mathcal{Q} , and \mathcal{D}_n with the dual of \mathcal{Q}_n .

2.3. Noncommutative symmetric functions. Define a new product \star and coproduct $\Delta_{\mathcal{D}}$ on \mathcal{D} by

$$(2.4) \quad X_\alpha \star X_\beta = X_{\alpha, \beta}$$

$$\Delta_{\mathcal{D}}(X_n) = \sum_{i+j=n} X_i \otimes X_j,$$

The definition of $\Delta_{\mathcal{D}}$ extends to every X_α by requiring $\Delta_{\mathcal{D}}(U \star V) = \Delta_{\mathcal{D}}(U) \star \Delta_{\mathcal{D}}(V)$. There is a natural isomorphism between $(\mathcal{D}, \star, \Delta_{\mathcal{D}})$ and the graded Hopf algebra of noncommutative symmetric functions [19], in which X_α is mapped to the complete symmetric function S^α .

The following result is part of Theorem 6.1 in [19] (cf. [27]).

Theorem 2.1. *$(\mathcal{D}, \star, \Delta_{\mathcal{D}})$ is the graded dual Hopf algebra of \mathcal{Q} under the pairing $\langle \cdot | \cdot \rangle$.*

Under the isomorphism between $(\mathcal{D}, \star, \Delta_{\mathcal{D}})$ and the Hopf algebra of noncommutative symmetric functions, the usual product on \mathcal{D} inherited from the group algebra is opposite to the internal product on noncommutative symmetric functions [19, Section 5]. The next result, which appears as Proposition 5.2 in [19], explicitly relates multiplication in the descent algebra to the product \star and coproduct $\Delta_{\mathcal{D}}$ (and thereby also \mathcal{Q} , by duality).

Proposition 2.2. For $r \geq 2$ let $\Delta_{\mathcal{D}}^r$ be defined inductively by $\Delta_{\mathcal{D}}^2 = \Delta_{\mathcal{D}}$ and $\Delta_{\mathcal{D}}^r = \Delta_{\mathcal{D}}^{r-1} \otimes I$, where I is the identity operator on \mathcal{D} . For any $G, F_1, \dots, F_r \in \mathcal{D}$, we have

$$G \cdot (F_1 \star \cdots \star F_r) = \sum_G (G_{(1)} \cdot F_1) \star \cdots \star (G_{(r)} \cdot F_r).$$

where $\Delta_{\mathcal{D}}^r(G) = \sum_G G_{(1)} \otimes \cdots \otimes G_{(r)}$ in Sweedler notation.

2.4. Lumping of random walks. Suppose that K is the transition probability matrix for a Markov chain with state space \mathfrak{S}_n . Suppose also that K has the property that

$$(2.5) \quad \sum_{\sigma \in \mathfrak{S}_n: D(\sigma)=\alpha} K(\pi, \sigma) = \sum_{\sigma \in \mathfrak{S}_n: D(\sigma)=\alpha} K(\tau, \sigma)$$

for any $\alpha \in \text{Comp}(n)$ and $\pi, \tau \in \mathfrak{S}_n$ such that $D(\pi) = D(\tau)$. In this case we may define a new Markov chain with state space $\text{Comp}(n)$ and transition probability matrix \overline{K} given by

$$(2.6) \quad \overline{K}(D(\pi), \beta) = \sum_{\sigma \in \mathfrak{S}_n: D(\sigma)=\beta} K(\pi, \sigma).$$

In other words, $\overline{K}(\alpha, \beta)$ is the probability that a permutation with descent composition α moves to some permutation with descent composition β in one step of the original Markov chain. We shall say that \overline{K} *lumps* K *by descent sets*. Lumping is discussed in §6.3 in Kemeny and Snell's book [23].

Lumping by descent sets occurs when a random walk on \mathfrak{S}_n is driven by a probability distribution that is constant on descent classes, as we now explain. Let $W = \sum_{\sigma \in \mathfrak{S}_n} W(\sigma)\sigma$ be a probability distribution on \mathfrak{S}_n such that $W \in \mathcal{D}_n$. Consider the Markov chain with state space \mathfrak{S}_n and transition probability matrix given by $K(\pi, \sigma\pi) = W(\sigma^{-1})$. Then the expression on the left-hand side of (2.5) represents the coefficient of π in $W \cdot Y_\alpha$ while the right-hand side represents the coefficient of τ . If $D(\pi) = D(\tau)$ then these coefficients are equal by the fact that \mathcal{D}_n is a subalgebra of $\mathbb{Q}[\mathfrak{S}_n]$. This means K can be lumped by descent sets, and the column of \overline{K}^m indexed by (n) encodes the m th convolution power of the distribution W :

$$(2.7) \quad \overline{K}^m(D(\pi), n) = W^{*m}(\pi).$$

Now we show how the eigenvalues and eigenvectors of the two matrices are related to each other.

Proposition 2.3. *Each eigenvector of \overline{K} gives rise to an eigenvector for K with the same eigenvalue. Moreover, linearly independent eigenvectors are thereby sent to eigenvectors that remain linearly independent.*

Proof. If the coordinate indexed by a particular descent class has value a_i in a chosen eigenvector of \overline{K} , then assigning value a_i/d for each permutation in the descent class, letting d be the size of the descent class, yields an eigenvector of \overline{K} . The fact that this is indeed an eigenvector of K follows again from the fact that K admits a lumping according to descent classes. \square

Proposition 2.4. *The matrix K is block diagonalizable, with one $(n! - 2^{n-1})$ by $(n! - 2^{n-1})$ block and each descent class D_i giving rise to a 1 by 1 block.*

Proof. The idea is to show how to decompose the vector space upon which K^T acts into subspaces each of which is carried to itself by K^T . For each descent class D_i , notice that K^T has an eigenvector by setting each coordinate indexed by an element of the descent class to 1 and all other coordinates to 0. On the other hand, we obtain an $(n! - 2^{n-1})$ -dimensional subspace also sent to itself by K^T by considering those vectors where the coordinates indexed by permutations in any chosen descent class D_i sum to 0 and all other coordinates are 0. It follows easily from the definition of lumping that each of these vectors is sent to a vector with the property that for any descent class, its coordinates which are indexed by permutations in that descent class sum to 0. \square

Question: When our random walk arises from a left action of the descent algebra, does this imply further structure on K ?

3. RANDOM WALKS RESULTING FROM ENDOMORPHISMS OF \mathcal{Q}

This section ties together results from [2], [19], and [29] to deduce a new consequence, namely that endomorphisms of quasisymmetric functions give rise to random walks under mild nonnegativity conditions.

Let $\Phi \in \text{End}(\mathcal{Q})$ and $n \geq 0$. Define $c_{\alpha,\beta}$, for $\alpha, \beta \in \text{Comp}(n)$ by

$$(3.1) \quad \Phi(F_\alpha) = \sum_{\beta \in \text{Comp}(n)} c_{\alpha,\beta} F_\beta.$$

Thus, $c_{\alpha,\beta} = \langle Y_\beta \mid \Phi(F_\alpha) \rangle$.

Suppose that the numbers $c_{\alpha,n}$, $\alpha \in \text{Comp}(n)$, are nonnegative and identically zero. Define a probability measure $\text{Prob}_\Phi : \mathfrak{S}_n \rightarrow \mathbb{R}$ by

$$(3.2) \quad \text{Prob}_\Phi(\pi) = \frac{c_{D(\pi),n}}{\sum_{\sigma \in \mathfrak{S}_n} c_{D(\sigma),n}}.$$

We will call this the QS^* -distribution corresponding to Φ because of its connection to Stanley's QS -distribution [34], as explained in Section 6.

Consider the random walk on \mathfrak{S}_n where π goes to $\sigma\pi$ with probability $\text{Prob}_\Phi(\sigma^{-1})$. Denote the corresponding transition probability matrix by K , so that

$$K(\pi, \sigma\pi) = \text{Prob}_\Phi(\sigma^{-1}).$$

Note that the transpose of K is the transition matrix of the left random walk on \mathfrak{S}_n driven by $\sum_{\sigma \in \mathfrak{S}_n} \text{Prob}_\Phi(\sigma)\sigma$, where one step takes π to $\sigma\pi$ with probability $\text{Prob}(\sigma)$.

Let $\lambda \in \mathbb{Q}$ be defined by $\Phi(M_1) = \lambda M_1$. Let Φ_n denote the restriction map $\Phi|_{\mathcal{Q}_n}$. If $\lambda \neq 0$, then define \overline{K} to be the transpose of the matrix representing $\frac{1}{\lambda^n} \Phi_n$ relative to the fundamental basis, namely

$$\overline{K}(\alpha, \beta) = \frac{1}{\lambda^n} c_{\alpha,\beta}.$$

for all $\alpha, \beta \in \text{Comp}(n)$. If we need to be explicit about n and Φ then we will write \overline{K}_n^Φ . The same goes for K .

Theorem 3.1. *Let $\Phi \in \text{End}(\mathcal{Q})$ and $n \geq 0$, and suppose that the numbers $c_{\alpha,n}$, $\alpha \in \text{Comp}(n)$, are nonnegative and not identically zero. Then $\lambda^n > 0$ and \overline{K} is a stochastic matrix. Furthermore, \overline{K} lumps K by descent sets.*

Let us illustrate Theorem 3.1 with an example with Stembridge's Θ -map before turning to the proof.

Example 3.2. Given $\alpha \in \text{Comp}(n)$ for some $n \geq 1$, let $\Lambda(\alpha) = \{i \in S_\alpha \mid i \neq 1 \text{ and } i-1 \neq S_\alpha\}$. For instance, if $\alpha = 1121134$ then $S_\alpha = \{1, 2, 4, 5, 6, 9\}$ and $\Lambda(\alpha) = \{4, 9\}$. Thus, if α is the descent composition of a permutation $\pi \in \mathfrak{S}_n$, then $\Lambda(\alpha)$ is the *peak set* of π , namely the set $\{i \in [2, n-1] \mid \pi_{i-1} < \pi_i > \pi_{i+1}\}$. According to [35, Proposition 3.5], Θ can be defined in terms of the fundamental basis by

$$\Theta(F_\alpha) = 2^{|\Lambda(\alpha)|+1} \sum_{\substack{\beta \in \text{Comp}(n) \\ \Lambda(\alpha) \subseteq S_\beta \Delta (S_{\beta+1})}} F_\beta$$

where $T+1 = \{t+1 \mid t \in T\}$ and Δ stands for the symmetric difference: $A \Delta B = (A-B) \cup (B-A)$. It is well known that Θ is a Hopf algebra homomorphism [2, Example 4.9].

The corresponding QS^* -distribution is given by

$$(3.3) \quad \text{Prob}_\Theta(\pi) = \begin{cases} \frac{1}{2^{n-1}} & \text{if the peak set of } \pi \text{ is empty} \\ 0 & \text{otherwise.} \end{cases}$$

This is the probability that a deck of n cards is in arrangement π after an inverse face up, face down shuffle [4]: Remove a subset of cards from the deck, letting all subsets have the same probability of being selected, and place the packet of removed cards face up on top of the remaining cards, which are face down. The row of \overline{K} indexed by (n) gives the distribution of descent sets after performing one face up, face down shuffle: Cut the deck into two packets according to the binomial distribution, flip the top packet over so the cards are facing up, then shuffle the two packets according to the Gilbert-Shannon-Reeds model; the probability of ending up with a permutation with descent composition α is $\overline{K}(n, \alpha)$. For instance,

$$\overline{K}_3^\Theta = \begin{array}{c|cccc} & 3 & 12 & 21 & 111 \\ \hline 3 & 1/4 & 1/4 & 1/4 & 1/4 \\ 12 & 1/4 & 1/4 & 1/4 & 1/4 \\ 21 & 0 & 1/2 & 1/2 & 0 \\ 111 & 1/4 & 1/4 & 1/4 & 1/4 \end{array}$$

Each entry, say $\overline{K}(21, 12) = 1/2$, can be explained in terms of lumping K by descents as follows. Pick any permutation $\pi \in \mathfrak{S}_3$ such that $D(\pi) = 21$, say $\pi = \mathbf{132}$ (short for $\pi_1 = 1, \pi_2 = 3, \pi_3 = 2$). Then $\overline{K}(21, 12)$ should be the probability a permutation $\sigma \in \mathfrak{S}_3$ chosen with probability $\text{Prob}_\Theta(\sigma)$ will have the property that $D(\sigma^{-1}\pi) = 12$.

Stembridge [35, Theorem 3.6] provides an alternate probabilistic interpretation of Θ : Given $\pi \in \mathfrak{S}_n$, independently assign a $+$ or $-$ sign to every π_i , letting each sign occur with probability $1/2$. Then the probability that the resulting signed permutation has descent composition β is $\overline{K}(D(\pi), \beta)$. We generalize Stembridge's result in Section 8.

The proof of Theorem 3.1 amounts to showing that Φ_n is, after normalization, dual to the operator on \mathcal{D}_n given by $W \mapsto (\sum_{\sigma \in \mathfrak{S}_n} \text{Prob}_\Phi(\sigma) \sigma) \cdot W$. We begin by setting up a bijection between $\text{End}(\mathcal{Q})$ and those series in $\widehat{\mathcal{D}}$ that encode characters of \mathcal{Q} . Recall that we have identified the dual vector space of \mathcal{Q} with $\widehat{\mathcal{D}}$, the set of formal infinite series $\sum_{n \geq 0} W_n$ such that $W_n \in \mathcal{D}_n$. Thus, a character of \mathcal{Q} is a series $W \in \widehat{\mathcal{D}}$ such that $\langle W | 1 \rangle = 1$ and $\langle W | FG \rangle = \langle W | F \rangle \langle W | G \rangle$ for all $F, G \in \mathcal{Q}$.

A character of fundamental importance is the series $X = \sum_{n \geq 0} X_n$, where X_n is the identity permutation in \mathfrak{S}_n (see (2.3)). This is called the *universal character*. It follows from the definition of X that

$$\langle X | M_\alpha \rangle = \langle X | F_\alpha \rangle = \begin{cases} 1 & \text{if } \alpha = () \text{ or } \alpha = (n) \\ 0 & \text{otherwise.} \end{cases}$$

Thus, for any quasisymmetric function $F(x_1, x_2, \dots)$,

$$\langle X | F(x_1, x_2, \dots) \rangle = F(1, 0, 0, \dots).$$

For $\Phi \in \text{End}(\mathcal{Q})$ and $W \in \widehat{\mathcal{D}}$, define the series $W^\Phi \in \widehat{\mathcal{D}}$ by

$$\langle W^\Phi | \cdot \rangle = \langle W | \Phi(\cdot) \rangle.$$

In the theory of combinatorial Hopf algebras, X satisfies a universal property [2, Theorem 4.1], of which the following is an immediate corollary:

Proposition 3.3. *The correspondence $\Phi \mapsto X^\Phi$ is a bijection between $\text{End}(\mathcal{Q})$ and the set of characters of \mathcal{Q} . Moreover,*

$$(3.4) \quad \Phi(M_\alpha) = \sum \langle X^\Phi | M_{\beta_1} \rangle \cdots \langle X^\Phi | M_{\beta_m} \rangle M_{(|\beta_1|, \dots, |\beta_m|)},$$

where the sum is over all sequences of compositions β_1, \dots, β_m such that $\beta_1 \cdots \beta_m = \alpha$.

Remark 3.4. It follows from Proposition 3.3 that the matrix relative to the monomial basis for Φ as a linear operator on \mathcal{Q}_n is lower triangular, provided that the ordering of the basis elements is a linear extension of the partial ordering on $\text{Comp}(n)$.

Next we develop some properties of characters to be used shortly. The following characterization is proven in [29, Theorem 3.2(ii)-(iii)].

Proposition 3.5. *A series $W \in \widehat{\mathcal{D}}$ is a character of \mathcal{Q} if and only if $\Delta_{\mathcal{D}}(W_n) = \sum_{i=0}^n W_i \otimes W_{n-i}$ for all n (i.e., W is group-like for $\Delta_{\mathcal{D}}$).*

We now derive a useful formula for left multiplication in \mathcal{D} by X^Φ .

Proposition 3.6. *For $\Phi \in \text{End}(\mathcal{Q})$ and $\alpha \in \text{Comp}(n)$,*

$$(3.5) \quad X^\Phi \cdot X_\alpha = (X_\alpha)^\Phi.$$

Proof. Let $\alpha = a_1 \dots a_k \in \text{Comp}(n)$. By applying Proposition 3.5 and then Proposition 2.2, we obtain

$$X^\Phi \cdot X_\alpha = X^\Phi \cdot (X_{a_1} \star \cdots \star X_{a_k}) = (X^\Phi \cdot X_{a_1}) \star \cdots \star (X^\Phi \cdot X_{a_k}).$$

Since X_{a_i} is the identity permutation in \mathfrak{S}_{a_i} , we have $X^\Phi \cdot X_{a_i} = (X^\Phi)_{a_i}$. For $\beta \in \text{Comp}(n)$,

$$\langle (X^\Phi)_{a_1} \star \cdots \star (X^\Phi)_{a_k} \mid M_\beta \rangle = \langle (X^\Phi)_{a_1} \otimes \cdots \otimes (X^\Phi)_{a_k} \mid \Delta_{\mathcal{Q}}^k(M_\beta) \rangle.$$

The right-hand side vanishes if $\beta \not\geq \alpha$. If $\beta \geq \alpha$, then there exist compositions β_1, \dots, β_k such that $\beta = \beta_1 \cdots \beta_k$ and $a_i = |\beta_i|$, and we have

$$\langle (X^\Phi)_{a_1} \otimes \cdots \otimes (X^\Phi)_{a_k} \mid \Delta_{\mathcal{Q}}^k(M_\beta) \rangle = \langle X^\Phi \mid M_{\beta_1} \rangle \cdots \langle X^\Phi \mid M_{\beta_k} \rangle.$$

According to (3.4), this is the coefficient of M_α in the monomial basis expansion of $\Phi(M_\beta)$. This coefficient is given by $\langle X_\alpha \mid \Phi(M_\beta) \rangle$, or equivalently $\langle (X_\alpha)^\Phi \mid M_\beta \rangle$.

We have proved that $X^\Phi \cdot X_\alpha$ and $(X_\alpha)^\Phi$ agree on the monomial basis, and hence that they are equal. \square

Proposition 3.7. *If $\Phi \in \text{End}(\mathcal{Q})$ then the character X^Φ satisfies*

$$(3.6) \quad \langle X^\Phi \cdot W \mid G \rangle = \langle W \mid \Phi(G) \rangle$$

for all $W \in \widehat{\mathcal{D}}$ and $G \in \mathcal{Q}$.

Proof. By Proposition 3.6, (3.6) holds for $W = X_\alpha$ for all α , hence it holds in general by linearity. \square

Remark 3.8. It is shown in [30, Lemma 5.1] that if $W \in \widehat{\mathcal{D}}$ is group-like for $\Delta_{\mathcal{D}}$ then the operator on \mathcal{D} given by $V \mapsto W \cdot V$ is a Hopf algebra endomorphism of $(\mathcal{D}, \star, \Delta_{\mathcal{D}})$. This assertion and its converse are direct consequences of Proposition 3.7 (together with Proposition 3.5).

The preceding result says that X^Φ , as an operator on \mathcal{D} acting by left multiplication, is dual to Φ . Since $\langle X^\Phi \mid F_\alpha \rangle = \langle X^\Phi \cdot Y_n \mid F_\alpha \rangle = \langle Y_n \mid \Phi(F_\alpha) \rangle = c_{\alpha, n}$, the homogeneous component of X^Φ of degree n is

$$(3.7) \quad (X^\Phi)_n = \sum_{\alpha \in \text{Comp}(n)} c_{\alpha, n} Y_\alpha = \sum_{\sigma \in \mathfrak{S}_n} c_{D(\sigma), n} \sigma.$$

Example 3.9. For the Θ -map, we have

$$(X^\Theta)_n = 2 \cdot (Y_n + Y_{(1, n-1)} + Y_{(1^2, n-2)} + \cdots + Y_{1^n}) = 2 \cdot \sum_{\sigma \in \mathfrak{S}_n: \Lambda(\sigma) = \emptyset} \sigma,$$

where $\Lambda(\sigma)$ stands for the peak set of σ . The fact that Θ is dual to X^Θ was noticed in [1, Remarks 7.11] and [6].

Proceeding with the proof of Theorem 3.1, we have

$$\sum_{\beta \in \text{Comp}(n)} c_{\alpha, \beta} Y_\beta = X^\Phi \cdot Y_\alpha = \sum_{\sigma, \tau \in \mathfrak{S}_n: D(\tau) = \beta} c_{D(\sigma), n} \cdot \sigma \tau = \sum_{\pi, \tau \in \mathfrak{S}_n: D(\tau) = \beta} c_{D(\pi \tau^{-1}), n} \cdot \pi,$$

which leads to the identity

$$(3.8) \quad c_{\alpha, \beta} = \sum_{\tau \in \mathfrak{S}_n: D(\tau) = \beta} c_{D(\pi \tau^{-1}), n}.$$

where π is any permutation such that $D(\pi) = \alpha$. In particular, $c_{\alpha,\beta} \geq 0$ for all $\alpha, \beta \in \text{Comp}(n)$. One consequence of (3.8) is $\sum_{\beta \in \text{Comp}(n)} c_{\alpha,\beta} = \sum_{\tau \in \mathfrak{S}_n} c_{D(\pi\tau^{-1}),n} = \sum_{\sigma \in \mathfrak{S}_n} c_{D(\sigma),n}$, which means every row of the matrix c has the same (positive) sum.

Let us show that the sum of the row indexed by 1^n is λ^n . We have

$$\Phi(M_{1^n}) = \Phi(F_{1^n}) = \sum_{\alpha \in \text{Comp}(n)} c_{1^n,\alpha} F_\alpha = \sum_{\alpha \in \text{Comp}(n)} c_{1^n,\alpha} \sum_{\beta \geq \alpha} M_\beta.$$

The coefficient of M_{1^n} in the last expression is $\sum_{\alpha \in \text{Comp}(n)} c_{1^n,\alpha}$. On the other hand, by (3.4) this coefficient is $\langle X^\Phi | M_1 \rangle^n = \lambda^n$.

An interesting consequence is the identity

$$(3.9) \quad \sum_{\sigma \in \mathfrak{S}_n} c_{D(\sigma),n} = \lambda^n,$$

which in turn implies $\lambda^n > 0$ and

$$(3.10) \quad \text{Prob}_\Phi(\pi) = \overline{K}(D(\pi), n).$$

To complete the proof of Theorem 3.1, it remains to show that \overline{K} lumps K by descents. This requires checking that for any $\beta \in \text{Comp}(n)$,

$$(3.11) \quad \sum_{\sigma \in \mathfrak{S}_n: D(\sigma)=\beta} c_{D(\pi\sigma^{-1}),n} = \sum_{\sigma \in \mathfrak{S}_n: D(\sigma)=\beta} c_{D(\tau\sigma^{-1}),n}$$

for all $\pi, \tau \in \mathfrak{S}_n$ such that $D(\pi) = D(\tau)$, and

$$\frac{1}{\lambda^n} c_{D(\pi),\beta} = \sum_{\sigma \in \mathfrak{S}_n: D(\sigma)=\beta} K(\pi, \sigma).$$

Both of these identities follow directly from (3.8), completing the proof.

3.1. Stationary distribution.

Theorem 3.10. *Suppose that the hypotheses of Theorem 3.1 are satisfied and that $c_{\alpha,n} > 0$ for some $\alpha \in \text{Comp}(n) \setminus \{n, 1^n\}$. Then K has a unique stationary distribution given by the uniform distribution on permutations, and \overline{K} has a unique stationary distribution equal to the distribution of descent sets in \mathfrak{S}_n .*

Proof. The proof has two parts, namely show (1) that for $w \in \mathbb{R}[\mathfrak{S}_n]$ in which each permutation has nonnegative coefficient between 0 and 1 so that these coefficients add up to 1, the Markov chain resulting from left action by w has a one dimensional space of fixed vectors spanned by the vector in which all coordinates are equal, provided that the set of permutations appearing with nonzero coefficient in w generate \mathfrak{S}_n , and (2) that any single descent class other than those containing only the identity or only the longest element in \mathfrak{S}_n will meet this condition of generating \mathfrak{S}_n . Our added assumption about $c_{\alpha,n}$ implies that there exists some $\tau \in \mathfrak{S}_n$, which is neither identity nor the longest permutation, such that $\text{Prob}_\Phi(\tau) > 0$. Since K^T is the matrix for the left action of $\sum_{\sigma \in \mathfrak{S}_n} \text{Prob}_\Phi(\sigma)\sigma$ on \mathfrak{S}_n , it follows from (1) and (2) that K^T , and hence K , has unique stationary distribution equal to the uniform distribution on permutations. Our assertion about \overline{K} follows from elementary properties of lumping.

First we prove (1). Each permutation σ appearing with nonzero coefficient a_σ in w gives a left action on \mathfrak{S}_n , so for each such σ we make a directed graph with $n!$ vertices, using directed edges to indicate for each element of \mathfrak{S}_n where it is sent by σ . Now combine these graphs for the various σ with nonzero a_σ , i.e. using one vertex set of size $n!$ and the union of all directed edges for all such σ . Choosing a set of σ that generate \mathfrak{S}_n implies that this directed graph has a directed path from each of its $n!$ vertices to all its other vertices. For u any eigenvector with eigenvalue 1, we have $wu = u$, which translates to an equation for each of the $n!$ coordinates in u . These equations are indexed by permutations, so consider the equation indexed by some $\gamma \in \mathfrak{S}_n$. For $u = \sum_{\sigma \in \mathfrak{S}_n} b_\sigma \sigma$, this equation may be written as

$$b_\gamma = \sum_{\sigma \in \mathfrak{S}_n} a_\sigma b_{\sigma^{-1}(\gamma)},$$

namely a sum in which the nonzero terms exactly come from permutations at the tails of arrows with head at γ . This expresses b_γ as a convex combination (i.e. a weighted average) of these coefficients $b_{\sigma^{-1}(\gamma)}$, implying b_γ is neither the smallest nor largest value among these coefficients unless all are equal. However, chasing the directed graph around, we see that every permutation is expressed as a convex combination of others in such a way that no coefficient b_γ may be smallest or largest among these coefficients, hence all must be equal, so we are done. If the permutations do not generate \mathfrak{S}_n , then the graph will have multiple components (resulting from multiple orbits in the left action on \mathfrak{S}_n), and we get a basis for the eigenspace with eigenvalue 1 by taking as basis vectors the sums over permutations in any one orbit.

Now to (2). Consider a permutation σ in our chosen descent class D . By assumption, there must be some values $j, j+1$ not appearing consecutively in the one-line notation expression for σ , this implies $(j, j+1)\sigma \in D$, implying $(j, j+1)$ is in the subgroup G_D of \mathfrak{S}_n generated by the elements of D . Thus, we get all adjacent transpositions except those $(i, i+1)$ for $i, i+1$ in consecutive positions in σ , but then consider a maximal segment of consecutive values appearing consecutively in σ . This segment must either be strictly increasing or strictly decreasing, and without loss of generality assume the former. Thus, the segment takes the form $i, i+1, \dots, i+r$, and by assumption we either have $i \neq 1$ or have $i+r \neq n$. Thus, we may swap either the values $i-1, i$ or the values $i+r, i+r+1$ to obtain another permutation π' in our descent class which has strictly shorter segment and which may be used to show either that the adjacent transposition $(i, i+1)$ is in G_D or else that $(i+r, i+r+1) \in G_D$. Continuing in this manner, one may show that all adjacent transpositions are in G_D , implying $G_D = \mathfrak{S}_n$, as desired. \square

3.2. Lumping by peak sets and other statistics. We establish variants of Theorem 3.1 and Theorem 3.10 from which the probabilistic interpretation of the Θ -map given in [8] can be deduced.

Let $\Phi \in \text{End}(\mathcal{Q})$ and $n \geq 0$. Let us say that Φ_n has the *partitioning property* if the following two conditions hold. First is that there exists an equivalence relation on $\text{Comp}(n)$ such that $\Phi(F_\alpha) = \Phi(F_\beta)$ whenever $\hat{\alpha} = \hat{\beta}$, where $\hat{\alpha}$ denotes the equivalence class containing α . Let E denote the set of equivalence classes and let $\phi_{\hat{\alpha}} = \Phi(F_\alpha)$ for all $\hat{\alpha} \in E$. The second condition is that $\{\phi_{\hat{\alpha}}\}_{\hat{\alpha} \in E}$ is a basis for $\Theta(\mathcal{Q}_n)$. If Φ_n has full rank then it trivially has the

partitioning property. A nontrivial example is the Θ -map: the equivalence class containing $\alpha \in \text{Comp}(n)$ is given by $\hat{\alpha} = \{\beta \in \text{Comp}(n) \mid \Lambda(\beta) = \Lambda(\alpha)\}$. An open question is whether there are any other nontrivial endomorphisms with the partitioning property.

Suppose that Φ_n has the partitioning property. Then define $d_{\hat{\alpha}, \hat{\beta}} \in \mathbb{Q}$ by

$$\Phi(\phi_{\hat{\alpha}}) = \sum_{\hat{\beta} \in E} d_{\hat{\alpha}, \hat{\beta}} \phi_{\hat{\beta}}$$

and if $\lambda^n \neq 0$ as well then define

$$\hat{K}(\hat{\alpha}, \hat{\beta}) = \frac{1}{\lambda^n} d_{\hat{\alpha}, \hat{\beta}}.$$

For $\hat{\beta} \in E$ define $Y_{\hat{\beta}} \in \mathcal{D}_n$ by

$$Y_{\hat{\beta}} = \sum_{\alpha \in \hat{\beta}} Y_{\alpha}.$$

It follows from the duality between Φ and X^{Φ} (Proposition 3.7) that

$$(3.12) \quad X^{\Phi} \cdot Y_{\hat{\beta}} = \sum_{\hat{\alpha} \in E} d_{\hat{\alpha}, \hat{\beta}} Y_{\hat{\alpha}}$$

A direct consequence is the following:

Proposition 3.11. *If Φ_n has the partitioning property then the subspace of \mathcal{D}_n spanned by $\{Y_{\hat{\alpha}}\}_{\hat{\alpha} \in E}$ is the right ideal $X^{\Phi} \cdot \mathcal{D}_n$.*

Specializing to the Θ -map, $X^{\Theta} \cdot \mathcal{D}_n$ is just Nyman's peak algebra [28], which was shown to be a right ideal in [30, 1] and implicitly in [24].

Further implications of (3.12) are the following variants of Theorems 3.1 and 3.10:

Theorem 3.12. *Suppose that Φ_n has the partitioning property and the numbers $c_{\alpha, n}, \alpha \in \text{Comp}(n)$ are nonnegative and not identically zero. Then \hat{K} is a stochastic matrix and it lumps K by equivalence classes.*

Theorem 3.13. *\hat{K} has a unique stationary distribution if and only if there is some $\alpha \in \text{Comp}(n) \setminus \{n, 1^n\}$ such that $c_{\alpha, n} \neq 0$. In this case the stationary distribution is the lumped version of the uniform distribution on permutations.*

These results can be interpreted for the Θ -map as follows, elucidating the results of [8, §3.1]. The map $\frac{1}{2^n} \Theta|_{\Pi_n}$ is dual to the left action of $\sum_{\sigma \in \mathfrak{S}_n} \text{Prob}_{\Theta}(\sigma) \sigma = \frac{1}{2^{n-1}} \sum_{\sigma \in \mathfrak{S}_n: \Lambda(\sigma) = \emptyset} \sigma$ on the right ideal $X^{\Theta} \cdot \mathcal{D}_n$. Thus $\frac{1}{2^n} \Theta|_{\Pi_n}$ gives rise to a random walk on peak sets in which the peak set of a permutation π steps to the peak set of $\sigma^{-1}\pi$ with probability $\text{Prob}_{\Theta}(\sigma)$. The (unique) stationary distribution is the distribution of peak sets of random permutations.

4. SPECTRAL DECOMPOSITION

This section describes the eigenvalues and gives a recursive process for constructing a basis of eigenvectors for all the nonzero eigenspaces. See Macdonald's book [26, Ch. VI Section 4] for an earlier instance of this type of recursive eigenvector construction arising in a different context. We show that our eigenvectors are primitive elements.

4.1. **Eigenvalues and eigenvectors.** Given $\Phi \in \text{End}(\mathcal{Q})$, let

$$\lambda_n = \langle X^\Phi \mid M_n \rangle \quad \text{and} \quad \lambda_\alpha = \lambda_{a_1} \cdots \lambda_{a_k}$$

for every $n \geq 0$ and every composition $\alpha = (a_1, \dots, a_k)$. By (3.4), λ_α is the coefficient of M_α in the monomial expansion of $\Phi(M_\alpha)$ and in particular $\lambda_1 = \lambda$.

Since Φ is triangular relative to the monomial basis (Remark 3.4), we obtain the following description of its eigenvalues.

Proposition 4.1. *For any $\Phi \in \text{End}(\mathcal{Q})$ and $n \geq 0$, the eigenvalues of Φ_n , taking into account multiplicities, are $(\lambda_\alpha)_{\alpha \in \text{Comp}(n)}$.*

Conversely, by Theorem 5.1 some of the eigenvalues can be arbitrarily prescribed:

Proposition 4.2. *For any list of rational numbers u_1, u_2, u_3, \dots , there exists $\Phi \in \text{End}(\mathcal{Q})$ such that $\lambda_i = u_i$ for all i .*

The main theorem of this section is as follows:

Theorem 4.3. *Let $\Phi \in \text{End}(\mathcal{Q})$ and $n \geq 0$. Suppose that for every $m \leq n$ such that $\lambda_m \neq 0$, we have*

$$(4.1) \quad \lambda_m \neq \lambda_\beta \text{ for all } \beta \in \text{Comp}(m) \setminus \{(m)\}.$$

Suppose also that $\text{rank}(\Phi_n) = \#\{\alpha \in \text{Comp}(n) \mid \lambda_\alpha \neq 0\}$. Then Φ_n is diagonalizable, and by duality so is the operator $(X^\Phi)_n$.

Example 4.4. For the Θ -map we have $\lambda_m = 0$ if m is even and $\lambda_\beta = 2^{\ell(\beta)} \neq 2 = \lambda_m$ whenever m is odd and $\beta \in \text{Comp}(m) \setminus \{(m)\}$. The dimension of $\Theta(\mathcal{Q}_n)$ equals the Fibonacci number f_n (with $f_0 = f_1 = f_2 = 1$) [35], or the number of compositions of n with only odd parts. Thus Θ_n is diagonalizable for all n .

Remark 4.5. The Hopf subalgebra $\Pi = \Theta(\mathcal{Q})$ of \mathcal{Q} is Stembridge's *peak algebra*, and the diagonalizability of the restriction of Θ to Π_n was proved in [8] by a direct argument.

More generally, a technique for diagonalizing operators on \mathcal{D} that correspond to various transformations of alphabets for noncommutative symmetric functions is described in [24, §3 & Note 5.20]. A prominent example is the $A \mapsto (1 - q)A$ transform, which yields the operator $W \mapsto X^\ominus \cdot W$ when $q = -1$ (cf. [6, Section 3]). The eigenvectors are constructed from a unique family of Lie idempotents. Here we describe a way to recursively construct eigenvectors (for a broader family of operators) that turn out to be Lie quasi-idempotents and satisfy a uniqueness property.

Our construction of eigenvectors is as follows. For each m such that $\lambda_m \neq 0$ and (4.1) holds, define $Z_m \in \mathcal{D}_m$ recursively by $\langle Z_m \mid M_m \rangle = 1$ and

$$(4.2) \quad \langle Z_m \mid M_\beta \rangle = \frac{1}{\lambda_m - \lambda_\beta} \sum_{\alpha < \beta} \langle Z_m \mid M_\alpha \rangle \langle X^\Phi \cdot X_\alpha \mid M_\beta \rangle \text{ if } \beta \in \text{Comp}(m) \setminus \{(m)\}.$$

For $\alpha = a_1 \dots a_k \in \text{Comp}(n)$ such that Z_{a_1}, \dots, Z_{a_k} are all defined, define $Z_\alpha \in \mathcal{D}_n$ by

$$Z_\alpha = Z_{a_1} \star \cdots \star Z_{a_k}.$$

Note that if Z_α is defined then $\lambda_\alpha \neq 0$.

Proposition 4.6. *If Z_α is defined then*

$$(4.3) \quad X^\Phi \cdot Z_\alpha = \lambda_\alpha Z_\alpha.$$

Proof. Since X^Φ is dual to Φ , it is an algebra map with respect to \star and so $X^\Phi \cdot Z_{a_1 \dots a_k} = (X^\Phi \cdot Z_{a_1}) \star \dots \star (X^\Phi \cdot Z_{a_k})$. Therefore it suffices to prove that $X^\Phi \cdot Z_n = \lambda_n Z_n$, or equivalently that $\langle X^\Phi \cdot Z_n \mid M_\beta \rangle = \lambda_n \langle Z_n \mid M_\beta \rangle$ for all $\beta \in \text{Comp}(n)$. This is done as follows:

$$\begin{aligned} \langle X^\Phi \cdot Z_n \mid M_\beta \rangle &= \left\langle X^\Phi \cdot \sum_{\alpha \in \text{Comp}(n)} \langle Z_n \mid M_\alpha \rangle X_\alpha \mid M_\beta \right\rangle \\ &= \sum_{\alpha \leq \beta} \langle Z_n \mid M_\alpha \rangle \langle X^\Phi \cdot X_\alpha \mid M_\beta \rangle \\ &= \lambda_\beta \langle Z_n \mid M_\beta \rangle + \sum_{\alpha < \beta} \langle Z_n \mid M_\alpha \rangle \langle X^\Phi \cdot X_\alpha \mid M_\beta \rangle \\ &= \lambda_\beta \langle Z_n \mid M_\beta \rangle + (\lambda_n - \lambda_\beta) \langle Z_n \mid M_\beta \rangle \\ &= \lambda_n \langle Z_n \mid M_\beta \rangle. \end{aligned}$$

In the second equality, the sum is over $\alpha \leq \beta$ because the operator X^Φ is triangular in the X -basis (see Remark 3.4). \square

To finish the proof of Theorem 4.3, note that because of our assumption about $\text{rank}(\Phi_n)$, it suffices to exhibit a set of linearly independent eigenvectors with cardinality equalling the number of nonzero eigenvalues counted with multiplicities. If (4.1) holds for every $m \leq n$ such that $\lambda_m \neq 0$, then Z_α is defined whenever $\lambda_\alpha \neq 0$. Moreover, the Z_α 's are linearly independent because of their triangular relation to the X -basis. This completes the proof.

Remark 4.7. The recursion (4.2) was designed to make the calculation in the previous proof go through. However, it is easy to see that (4.2) is also a necessary condition for Z_n to be an eigenvector with eigenvalue λ_n . The following makes this precise.

Proposition 4.8. *Let $\Phi \in \text{End}(\mathcal{Q})$ and $m \geq 0$. Suppose that $\lambda_m \neq 0$ and (4.1) holds, and that Z'_m is any element of \mathcal{D}_m satisfying $X^\Phi \cdot Z'_m = \lambda_m Z'_m$ and $\langle Z'_m \mid M_m \rangle \neq 0$. Then $Z'_m = \langle Z'_m \mid M_m \rangle Z_m$.*

An explicit, non-recursive formula for Z_n for an interesting class of endomorphisms is given in Proposition 7.3. In Example 5.3 we give a generic formula for Z_1, Z_2 , and Z_3 that holds for arbitrary endomorphisms.

4.2. Primitive elements. An element $W \in \mathcal{D}_n$ is called a *primitive element* if $\Delta_{\mathcal{D}}(W) = 1 \otimes W + W \otimes 1$. If W is primitive then it is a Lie quasi-idempotent in the sense of [24, Theorem 3.1] (cf. [29, Theorems 3.1 and 3.2]).

Proposition 4.9. *Let $\Phi \in \text{End}(\mathcal{Q})$ and $n \geq 0$, and suppose that Z_m is defined for all $m \leq n$. Then Z_n is a primitive element.*

Proof. Let $(Z_\alpha^*)_{\alpha \in \text{Comp}(n)} \subseteq \mathcal{Q}$ denote the dual basis of $(Z_\alpha)_{\alpha \in \text{Comp}(n)}$, so that $\langle Z_\alpha \mid Z_\beta^* \rangle = \delta_{\alpha, \beta}$. Then (Z_α^*) form a basis of eigenvectors for Φ_n . For any pair of compositions α, β such

that $|\alpha| + |\beta| = n$, we have

$$\langle \Delta_{\mathcal{D}}(Z_n) \mid Z_{\alpha}^* \otimes Z_{\beta}^* \rangle = \langle Z_n \mid Z_{\alpha}^* Z_{\beta}^* \rangle$$

by the Hopf algebraic duality between \mathcal{D} and \mathcal{Q} . Notice that $Z_{\alpha}^* Z_{\beta}^*$ is an eigenvector of Φ with eigenvalue $\lambda_{\alpha} \lambda_{\beta} = \lambda_{\alpha \cdot \beta}$. This eigenvalue equals λ_n if and only if $\alpha = (n)$ or $\beta = (n)$. Therefore the expansion of $Z_{\alpha}^* Z_{\beta}^*$ in the basis (Z_{γ}^*) involves the element Z_n^* if and only if α or β is (n) . \square

In [24, Theorem 3.16] it is shown how to obtain a family of orthogonal idempotents of the descent algebra from a sequence of Lie quasi-idempotents. In particular, such a family can be obtained from our sequence $(Z_n)_{n \geq 0}$. See [10, p. 919 Remark] for a related result on obtaining orthogonal idempotents of the descent algebra.

5. DESCRIBING ENDOMORPHISMS BY EXPLICIT MATRICES

We will describe how endomorphisms of \mathcal{Q} can be represented by triangular matrices with polynomial entries. An algorithm is suggested for computing these polynomials.

5.1. Matrices representing endomorphisms. A composition is called *Lyndon* if it is lexicographically smaller than all of its nontrivial cyclic rearrangements. Let \mathcal{L} denote the set of Lyndon compositions. The set $\{M_{\alpha} \mid \alpha \in \mathcal{L}\}$ freely generates \mathcal{Q} as an algebra, that is, $\mathcal{Q} = \mathbb{Q}[M_{\alpha} \mid \alpha \in \mathcal{L}]$ [22, §6, Example 1] (cf. [27, Corollary 2.2]). For example, the composition 21 is not Lyndon, but we can express M_{21} uniquely as a polynomial in the monomial functions indexed by Lyndon compositions:

$$M_{21} = M_1 M_2 - M_{12} - M_3.$$

We call this the Lyndon expansion of M_{21} .

Recall from (3.4) that the entries of the matrix for Φ relative to the monomial basis are polynomials in the values $\langle X^{\Phi} \mid M_{\alpha} \rangle$. Since $\langle X^{\Phi} \mid \cdot \rangle : \mathcal{Q} \rightarrow \mathbb{Q}$ is an algebra map, $\langle X^{\Phi} \mid M_{\alpha} \rangle$ is a polynomial in the values $\langle X^{\Phi} \mid M_{\beta} \rangle$ for $\beta \in \mathcal{L}$. To describe these polynomials, we introduce commutative variables u_{α} , $\alpha \in \mathcal{L}$ and define $A_n(\beta, n)$, where $\beta \in \text{Comp}(n)$, to be the polynomial obtained by substituting u_{α} for M_{α} in the Lyndon expansion of M_{β} . For instance, $A(21, 3) = u_1 u_2 - u_{12} - u_3$.

For $n \geq 1$ and $\alpha, \beta \in \text{Comp}(n)$, let $A_n(\beta, \alpha) = 0$ if $\alpha \not\leq \beta$ and

$$A_n(\beta, \alpha) = A_{a_1}(\beta_1, a_1) A_{a_2}(\beta_2, a_2) \cdots A_{a_k}(\beta_k, a_k)$$

if $\alpha \leq \beta$, where β_1, \dots, β_k are the compositions such that $\beta = \beta_1 \cdots \beta_k$ and $\alpha = (|\beta_1|, |\beta_2|, \dots, |\beta_k|)$. In addition, we set $A_0 = (1)$. The matrices A_1, A_2 , and A_3 are

$$A_1 = \begin{array}{c|c} & 1 \\ \hline 1 & u_1 \end{array} \quad A_2 = \begin{array}{c|cc} & 2 & 11 \\ \hline 2 & u_2 & 0 \\ 11 & \frac{1}{2}(u_1^2 - u_2) & u_1^2 \end{array}$$

$$(5.1) \quad A_3 = \begin{array}{c|cccc} & 3 & 12 & 21 & 111 \\ \hline 3 & u_3 & 0 & 0 & 0 \\ 12 & u_{12} & u_1 u_2 & 0 & 0 \\ 21 & u_1 u_2 - u_{12} - u_3 & 0 & u_1 u_2 & 0 \\ 111 & \frac{1}{6}u_1^3 - \frac{1}{2}u_1 u_2 + \frac{1}{3}u_3 & \frac{1}{2}(u_1^2 - u_2)u_1 & \frac{1}{2}(u_1^2 - u_2)u_1 & u_1^3 \end{array}$$

The preceding discussion leads to this refinement of Proposition 3.3:

Theorem 5.1. *For any assignment of rational numbers to the variables u_α , the linear operator Φ on \mathcal{Q} defined by*

$$(5.2) \quad \Phi(M_\beta) = \sum_{\alpha \leq \beta} A_n(\beta, \alpha) M_\alpha \quad \text{for all } n \geq 0 \text{ and } \beta \in \text{Comp}(n)$$

is an endomorphism of \mathcal{Q} . Moreover, every endomorphism of \mathcal{Q} has the form (5.2).

Example 5.2. We apply Theorem 5.1 to construct an endomorphism that satisfies the nonnegativity hypotheses of Theorem 3.1 and hence gives rise to a QS^* -distribution.

Let $c_{\alpha, \beta}$ be as in (3.1). By a change of basis,

$$c_{\alpha, n} = \sum_{\beta \geq \alpha} A_n(\beta, n).$$

For $n = 3$, this gives

$$\begin{aligned} c_{3,3} &= \frac{1}{6}u_1^3 + \frac{1}{2}u_1 u_2 + \frac{1}{3}u_3 & c_{12,3} &= \frac{1}{6}u_1^3 - \frac{1}{2}u_1 u_2 + u_{12} + \frac{1}{3}u_3 \\ c_{21,3} &= \frac{1}{6}u_1^3 + \frac{1}{2}u_1 u_2 - u_{12} - \frac{2}{3}u_3 & c_{111,3} &= \frac{1}{6}u_1^3 - \frac{1}{2}u_1 u_2 + \frac{1}{3}u_3 \end{aligned}$$

Let us choose a specialization of the variables so that the $c_{\alpha,3}$ are nonnegative and not identically zero. For instance, set $u_1 = 2, u_2 = 1/2, u_3 = 2, u_{12} = -1$, and assign arbitrary values to all other u_α . Then $c_{3,3} = 5/2, c_{12,3} = 1/2, c_{21,3} = 3/2, c_{111,3} = 3/2$, and the corresponding QS^* -distribution is (using (3.10))

$$\sum_{\sigma \in \mathfrak{S}_3} \text{Prob}_\Phi(\sigma) \sigma = \frac{1}{2^3} \cdot \sum_{\alpha \in \text{Comp}(3)} c_{\alpha,3} Y_\alpha = \frac{5}{16} \cdot \mathbf{123} + \frac{1}{16} \cdot (\mathbf{213} + \mathbf{312}) + \frac{3}{16} \cdot (\mathbf{132} + \mathbf{231} + \mathbf{321}).$$

We can quickly find the eigenvalues for the transition probability matrix \overline{K} of the lumped random walk by reading off the diagonal entries of A_n and multiplying by $1/2^3$; they are $1, 1/4, 1/8, 1/8$.

Example 5.3. For an arbitrary endomorphism, if the associated eigenvectors Z_1, Z_2, Z_3 exist (see § 4.1), then they are given by

$$\begin{aligned} Z_1 &= X_1, & Z_2 &= X_2 - \frac{1}{2}X_{11}, \\ Z_3 &= X_3 + \frac{u_{12}}{u_3 - u_1 u_2} X_{12} + \frac{u_1 u_2 - u_3 - u_{12}}{u_3 - u_1 u_2} X_{21} + \frac{1}{3}X_{111}. \end{aligned}$$

5.2. Algorithm for computing A_n . One might hope for a reasonable algorithm for computing the polynomials appearing in the matrix A_n , which amounts to computing the polynomial expansion of an arbitrary M_α in terms of those M_β with $\beta \in \mathcal{L}$. However, the Lyndon expansion for M_α is often exponential in length, precluding the existence of an efficient algorithm.

An inefficient recursive algorithm is as follows: for $\alpha = (a_1, \dots, a_l)$ choose the earliest k such that $(a_k, a_{k+1}, \dots, a_{l-1}, a_l)$ is Lyndon. Then $M_{(a_1, \dots, a_{k-1})} M_{(a_k, \dots, a_l)}$ is a sum over all ways of quasi-shuffling (a_1, \dots, a_{k-1}) with (a_k, \dots, a_l) , in the sense of [22]. This gives a straightening law on monomial quasisymmetric functions which may be used to reduce any monomial quasisymmetric function into a sum of products of Lyndon ones in a finite number of steps; one summand in $M_{(a_1, \dots, a_{k-1})} M_{(a_k, \dots, a_l)}$ is M_α while all other summands either have strictly fewer parts than α or else are lexicographically smaller of the same length as α ; in either case, they are closer to Lyndon.

6. CONNECTIONS TO THE QS-DISTRIBUTION AND RANDOM WALKS OF BIDIGARE, HANLON, AND ROCKMORE

Throughout this section, let (r_1, r_2, \dots) be an infinite sequence of nonnegative rational numbers summing to 1. The *QS-distribution* on \mathfrak{S}_n may be defined [34, Theorem 2.1] as the probability distribution on \mathfrak{S}_n in which a permutation π is selected with probability

$$\text{Prob}_{QS}(\pi) = F_{D(\pi^{-1})}(r_1, r_2, \dots).$$

The link between the *QS-distribution* and the *QS*-distribution* comes from the observation that the evaluation map on \mathcal{Q} sending x_i to r_i for all i is a character, and so by Proposition 3.7 there is a unique $\Phi \in \text{End}(\mathcal{Q})$ such that

$$(6.1) \quad \langle X^\Phi \mid G(x_1, x_2, \dots) \rangle = G(r_1, r_2, \dots) \quad \text{for all } G \in \mathcal{Q}.$$

This leads to the following result:

Theorem 6.1. *There exists $\Phi \in \text{End}(\mathcal{Q})$ such that for every permutation π ,*

$$(6.2) \quad \text{Prob}_\Phi(\pi) = \text{Prob}_{QS}(\pi^{-1})$$

In other words, the transition probability matrix of the left random walk on \mathfrak{S}_n driven by the QS-distribution is K_n^Φ , with $K_n^\Phi(\pi, \sigma\pi) = \text{Prob}_{QS}(\sigma)$ for all $\sigma, \pi \in \mathfrak{S}_n$.

Proof. Choose Φ so that (6.1) holds. By (3.7) and (3.10),

$$F_{D(\pi)}(r_1, r_2, \dots) = \langle X^\Phi \mid F_{D(\pi)} \rangle = \lambda^n \text{Prob}_\Phi(\pi).$$

We have $\lambda = \langle X^\Phi \mid M_1 \rangle = M_1(r_1, r_2, \dots) = \sum r_i = 1$, completing the proof. \square

Remark 6.2. The *QS-distribution* is an instance of a transformation of alphabets in the context of noncommutative symmetric functions [24]. This viewpoint is developed in further in [15, Section 3.6], where the main results of [34] are deduced from the theory of free quasisymmetric functions.

Let us describe more explicitly the entries of K and \overline{K} . Let $\delta_1, \delta_2, \dots$ be i.i.d. random variables such that

$$\text{Prob}(\delta_i = j - 1) = r_j \quad \text{for } j \text{ a positive integer.}$$

Given a sequence of distinct numbers $\pi = (\pi_1, \dots, \pi_n)$, let $\text{st}(\pi)$ denote the standardization of π , that is, the unique permutation $(\sigma_1, \dots, \sigma_n) \in \mathfrak{S}_n$ such that for all $i, j \leq n$, $\pi_i < \pi_j$ if and only if $\sigma_i < \sigma_j$. For instance, $\text{st}((-2, 1, 3, -4)) = (2, 3, 4, 1)$.

Theorem 6.3. *Let Φ be as in Theorem 6.1. For all $\sigma, \tau \in \mathfrak{S}_n$ and $\beta \in \text{Comp}(n)$, we have*

$$K_n^\Phi(\sigma, \tau) = \text{Prob}(\text{st}((\sigma_1 + \delta_1 n, \sigma_2 + \delta_2 n, \dots, \sigma_n + \delta_n n)) = \tau)$$

and

$$\overline{K}_n^\Phi(D(\sigma), \beta) = \text{Prob}(D((\sigma_1 + \delta_1 n, \sigma_2 + \delta_2 n, \dots, \sigma_n + \delta_n n)) = \beta).$$

Proof. Let $\pi = \sigma\tau^{-1}$. By Theorem 6.1,

$$K_n^\Phi(\sigma, \tau) = F_{D(\pi)}(r_1, r_2, \dots) = \sum_{\substack{i_1 \leq i_2 \leq \dots \leq i_n \\ i_k \in S_{D(\pi)} \implies i_k < i_{k+1}}} r_{i_1} \cdots r_{i_n}.$$

In other words, $K(\sigma, \tau)$ is the probability that

$$(6.3) \quad \delta_1 \leq \delta_2 \leq \dots \leq \delta_n \quad \text{and} \quad \delta_i < \delta_{i+1} \quad \text{whenever } i \in S_{D(\pi)}.$$

Now consider instead the probability that $\text{st}((\sigma_1 + \delta_1 n, \sigma_2 + \delta_2 n, \dots, \sigma_n + \delta_n n)) = \tau = \pi^{-1}\sigma$. Let i_1, \dots, i_n be indices such that $\sigma_{i_j} = \pi_j$ for all j . Multiplying σ on the left by π^{-1} has the effect of replacing the occurrence of σ_{i_j} in $(\sigma_1, \dots, \sigma_n)$ by the value j . The resulting permutation equals $\text{st}((\sigma_1 + \delta_1 n, \sigma_2 + \delta_2 n, \dots, \sigma_n + \delta_n n))$ if and only if $\pi_1 + \delta_{i_1} n < \pi_2 + \delta_{i_2} n < \dots < \pi_n + \delta_{i_n} n$, and this occurs if and only if

$$(6.4) \quad \delta_{i_1} \leq \delta_{i_2} \leq \dots \leq \delta_{i_n} \quad \text{and} \quad \delta_{i_j} < \delta_{i_{j+1}} \quad \text{whenever } j \in S_{D(\pi)}.$$

Since the δ_i 's are i.i.d., events (6.3) and (6.4) occur with equal probability. The formula for \overline{K} follows via lumping. \square

The eigenvalues of \overline{K} can be determined immediately from Proposition 4.1 (cf. [34, Theorem 2.2] and references thereafter).

Proposition 6.4. *Let Φ be as in Theorem 6.1. For any $n \geq 0$, the eigenvalues of \overline{K} are the power sum symmetric functions $p_\mu(r_1, r_2, \dots)$ where μ ranges over all partitions of n . The multiplicity of $p_\mu(r_1, r_2, \dots)$ is the number of different compositions obtainable by rearranging the parts of the partition μ .*

Bidigare, Hanlon, and Rockmore (BHR) introduced a class of random walks on chambers of central hyperplane arrangements [5], generalizing many shuffling and sorting schemes like the Tsetlin Library. Their work was developed further by Brown and Diaconis [4] and Brown [10, 11]. In some cases a BHR random walk is isomorphic to a right random walk on a finite reflection group. For instance Stanley showed that the transpose of the right random walk on \mathfrak{S}_n driven by the QS -distribution is an instance of a BHR random walk on the chambers of the braid arrangement [34]. More generally, Theorem 8 of [10] implies that if $W = \sum_{\sigma \in \mathfrak{S}_n} W(\sigma)\sigma$ is a probability distribution on \mathfrak{S}_n such that $W \in \mathcal{D}_n$ and the

expansion of W in the basis $\{X_\alpha\}$ has nonnegative coefficients, then right multiplication by W is isomorphic to a BHR random walk. Such a distribution W will be called a *BHR-distribution*.

Not every QS^* -distribution is a BHR-distribution. The Θ -map provides one family examples: for instance when $n = 3$,

$$(X^\Theta)_3 = 2^3 \cdot \sum_{\sigma \in \mathfrak{S}_3} \text{Prob}_\Theta(\sigma)\sigma = 2(Y_3 + Y_{12} + Y_{111}) = 2(X_3 - X_{21} + X_{111}).$$

The endomorphism Φ considered in Example 5.2 provides another example:

$$(X^\Phi)_3 = 2^3 \cdot \left[\frac{5}{16}Y_3 + \frac{1}{16}Y_{12} + \frac{3}{16}(Y_{21} + Y_{111}) \right] = 2X_3 - X_{12} + \frac{3}{2}X_{111}.$$

Note that BHR random walks on \mathfrak{S}_n are right random walks, whereas in this paper we are dealing with left random walks (encoded by K) and their lumped versions (encoded by \overline{K}). This distinction might not matter much in practice, since there is no difference between left and right random walks if we start at the identity permutation. Also, all of the associated matrices—left or right, lumped or not—that arise from a BHR-distribution on \mathfrak{S}_n have the same eigenvalues, thanks to [10, Theorem 8] (cf. [24, Theorem 3.12]).

The references [5, 12, 10] all give bounds on rates of convergence to the stationary distribution. Here we will not pursue the problem of estimating the convergence rates of K or \overline{K} , although it would be interesting to see whether any techniques from those papers could be adapted to our setting.

7. CONNECTIONS TO a -SHUFFLES AND THE TCHEBYSHEV OPERATOR

7.1. a -shuffles. Let a be a positive integer. An a -*shuffle* of a deck of cards involves cutting the deck into a (possibly empty) packets according to the multinomial distribution and then letting cards fall one at a time from the bottoms of the packets into new pile, where the probability that the bottom card from a particular packet falls is proportional to the current size of the packet. When $a = 2$ this is just the standard Gilbert-Shannon-Reeds (GSR) model of riffle shuffling. See [4] details.

A well-known formula due to Bayer and Diaconis [4] for the probability that the deck is in arrangement π after an a -shuffle is

$$(7.1) \quad \text{Prob}_a(\pi) = \binom{n + a - d(\pi^{-1}) - 1}{n} / a^n,$$

where $d(\pi)$ is the number of descents in π .

Let Ψ_a denote the unique endomorphism of \mathcal{Q} such that $X^{\Psi_a} = X \star \cdots \star X$ (a terms), where X is the universal character introduced in Section 3. This convolution character is discussed in [2, Example 4.7], where an explicit formula for Ψ_a is given.

Proposition 7.1. *For every permutations π ,*

$$(7.2) \quad \text{Prob}_{\Psi_a}(\pi) = \text{Prob}_a(\pi^{-1}) = \text{Prob}_{QS}(\pi^{-1}),$$

where the QS -distribution has $r_1 = r_2 = \cdots = r_a = \frac{1}{a}$ and $r_i = 0$ for $i > a$.

Proof. The second equality was shown by Stanley [34]. Using (3.4) it is easy to obtain a formula for Ψ_a in the monomial basis (or see [2, (4.5)]), from which it follows that

$$\overline{K}_n^{\Psi_a}(\beta, n) = \binom{n+a-\ell(\beta)}{n} / a^n.$$

This is equivalent to the formula of Bayer and Diaconis, as $\ell(D(\pi)) = d(\pi) + 1$. \square

Proposition 7.1 can also be inferred from [15, §3.6].

The eigenvalues and diagonalizability of operators associated with a -shuffles are well known; see, for instance, [4, 5, 10]. Noting that $\langle X^{\Psi_a} \mid M_n \rangle = a$, the results of Section 4 imply the following:

Proposition 7.2. *For all $n \geq 0$, $(\Psi_a)_n$ is diagonalizable. Its eigenvalues, with multiplicities, are $(a^{\ell(\alpha)})_{\alpha \in \text{Comp}(n)}$.*

It is also possible to determine an explicit formula for the eigenvectors:

Proposition 7.3. *For every $n \geq 0$, the unique eigenvector Z_n such that $X^{\Psi_a} \cdot Z_n = aZ_n$ and $\langle Z_n \mid M_n \rangle = 1$ is given by*

$$(7.3) \quad Z_n = \sum_{\beta \in \text{Comp}(n)} \frac{(-1)^{\ell(\beta)-1}}{\ell(\beta)} X_\beta.$$

Proof. For compositions $\alpha \leq \beta$ as above, let

$$g(\beta, \alpha) = \binom{a}{\ell(\beta_1)} \cdots \binom{a}{\ell(\beta_k)}$$

where $\beta = \beta_1 \cdots \beta_k$ and $\alpha = (|\beta_1|, \dots, |\beta_k|)$. It follows from (3.4) that $\langle X^{\Psi_a} \cdot X_\beta \mid M_\alpha \rangle = g(\beta, \alpha)$. If (7.3) holds then

$$X^{\Psi_a} \cdot Z_n = \sum_{\beta \in \text{Comp}(n)} \frac{(-1)^{\ell(\beta)-1}}{\ell(\beta)} \sum_{\gamma \geq \beta} g(\gamma, \beta) X_\gamma = \sum_{\gamma \in \text{Comp}(n)} X_\gamma \left(\sum_{\beta \leq \gamma} \frac{(-1)^{\ell(\gamma)-1}}{\ell(\gamma)} g(\gamma, \beta) \right).$$

So, to conclude that $X^{\Psi_a} \cdot Z_n = aZ_n$ it suffices to prove

$$(7.4) \quad \sum_{\beta \leq \gamma} \frac{(-1)^{\ell(\gamma)-1}}{\ell(\gamma)} g(\gamma, \beta) = a \frac{(-1)^{\ell(\gamma)-1}}{\ell(\gamma)}.$$

We may assume without loss of generality that $\gamma = 1^n$. In this case (7.4) becomes

$$\sum_{(c_1, \dots, c_h) \models n} \frac{(-1)^{h-1}}{h} \binom{a}{c_1} \cdots \binom{a}{c_h} = a \frac{(-1)^{n-1}}{n},$$

which is verified by equating the coefficients of x^n in

$$\ln(1 + ((1+x)^a - 1)) = a \ln(1+x).$$

\square

Remark 7.4. Under the isomorphism between $(\mathcal{D}, \star, \Delta_{\mathcal{D}})$ and the Hopf algebra of noncommutative symmetric functions, $n \cdot Z_n$ goes to the noncommutative power sum symmetric function of the second kind indexed by n (see [19] for definitions).

Let $\{P_{\alpha}\} \subseteq \mathcal{Q}_n$ be the dual basis of $\{Z_{\alpha}\}$. Thus, $\langle Z_{\alpha} | P_{\beta} \rangle = \delta_{\alpha, \beta}$ and

$$\Psi_a(P_{\beta}) = a^{\ell(\beta)} P_{\beta}.$$

This basis was introduced in [27]. An explicit formula for P_{α} is obtained as follows. For compositions $\alpha = (a_1, \dots, a_k)$ and $\beta = (b_1, \dots, b_{\ell})$ such that $\alpha \leq \beta$, let $\beta_1, \beta_2, \dots, \beta_k$ be the sequence of compositions such that $\beta = \beta_1 \cdots \beta_k$ and $|\beta_i| = a_i$ for all i . Let $f(\beta, \alpha) = \ell(\beta_1)! \cdots \ell(\beta_k)!$. Then we have

$$(7.5) \quad P_{\beta} = \sum_{\alpha \leq \beta} \frac{1}{f(\beta, \alpha)} M_{\alpha}.$$

This is [27, Formula (2.12)].

7.2. Tchebyshev operator of the second kind. The Tchebyshev operator (of the second kind) was introduced by Hetyei [21] as an operator on ab -words that encodes how the flag f -vector of a graded poset changes when the poset undergoes a certain combinatorial transformation. The Tchebyshev polynomials of the second kind can be obtained by a suitable specialization of the operator. Ehrenborg and Readdy [18] introduced an equivalent operator U on \mathcal{Q} and showed that it is a Hopf endomorphism. They showed that U is diagonalizable and determined the eigenvalues and eigenvectors. Here we explain how the spectra can be deduced from our results and how U encodes riffle shuffles.

Given a graded poset P of rank $n \geq 0$ with unique minimal and maximal elements $\hat{0}$ and $\hat{1}$, respectively, define $F(P) \in \mathcal{Q}_n$ by $F(P) = 1$ if $n = 0$ and

$$F(P) = \sum_{\hat{0}=t_0 < t_1 < \dots < t_k = \hat{1}} M_{(\rho(t_0, t_1), \rho(t_1, t_2), \dots, \rho(t_{k-1}, t_k))}$$

if $n \geq 1$, where the sum is over all chains in P from $\hat{0}$ to $\hat{1}$. The definition of $F(P)$ is due to Ehrenborg [16]. See [32] for background on posets. The *Tchebyshev operator of the second kind* [18] is the linear operator U on \mathcal{Q} satisfying

$$U(F(P)) = \sum_{\hat{0}=t_0 < t_1 < \dots < t_k = \hat{1}} \langle G | F([t_0, t_1]) \rangle \cdots \langle G | F([t_{k-1}, t_k]) \rangle \cdot M_{(\rho(t_0, t_1), \dots, \rho(t_{k-1}, t_k))}$$

for every graded poset P , where $G \in \hat{\mathcal{D}}$ is the character given by $\langle G | F(P) \rangle = \#P$.

Proposition 7.5. *We have*

$$(7.6) \quad U = \Psi_2.$$

Proof. It is straightforward to check that $G = X \star X$, using the fact that $P \mapsto F(P)$ defines a Hopf algebra homomorphism from the Hopf algebra of graded posets to \mathcal{Q} [16] and that \mathcal{Q} is spanned by the $F(P)$ as P ranges over all graded posets [9]. Comparing (3.4) with the definition of U , we get $U \circ F = \Psi_2 \circ F$, which implies $U = \Psi_2$ since the $F(P)$ span \mathcal{Q} . \square

Formulas for the eigenvalues and eigenvectors are given in Proposition 7.2 and (7.5) (cf. [18, Theorem 10.10]). Also, Proposition 7.1 yields the following probabilistic interpretation:

Corollary 7.6. *The coefficient of F_α in the fundamental-basis expansion of $\frac{1}{2^{nk}}U^k(F_n)$ is the probability of ending up with some permutation with descent composition $\alpha \in \text{Comp}(n)$ after performing k riffle shuffles.*

A natural question is whether there is a transformation of graded posets for which the operator Ψ_m , $m > 2$, plays the role analogous to the Tchebyshev operator U .

8. A RANDOM WALK ON ab -WORDS

The cd -index is a remarkably convenient encoding for the flag f -vector of any Eulerian poset (e.g. Bruhat order), namely the vector counting chains through various rank sets (e.g. flags of faces of specified dimensions in a regular cell decomposition of a sphere). The ab -index is a vector which makes sense for all graded posets, and has as its basis all words in the two noncommuting variables a and b . The cd -index is derived from the ab -index by the substitutions $c = a + b$ and $d = ab + ba$. There is an extensive literature using cd -index to study which vectors may arise as flag f -vectors. See e.g. [7] for further background.

In [17], Ehrenborg introduced a family of linear operators ω_r which act on ab -words as follows. Given an ab -word, replace each appearance of ab by $r(ab - (r-1)ba)$. Now send each remaining a from the original word to $a + (r-1)b$ and send each remaining b from the original word to $b + (r-1)a$. This operator, which generalizes the ω map of [7], was designed to compute flag vectors of the r -Birkhoff transform, perhaps most notably including a family of complex regular polytopes known as Shephard's generalized orthotopes. Ehrenborg showed that the operator $r\omega_r$ is isomorphic to a Hopf algebra endomorphism $\vartheta_r \in \text{End}(\mathcal{Q})$ and he made some conjectures about the spectrum. This section proves Ehrenborg's conjectures. We will then show that ϑ_r is dual to the q -bracketing operators studied in [24]. Finally, we give a probabilistic interpretation of ϑ_r that generalizes a theorem of Stembridge [35].

First let us define ϑ_r . Given an ab -word $u = u_1u_2 \cdots u_n$, where each u_i is a or b , we define $S_u \subseteq [n]$ by putting $i \in S_u$ if and only if $u_i = b$. For instance $S_{abaab} = \{2, 5\}$. The correspondence $u \leftrightarrow S_u$ is a bijection between ab -words of length n and subsets of $[n]$. This bijection induces an isomorphism, denoted γ , from the vector space of ab -words onto $\bigoplus_{i=1}^{\infty} \mathcal{Q}_i$, given by $\gamma(u) = F_{\text{co}(S_u)}$. Define ϑ_r to be the linear operator on \mathcal{Q} given by $\vartheta_r(F) = \gamma(r\omega_r(\gamma^{-1}(F)))$ if $F \in \bigoplus_{i=1}^{\infty} \mathcal{Q}_i$ and $\vartheta_r(1) = 1$. Ehrenborg [17] proved that ϑ_r is a Hopf algebra endomorphism of \mathcal{Q} and that ϑ_2 is Stembridge's map Θ , and he made the follow conjecture on the spectrum:

Proposition 8.1. *For all $n \geq 0$ and $r \in \mathbb{R}$, $(\vartheta_r)_n$ is diagonalizable and its eigenvalues, counting multiplicities, are*

$$\lambda_\alpha = (1 - (1 - r)^{a_1})(1 - (1 - r)^{a_2}) \cdots (1 - (1 - r)^{a_k})$$

for $\alpha = (a_1, a_2, \dots, a_k) \in \text{Comp}(n)$.

Proof. For any ab -word u , $r^n \overline{K}(\text{co}(S_u), n)$ is the coefficient of a^{n-1} in $r \cdot \omega_r(u)$ (here $\overline{K} = \overline{K}_n^{\vartheta_r}$). This coefficient is straightforward to compute using the definition of ω_r : it is $r(r-1)^k$ if u

has the form $b^k a^{n-1-k}$ and 0 otherwise. It follows that

$$(8.1) \quad \overline{K}(\alpha, n) = \begin{cases} \frac{(r-1)^k}{r^{n-1}} & \text{if } \alpha = (1^k, n-k) \text{ for } 0 \leq k \leq n-1, \\ 0 & \text{otherwise.} \end{cases}$$

Note that

$$\vartheta_r(M_n) = \vartheta_r \left(\sum_{\alpha \in \text{Comp}(n)} (-1)^{\ell(\alpha)-1} F_\alpha \right) = \sum_{\gamma \in \text{Comp}(n)} M_\gamma \sum_{\substack{\alpha \in \text{Comp}(n) \\ \beta \leq \gamma}} (-1)^{\ell(\alpha)-1} r^n \overline{K}(\alpha, \beta).$$

Since λ_n is the coefficient of M_n in the monomial expansion of $\vartheta_r(M_n)$, we have

$$\lambda_n = \sum_{\alpha \in \text{Comp}(n)} (-1)^{\ell(\alpha)-1} r^n \overline{K}(\alpha, n) = \sum_{k=0}^{n-1} (-1)^k r (r-1)^k = 1 - (1-r)^n.$$

The formula for the eigenvalues now follows from Proposition 4.1 and the diagonalizability follows from Theorem 4.3. \square

Remark 8.2. If $r \geq 1$ and $\pi \in \mathfrak{S}_n$ then $\text{Prob}_{\vartheta_r}(\pi)$ is the probability that a deck of n cards is in arrangement π after performing an inverse r -weighted face up, face down shuffle: Remove a subset of cards from the deck to form a new packet, where a card is selected independently for removal with probability $1 - \frac{1}{r}$ and the cards are kept in the same relative order, and then place the new packet face up on top of the pile of remaining cards, which are face down.

In [24] it was shown that the $A \mapsto (1-q)A$ transform on noncommutative symmetric functions is equivalent to the left action on the descent algebra \mathcal{D}_n by

$$\eta_q = (1-q) \cdot \sum_{k=0}^{n-1} (-q)^k Y_{(1^k, n-k)}.$$

This action was shown to be diagonalizable and formulas for the eigenvalues and eigenvectors were given. The eigenvalues turn out to be the same as the eigenvalues for ϑ_{1-q} . This is explained by the following result.

Proposition 8.3. *The restriction of ϑ_r to \mathcal{Q}_n is dual to the left action of η_{1-r} on \mathcal{D}_n ; that is,*

$$\langle \eta_{1-r} \cdot W \mid G \rangle = \langle W \mid \vartheta_r(G) \rangle$$

for all $W \in \mathcal{D}_n$ and $G \in \mathcal{Q}_n$. In other words, $(X^{\vartheta_r})_n = \eta_{1-r}$.

Proof. Using the formula for $\overline{K}(\alpha, n)$ derived in the proof of Proposition 8.1, we have

$$\eta_{1-r} = r \sum_{k=0}^{n-1} (r-1)^k Y_{(1^k, n-k)} = \sum_{\alpha \in \text{Comp}(n)} \overline{K}(\alpha, n) Y_\alpha = (X^{\vartheta_r})_n.$$

Applying Proposition 3.7 completes the proof.

This result may also be proven by constructing an explicit isomorphism from ϑ_r to the dual of η_{1-r} . Consider an ab -word upon which Ehrenborg's operator acts. Send each "a" to a descent and send each "b" to an ascent. Send r to $1-q$. Each "ab" now corresponds to a valley, and for each set S , the allowable sets T in the image are as follows. Each

valley must be sent either to a valley or to a peak for a set to be allowable, but this is the only requirement. Now for each T , the coefficient $r^m(r-1)^n$ is obtained as follows. The exponent m counts valleys in S , just as the exponent for $1-q$ in η_q does. The exponent n counts discrepancies between S and T , with each valley/peak combination counted as a single discrepancy. Thus, we obtain the following formula, which is the dual version of the formula for the operator η_q proven in [24, Proposition 5.41]:

$$\vartheta_{1-q}(F_\beta) = (1-q)^{hl(\beta)} \sum_{\Lambda(\beta) \subseteq S_\alpha \Delta(S_{\alpha+1})} (-q)^{b(S_\alpha, S_\beta)} F_\alpha,$$

where β can be written as a concatenation $\beta = \beta_1 \cdots \beta_{hl(\beta)}$ of hook compositions $\beta_i = (1^k, l)$ and $b(S, T) = |(1 + (S \setminus T)) \cup (T \setminus S)|$. \square

Given $r > 1$, let $\delta_{r,1}, \delta_{r,2}, \dots$ be i.i.d. random variables taking on values in $\{1, -1\}$ such that

$$\text{Prob}(\delta_{r,i} = 1) = \frac{1}{r}, \quad \text{and} \quad \text{Prob}(\delta_{r,i} = -1) = 1 - \frac{1}{r}.$$

Theorem 8.4. *For all $\sigma, \tau \in \mathfrak{S}_n$ and $\beta \in \text{Comp}(n)$, we have*

$$K_n^{\vartheta_r}(\sigma, \tau) = \text{Prob}(\text{st}((\delta_{r,1}\sigma_1, \dots, \delta_{r,n}\sigma_n)) = \tau)$$

and

$$\overline{K}_n^{\vartheta_r}(D(\sigma), \beta) = \text{Prob}(D((\delta_{r,1}\sigma_1, \dots, \delta_{r,n}\sigma_n)) = \beta).$$

Proof. Left multiplying σ by $Y_{(1^k, n-k)}$ has the effect of negating all possible choices of $k-1$ values chosen from $[2, n]$ within the permutation σ , then standardizing values to obtain a permutation, since this negation reverses relative order of the values being negated and makes them smaller than all other values. Notice also that negating the value 1 has no impact on the relative order of values. The sum

$$\sum_{k=0}^{n-1} (r-1)^k Y_{(1^k, n-k)}$$

may therefore be viewed as a sum over all choices of which values in $[2, n]$ to negate, with k recording the number of values other than 1 being negated. If each value in $[n]$ is independently negated with probability $1 - \frac{1}{r}$, then

$$\sum_{k=0}^{n-1} \frac{(r-1)^k (1)^{n-1-k}}{r^{n-1}} Y_{(1^k, n-k)}$$

is a sum over all possible subsets of $[n]$ to be negated, with each possibility multiplied by its probability of being chosen. \square

Setting $r = 2$ in Theorem 8.4 yields Theorem 3.6 of Stembridge [35].

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