

Multi-factor Option Pricing

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Abstract

Multi-factor models provide considerable flexibility in representing the dynamic behavior of asset prices. European options can be written in terms of the prices of a set of (possibly artificial) assets and probabilities that the option expires in-the-money. The values of these terms must be evaluated with respect to appropriately transformed factor processes. For certain classes of models, all of the required terms can be computed by solving a system of ordinary differential equations, making the approach computationally tractable. The flexibility of the framework is illustrated by providing explicit solutions for pricing options on dividend paying spot assets, on futures written on spot assets, on zero-coupon bonds and zero-coupon bond futures.

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Recent empirical work has demonstrated the limitations of single factor models for capturing important aspects of asset price behavior. In stock price modeling, volatility smile effects seem best captured with the inclusion of an stochastic volatility process. In fixed income modeling, multiple factors are required to capture changes in the slope and curvature of the term structure. In commodity price modeling, incorporating the effect of stochastic convenience yields and interest rates significantly improves model fit to observed prices.

With the development of multi-factor models for asset prices comes the need to price options written on these assets. Option pricing formulas have been developed for a number of specific multi-factor models. Examples include the two-factor stochastic volatility option pricing models of Hull and White and Heston. Recently Miltersen and Schwartz developed a general framework for pricing options in a multi-factor Gaussian setting. Chen and Scott discuss pricing interest rate options for multi-factor Cox-Ingersoll-Ross type models. All of these models fall into the class of affine models. Beaglehole and Tenney and Jamshidian(1996) develop option pricing formulas for fixed income assets in the class of quadratic asset pricing models.

In this paper a simple characterization of European option values is presented for general multi-factor models. It is first presented for options on non-dividend paying assets and then generalized for dividend-paying assets. In the former case it is shown that an option can be evaluated in term of a bond price, the price of the underlying

asset and values that represent the probability that the option expires in-the-money under two alternative measures. With dividend-paying assets an additional term is required that represents the value of an artificial bond. Although the approach has been utilized before in specific cases, a simple and unified treatment applicable to a wide variety of option pricing problems is provided here.

The main computational problem for pricing options in the multi-factor setting is that, except in rare cases such as the Gaussian model, it is difficult to compute the required probabilities. Although the probabilities can be expressed in terms of a partial differential equation, the curse of dimensionality can make these difficult and costly to solve. There are two classes of models, however, for which the curse of dimensionality can be circumvented while still maintaining a considerable degree of flexibility. With both classes of models the characteristic functions associated with the desired probability functions can be expressed as a system of ordinary differential equations, which are easily solved. The probability values can then be recovered using the inverse Fourier transform. All of the option pricing models mentioned above, as well as many unexplored models, fit into the affine diffusion class and can be evaluated using the approach described here.

The discussion proceeds in three main sections. First, a formula for a European option on a dividend protected asset is developed that extends previous treatments by explicitly considering the effect of interest rate risk on the value of the option. The

basic valuation formula is then extended to assets that pay dividends; the general formula can also be used to price options on futures. The next section specializes this formula to the case of affine asset pricing models and provides a computationally tractable approach to pricing options based on multi-factor models. This is followed by an analogous section describing the quadratic asset pricing model. The paper ends with some brief concluding comments.

Pricing Options on Dividend-Protected Assets

A multi-factor diffusion process, x , is taken as the starting point of the modeling effort. Under the risk-neutral measure, the underlying factor process can be described by the stochastic differential equation

$$dx = \mu(x, t)dt + \sigma(x, t)dW,$$

where z is a n -vector of independent standard Weiner processes. The process x is composed of n factors, that may or may not be unobservable. In some cases, one of the factors may be the price of an asset on which the option of interest is written; in other cases, however, the price of the underlying asset may be a time-varying function of the factors, e.g., in pricing options on bonds or on futures.

Consider the value of a put option, $V(x, t; T)$, with strike price K , expiring at time T , that is written on a non-dividend paying asset with price process $S(x, t)$.

The basic result of this section is that value of the option can be written as

$$V(x, t; T) = B(x, t; T)K P^k(x, t; T) - S(x, t)P^s(x, t; T), \quad (1)$$

where B is the price of a risk-free bond paying one unit of account at time T , and where P^k and P^s equal the probability of expiring in-the-money evaluated under two alternative measures. This result is demonstrated by verifying the standard arbitrage condition¹

$$0 = V_t + V_x \mu + \frac{1}{2} \text{trace}(V_{xx} \Sigma) - rV, \quad (2)$$

where $\Sigma = \sigma \sigma^\top$ and r is the risk-free rate of interest (possibly a function of x). For a specific asset that pays $h(x)$ units of account at time T , it must also be verified that the terminal arbitrage condition $V(x, T; T) = h(x)$ is satisfied. For a put option the terminal condition is $h(x) = (K - S)1_{S \leq K}$, where 1_c equals 1 if c is true and 0 if not. For a pure discount bond the terminal condition is $h(x) = 1$.

Another well-known result (Cox and Miller) used below is that the function $f(x, t) = E_t[g(x(T))]$ satisfies the Kolmogorov backwards equation

$$0 = f_t + f_x \mu + \frac{1}{2} \text{trace}(f_{xx} \Sigma),$$

together with the terminal condition $f(x, T) = h(x)$. In particular, when $h(x) = 1_{S \leq K}$, $f(x, t)$ represents the time t conditional probability that the option expires in-the-money.

¹See Duffie for discussion, including regularity conditions.

To verify that (1) satisfies the arbitrage conditions, first substitute it and its derivatives into (2):²

$$\begin{aligned}
V_t &+ V_x \mu + \frac{1}{2} \text{trace}(V_{xx} \Sigma) - rV \\
&= \left[KP^k B_t + KBP_t^k - P^s S_t - SP_t^s \right] \\
&+ \left[KP^k B_x + KBP_x^k - P^s S_x - SP_x^s \right] \mu \\
&+ \frac{1}{2} \text{trace} \left(\left[KP^k B_{xx} + KB_x^\top P_x^k + KBP_{xx}^k + KP_x^{k\top} B_x \right. \right. \\
&\quad \left. \left. - P^s S_{xx} - S_x^\top P_x^s - SP_{xx}^s - P_x^{s\top} S_x \right] \Sigma \right) \\
&- r(BKP^k - SP^s).
\end{aligned}$$

The terms in this expression can be rearranged in the following way:

$$\begin{aligned}
&KP^k \left[B_t + B_x \mu + \frac{1}{2} \text{trace}(B_{xx} \Sigma) - rB \right] \\
&+ KB \left[P_t^k + P_x^k \left(\mu + \Sigma \frac{B_x^\top}{B} \right) + \frac{1}{2} \text{trace}(P_{xx}^k \Sigma) \right] \\
&- P^s \left[S_t + S_x \mu + \frac{1}{2} \text{trace}(S_{xx} \Sigma) - rS \right] \\
&- S \left[P_t^s + P_x^s \left(\mu + \Sigma \frac{S_x^\top}{S} \right) + \frac{1}{2} \text{trace}(P_{xx}^s \Sigma) \right].
\end{aligned}$$

Each of the bracketed terms is equal to zero and hence the arbitrage condition is satisfied. The first and third terms are arbitrage equations for the bond price and the price of the underlying asset, respectively, and are therefore identically zero. The second and fourth terms in brackets are Kolmogorov backwards equations applied to

²Subscripts on functions represent partial derivatives, with the convention that V_x is a $1 \times n$ vector.

factor processes with altered drift rates. For P^k the drift rate is altered to reflect its covariance with the bond price

$$\mu^k = \mu + \Sigma \frac{B_x^\top}{B}$$

($P_x^k \Sigma B_x^\top$ is the instantaneous covariance between P^k and B). For P^s , the drift is adjusted for its covariance with the price of the underlying asset:³

$$\mu^{sn} = \mu + \Sigma \frac{S_x^\top}{S}$$

It remains to verify that the terminal arbitrage condition is satisfied. Given that $B(x, T; T) = 1$ and $P^k(x, T; T) = P^s(x, T, T) = 1_{S \leq K}$, this can be verified by inspection.

The basic result (1) simplifies when the bond price and the price of the underlying asset are stochastically independent, as would be true if the bond price is treated as deterministic. Such an assumption underlies a number of standard option pricing models, including the Black-Scholes model and the stochastic volatility models of Hull and White and of Heston. In this case, the covariance term $P_x^k \Sigma B_x^\top$ is zero and thus one can set $\mu^k = \mu$.

³The superscript sn is meant to distinguish this drift term for non-dividend paying assets from the more general form presented in the next section.

Pricing Options on Dividend-Paying Assets

Suppose now that the underlying asset pays a proportional dividend, $w(x, t)$. The arbitrage condition for the asset's price must now be altered to reflect this:

$$0 = wS + S_t + S_x\mu + \frac{1}{2}\text{trace}(S_{xx}\Sigma) - rS. \quad (3)$$

In this case the basic result (1) must be modified to account for the fact that the holder of the underlying asset receives the dividend stream but the holder of the option does not. The modification involves an additional asset, $D(x, t; T)$, that can be thought of as a bond with a value based on the dividend rate w rather than on the spot interest rate r . Specifically,

$$V(x, t; T) = B(x, t; T)KP^k(x, t; T) - D(x, t; T)S(x, t)P^s(x, t; T). \quad (4)$$

Verification proceeds much as before. First substitute the appropriate terms into (3):

$$\begin{aligned} V_t &+ V_x\mu + \frac{1}{2}\text{trace}(V_{xx}\Sigma) - rV \\ &= \left[KP^k B_t + KBP_t^k - SP^s D_t - DP^s S_t - DSP_t^s \right] \\ &+ \left[KP^k B_x + KBP_x^k - SP^s D_x - DP^s S_x - DSP_x^s \right] \mu \\ &+ \frac{1}{2}\text{trace} \left(\left[KP^k B_{xx} + KB_x^\top P_x^k + KBP_{xx}^k + KP_x^k{}^\top B_x \right. \right. \\ &\quad \left. \left. - SP^s D_{xx} - SD_x^\top P_x^s - P^s D_x^\top S_x \right. \right. \\ &\quad \left. \left. - DP^s S_{xx} - DS_x^\top P_x^s - P^s S_x^\top D_x \right. \right. \\ &\quad \left. \left. - DSP_{xx}^s - DP_x^s{}^\top S_x + SP_x^s{}^\top D_x \right] \Sigma \right) \\ &- r(BKP^k - DSP^s). \end{aligned}$$

Then add and subtract the term $DSP^s w$ before rearranging terms:

$$\begin{aligned}
& KP^k \left[B_t + B_x \mu + \frac{1}{2} \text{trace}(B_{xx} \Sigma) - rB \right] \\
& + KB \left[P_t^k + P_x^k \left(\mu + \Sigma \frac{B_x^\top}{B} \right) + \frac{1}{2} \text{trace}(P_{xx}^k \Sigma) \right] \\
& - SP^s \left[D_t + D_x \left(\mu + \Sigma \frac{S_x^\top}{S} \right) + \frac{1}{2} \text{trace}(D_{xx} \Sigma) - wD \right] \\
& - DP^s \left[wS + S_t + S_x \mu + \frac{1}{2} \text{trace}(S_{xx} \Sigma) - rS \right] \\
& - DS \left[P_t^s + P_x^s \left(\mu + \Sigma \left(\frac{S_x^\top}{S} + \frac{D_x^\top}{D} \right) \right) + \frac{1}{2} \text{trace}(P_{xx}^s \Sigma) \right].
\end{aligned}$$

The first two terms in brackets are the same as before. The third term in brackets now represents the arbitrage condition for D , a bond maturing at time T that is evaluated “as if” the risk free rate of return is $w(x, t)$ and the drift term on x is

$$\mu^d = \mu + \Sigma \left(\frac{S_x^\top}{S} \right).$$

Also the fifth term involves an additional modification of the drift used to evaluate P^s :

$$\mu^s = \mu + \Sigma \left(\frac{S_x^\top}{S} + \frac{D_x^\top}{D} \right).$$

It is easy to verify that the previous result is indeed a special case in which the dividend rate is zero. In this case the artificial bond price, D , is the current value of one unit of account at time T when the spot interest rate is zero, which is identically equal to one.

The general expression (4) can be used to price options on futures by recognizing that a futures price is equal to the price of an asset that pays a dividend rate equal

to the spot interest rate, r (Cox et al.). Hence the artificial bond is evaluated at the true spot interest rate, but with a transformed drift term. Furthermore, if the futures price and the interest rate are stochastically independent, the covariance term in μ^d can be ignored (as can the D_x term in μ^s). The option valuation formula then becomes

$$V = B(KP^k - SP^s),$$

which is recognizable, for example, as a generalization of the formula for options on futures given by Black.

The general method can now be summarized. Observe $S(x, t)$ and $B(x, t; T)$ or compute them using the usual arbitrage conditions with the risk-neutral drift μ . Compute the value of the artificial bond $D(x, t; T)$ using $w(x, t)$ as the spot interest rate and μ^d as the drift term. Compute the probability, $P^k(x, t; T)$, of expiring in-the-money with drift μ^k and the probability, $P^s(x, t; T)$, of expiring in-the-money with drift μ^s .

It is easy to extend these results to pricing call options. The time T payout on a call can be written as $S(1 - \delta_{S \leq K}) - K(1 - \delta_{S \leq K})$; thus the same PDE holds but with a different boundary condition. It is easy to verify that the PDE and the boundary conditions are satisfied at

$$DS(1 - P^s) - BK(1 - P^k).$$

This in turn yields the put-call parity relationship:

$$\text{Call-Put} = DS - BK.$$

The main difficulty with implementing the this approach to valuation lies in computing the probability terms. In general this requires the solution of multi-state partial differential equations. In some cases, however, this can be avoided. The next two sections discuss two such cases, the so-called affine and quadratic asset pricing models.

Option Valuation for Affine Asset Pricing Models

Affine models have been used widely, particularly in the pricing of fixed income assets. A general description of the affine model in this context is provided in Duffie and Kan, with further developments in Dai and Singleton. The commodity price models of Schwartz are Gaussian special cases of the affine class, with option pricing discussed in Miltersen and Schwartz. Also some stochastic volatility option pricing models are based on the affine model, including Hull and White, Heston and Chen and Scott.

An affine diffusion is a diffusion in which the instantaneous mean and variance are both affine in the levels of the factors. In stochastic differential equation form they

can be represented by

$$dx = [a(t) + A(t)x]dt + C(t)\text{diag}\left(\sqrt{b(t) + B(t)x}\right) dW. \quad (5)$$

To facilitate navigating the notation used here, the following conventions are used with coefficients: capital letters denote matrices, lower case letters denote vectors and the subscript 0 is used on a scalar coefficient in association with a coefficient vector, e.g., in an expression such as $g_0 + x^\top g$. All coefficients may be functions of time or, where appropriate, time-to-maturity.

Consider a time T maturing asset which depends on the state variables. The asset has a terminal payout at time T of $\exp(h_0(T) + h(T)x(T))$. Prior to time T it pays a dividend at rate $w(t) = w_0(t) + w(t)x(T)$. The default-free short interest rate is $r_0(t) + r(t)x(T)$. Using standard no-arbitrage arguments the value of such an asset, $V(x, t; T)$, satisfies

$$0 = V_t + V_x(a + Ax) + \frac{1}{2}\text{trace}\left(C\text{diag}(b + Bx)C^\top V_{xx}\right) - (g_0 + gx)V, \quad (6)$$

where $g_0(t) = r_0(t) - w_0(t)$ and $g(t) = r(t) - w(t)$, subject to the boundary condition

$$V(x, T; T) = \exp\left(h_0(T) + h(T)x(T)\right)$$

(note: g , h , w , and r are all $1 \times n$). These specifications define what has been called the affine asset pricing model (e.g. Singleton).

The solution to this problem has the exponential affine form

$$V(x, t; T) = \exp(\beta_0(T - t) + \beta(T - t)x(t))$$

where β_0 and β satisfy the system of (Riccati) differential equations

$$\beta'_0(\tau) = \beta(\tau)a + \frac{1}{2}\beta(\tau)C\text{diag}\left(\beta(\tau)C\right)a - g_0$$

and

$$\beta'(\tau) = \beta(\tau)A + \frac{1}{2}\beta(\tau)C\text{diag}\left(\beta(\tau)C\right)B - g$$

(the parameters a, A, b, B, C, g_0, g are all evaluated at $t = T - \tau$). The initial conditions for these differential equations are $\beta_0(0) = h_0(T)$ and $\beta(0) = h(T)$. To facilitate exposition, we will denote the functional operator that maps the model parameters into the solution parameters by

$$[\beta, \beta_0] = \text{expaff}(a, A, b, B, C, g, g_0, h, h_0, T).$$

The distinguishing feature of the exponential affine form is that the partial elasticities are constant in x :

$$\frac{V_x}{V} = \beta.$$

This implies that the transformed drifts (μ^k, μ^s and μ^d) discussed in the previous section have the form

$$\begin{aligned} \mu + \Sigma \frac{V_x}{V}^\top &= a + Ax + C\text{diag}(b + Bx)C^\top \beta^\top \\ &= [a + C\text{diag}(\beta C)b] + [A + C\text{diag}(\beta C)B]x. \end{aligned}$$

Thus the class of affine diffusions is closed under the form of drift transformations relevant for option pricing.

To use this framework to price options, we assume that the log of the underlying asset price, the short interest rate and the dividend rate on the underlying asset are all affine functions of the state vector. The terms S , B , and D in (4) can be computed using the exponential affine operator. The probabilities P^k and P^f are not of the exponential affine form and therefore require different treatment. Heston showed, however, that P^k and P^s can be computed by first solving a related expectation, that of the associated characteristic function for the log of the asset price, $\ln S(t) = h_0(t) + h(t)x(t)$:

$$\phi^c(\omega) = E_t \left[e^{i\omega(h_0(T) + h(T)x(t))} \right],$$

for $c = k$ and s , and then using an inversion relationship to recover the desired probabilities.

The characteristic function is an expectation of an exponential affine function of x and thus has exponential affine form

$$\phi^c(\omega) = e^{\beta_0^c(T-t) + \beta^c(T-t)x(t)}$$

with

$$[\beta^c, \beta_0^c] = \text{expaff}(\tilde{a}, \tilde{A}, b, B, C, 0, 0, i\omega h(T), i\omega h_0(T), T).$$

Note that the functions β^c and β_0^c are generally complex valued (except in the Gaussian case where $B = 0$). The values of \tilde{a} and \tilde{A} used depend on the specific pricing problem being solved, as discussed further below.

The probability functions P^k and P^s can then be calculated using the inverse Fourier transform:

$$P^c(x, t; T) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \frac{\sin(\omega \ln K)}{\omega} \mathbf{Re} \phi^c(\omega) - \frac{\cos(\omega \ln K)}{\omega} \mathbf{Im} \phi^c(\omega) d\omega \quad (7)$$

(Stuart and Ord., eq. 4.14, p. 127). Utilizing standard trigonometric identities,⁴ the probability function can thus be expressed as

$$P^c(x, t; T) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \frac{\exp(\mathbf{Re}(\beta_0^c + \beta^c x)) \sin(\omega \ln K - \mathbf{Im}(\beta_0^c + \beta^c x))}{\omega} d\omega. \quad (8)$$

The approach transforms the pricing problem from solving a multi-factor PDE to solving the system of ODEs for β_0 and β and then performing numerical integration. In some cases, such as Heston's model, an explicit solution for the characteristic function can be found, and only numerical integration need be performed. Also in the time-homogeneous Gaussian ($B = 0$) case, an analytic solution is immediately available and the option value can be solved explicitly using the Black-Scholes formula. In the general affine diffusion case, however, an explicit solution does not exist. Computing the option value thus requires solving the system of ODEs numerically at a number of values of ω and using these values to obtain approximate P^k and P^s with numerical quadrature. Although somewhat involved, this is nonetheless computationally simpler than solving multi-state PDEs.

Summarizing, S , B , and D are all exponential affine functions. P^k and P^s are computed by applying Fourier inversion to the characteristic functions $\phi^k(\omega)$ and

⁴ $\exp(x + iy) = \exp(x)(\cos(y) + i \sin(y))$ and $\sin(x) \cos(y) - \cos(x) \sin(y) = \sin(x - y)$.

$\phi^s(\omega)$, which are exponential affine functions. The basic approach then will involve computing the parameters of various exponential affine functions and, in the case of P^k and P^s , using these parameters to numerically integrate the Fourier inversion of the associated characteristic function.

The flexibility of the approach is illustrated with applications to options on dividend paying assets, on futures, on bonds and on bond futures. In all cases, for an option expiring at time T^o , we have

$$S = \exp(\beta_0^S + \beta^S x)$$

$$B = \exp(\beta_0^B + \beta^B x)$$

$$D = \exp(\beta_0^D + \beta^D x)$$

$$\phi^k(\omega) = \exp(\beta_0^k + \beta^k x)$$

$$\phi^s(\omega) = \exp(\beta_0^s + \beta^s x)$$

where

$$[\beta^B, \beta_0^B] = \text{expaff}(a, A, b, B, C, r_0, r, 0, 0, T^o)$$

$$[\beta^D, \beta_0^D] = \text{expaff}(a + \Psi^D b, A + \Psi^D B, b, B, C, g_0, g, 0, 0, T^o)$$

$$[\beta^k, \beta_0^k] = \text{expaff}(a + \Psi^k b, A + \Psi^k B, b, B, C, 0, 0, i\omega\beta_0^S, i\omega\beta^S, T^o)$$

$$[\beta^s, \beta_0^s] = \text{expaff}(a + \Psi^s b, A + \Psi^s B, b, B, C, 0, 0, i\omega\beta_0^S, i\omega\beta^S, T^o)$$

and

$$\Psi^D = C \text{diag}(\beta^S C)$$

$$\Psi^k = C \text{diag}(\beta^B C)$$

$$\Psi^s = C \text{diag}((\beta^S + \beta^D C)).$$

Unless otherwise noted, the parameters $a, A, b, B, C, g_0, g, h_0,$ and h are evaluated at t and the various β_0 and β functions are evaluated at $\tau = T^o - t$. The various models are distinguished by the parameters defining the underlying asset S (β_0^S and β^S) and the rate of return on the dividend bond D (g_0 and g).

Consider first options on a spot asset (stock, commodity, etc.) with a log price process given by $S(t) = h_0(t) + h(t)x(t)$ and a dividend rate of $w_0(t) + w(t)x(t)$. An option on the asset itself can be computed using

$$[\beta^S(\tau), \beta_0^S(\tau)] = [h(T^o - \tau), h_0(T^o - \tau)]$$

and

$$[g, g_0] = [w, w_0].$$

A futures price is equal to the value of an asset with a dividend rate equal to the short risk free rate (Cox et al.). Thus, the futures price on a contract maturing at time T^f written on the spot asset is given by

$$F(x, t; T) = e^{\beta_0^F(T^f - t) + \beta^F(T^f - t)x(t)}$$

with

$$[\beta^F, \beta_0^F] = \text{expaff}(a, A, b, B, C, 0, 0, h(T^f), h_0(T^f), T^f).$$

The value of an option written on such a futures contract takes $S = \ln F$ and is computed using

$$[\beta^S(\tau), \beta_0^S(\tau)] = [\beta^F(\tau + T^f - T^o), \beta_0^F(\tau + T^f - T^o)]$$

and

$$[g, g_0] = [r, r_0].$$

For pricing options on bonds, consider an option written on a default-free zero-coupon bond expiring at time T^z . The bond (zero) price is

$$Z(x, t; T^z) = e^{\beta_0^Z(T^z-t) + \beta^Z(T^z-t)x(t)}$$

with

$$[\beta^Z, \beta_0^Z] = \text{expaff}(a, A, b, B, C, r_0, r, 0, 0, T^z).$$

Here $S = \ln Z$ and the option can be priced using

$$[\beta^S(\tau), \beta_0^S(\tau)] = [\beta^Z(\tau + T^z - T^o), \beta_0^Z(\tau + T^z - T^o)]$$

and

$$[g, g_0] = [0, 0]$$

(the bond pays no dividends so the dividend bond, D , is identically equal to 1).

Finally, consider the futures price of a contract that matures at time T^f , written on a bond maturing at time T^z ($T^o < T^f < T^z$). The bond futures price is

$$F^z(x, t; T) = e^{\beta_0^{FZ}(T^f-t) + \beta^{FZ}(T^f-t)x(t)}$$

where

$$[\beta^{FZ}, \beta_0^{FZ}] = \text{expaff}(a, A, b, B, C, 0, 0, \beta^Z(T^z - T^f), \beta_0^Z(T^z - T^f), T^f).$$

Here $S = \ln F^z$ and the option value is computed using

$$[\beta^S(\tau), \beta_0^S(\tau)] = [\beta^{FZ}(\tau + T^f - T^o), \beta_0^{FZ}(\tau + T^f - T^o)]$$

and

$$[g, g_0] = [r, r_0].$$

The various formulas are summarized in Table 1.

It can be seen that the basic requirement for evaluating options in this framework is a module for computing the β_0 and β functions. As these are solutions to systems of ordinary differential equations with well-defined initial values, this step is relatively straightforward (MATLAB code is available at the author's website).

Option Valuation for

Quadratic Asset Pricing Models

The affine asset pricing model provides a convenient framework for pricing basic financial assets by avoiding the need to explicitly solve multidimensional PDEs. The less well known quadratic asset pricing model also has this property. Beaglehole and Tenney and Jamshidian discuss this class of models. A special case is the model of Constantinides, which is discussed and extended by Ahn et al.

Table 1. Summary of Pricing Formulas – Affine Model

Futures maturing at T^f on a spot asset:

$$[\beta^F, \beta_0^F] = \text{expaff}(a, A, b, B, C, 0, 0, h(T^f), h_0(T^f), T^f)$$

Bond maturing at T^z :

$$[\beta^Z, \beta_0^Z] = \text{expaff}(a, A, b, B, C, r, r_0, 0, 0, T^z)$$

Futures maturing at T^f on a bond maturing at T^z :

$$[\beta^{FZ}, \beta_0^{FZ}] = \text{expaff}(a, A, b, B, C, 0, 0, \beta^Z(T^z - T^f), \beta_0^Z(T^z - T^f), T^f)$$

Option on a spot asset:

$$[\beta^S(\tau), \beta_0^S(\tau)] = [h(T^o - \tau), h_0(T^o - \tau)]$$

$$[g, g_0] = [w_0, w]$$

Option on a futures on a spot asset:

$$[\beta^S(\tau), \beta_0^S(\tau)] = [\beta^F(\tau + T^f - T^o), \beta_0^F(\tau + T^f - T^o)]$$

$$[g, g_0] = [r, r_0]$$

Option on a bond:

$$[\beta^S(\tau), \beta_0^S(\tau)] = [\beta^Z(\tau + T^z - T^o), \beta_0^Z(\tau + T^z - T^o)]$$

$$[g, g_0] = [0, 0]$$

Option on a bond futures:

$$[\beta^S(\tau), \beta_0^S(\tau)] = [\beta^{FZ}(\tau + T^f - T^o), \beta_0^{FZ}(\tau + T^f - T^o)]$$

$$[g, g_0] = [r, r_0]$$

The quadratic model takes the short rate, dividend process and terminal payoff function to be quadratic functions of a multivariate Gaussian process. The vector of underlying factors is described by

$$dx = (a + Ax)dt + dz$$

(the normalization that the instantaneous covariance is the identity matrix is for convenience only). Let the short rate of interest be given by

$$r_0(t) + r(t)x + x^\top R(t)x.$$

The time T payoff function on the asset of interest is

$$\exp\left(h_0(T) + h(T)x + x^\top H(T)x\right)$$

and the dividend rate is

$$w_0(t) + w(t)x + x^\top W(t)x.$$

With no loss of generality, take R , G and W to be symmetric matrices.

Analogous to (6), the fundamental pricing PDE for this asset is

$$0 = V_t + V_x(a + Ax) + \frac{1}{2}\text{trace}(V_{xx}) - (g_0 + gx + x^\top Gx)V,$$

with terminal condition

$$V(x, T; T) = \exp\left(h_0(T) + h(T)x + x^\top H(T)x\right),$$

where $g_0 = r_0 - w_0$, $g = r - w$ and $G = R - W$. The general solution to this pricing PDE has the form

$$V(x, t; T) = \exp\left(\beta_0(\tau) + \beta(\tau)\beta + x^\top \mathcal{B}(\tau)x\right),$$

where, as before, $\tau = T - t$. The functions β_0 , β and \mathcal{B} solve the following system of ODEs:

$$\beta_0' = \beta^\top a + \text{trace}(\mathcal{B}) + \frac{1}{2}\beta^\top \beta - g_0$$

$$\beta' = (A^\top + 2\mathcal{B})\beta + 2\beta a - g$$

and

$$\mathcal{B}' = 2\mathcal{B}^2 + \mathcal{B}A + A^\top \mathcal{B} - G.$$

The initial conditions on these ODEs are $\beta_0(0) = h_0(T)$, $\beta(0) = h(T)$ and $\mathcal{B}(0) = H(T)$. \mathcal{B} is symmetric and hence only the lower part needs be explicitly computed:

$$\text{vech}(\mathcal{B}') = \text{vech}\left(2\mathcal{B}^2 + \beta A + A^\top \mathcal{B} - G\right).$$

As in the previous section, this is a system of Ricatti equations. More specifically, the $n(n+1)/2$ ODEs defining $\text{vech}(\mathcal{B})$ are Ricatti equations, the n ODEs defining β are linear (with non-constant coefficients) and the equation defining β_0 is an ordinary integral in τ . As before, it is convenient to define an operator that maps the parameters of the underlying model into the solution functions:

$$[\mathcal{B}, \beta, \beta_0] = \text{expquad}(a, A, G, g, g_0, H, h, h_0, T).$$

Using this operator, the same collection of assets discussed in the previous section can be priced.

With the quadratic asset pricing model, the partial elasticities are affine in x ,

$$\frac{V_x}{V} = \beta + 2x^\top \mathcal{B}.$$

This implies that the transformed drifts (μ^k , μ^s and μ^d) used in the option pricing formulas have the form

$$\mu + \Sigma \frac{V_x^\top}{V} = [a + \beta^\top] + [A + 2\mathcal{B}]x.$$

Hence, the underlying factors are still Gaussian after transforming their drifts.

Using the same approach as in the previous section, for an option expiring at time T^o , we have

$$S = \exp(\beta_0^S + \beta^S x + x^\top \mathcal{B}^S x)$$

$$B = \exp(\beta_0^B + \beta^B x + x^\top \mathcal{B}^B x)$$

$$D = \exp(\beta_0^D + \beta^D x + x^\top \mathcal{B}_0^D x)$$

$$\phi^k(\omega) = \exp(\beta_0^k + \beta^k x + x^\top \mathcal{B}^k x)$$

$$\phi^s(\omega) = \exp(\beta_0^s + \beta^s x + x^\top \mathcal{B}^s x)$$

where

$$[\mathcal{B}^B, \beta^B, \beta_0^B] = \text{expquad}(a, A, R, r, r_0, 0, 0, 0, T^o)$$

$$[\mathcal{B}^D, \beta^D, \beta_0^D] = \text{expquad}(a + (\beta^S)^\top, A + 2\mathcal{B}^S, G, g, g_0, 0, 0, 0, T^o)$$

$$[\mathcal{B}^k, \beta^k, \beta_0^k] = \text{expquad}(a + (\beta^B)^\top, A + 2\mathcal{B}^B, 0, 0, 0, i\omega \mathcal{B}^S, i\omega \beta^S, i\omega \beta_0^S, T^o)$$

$$[\mathcal{B}^s, \beta^s, \beta_0^s] = \text{expquad}(a + (\beta^S + \beta^D)^\top, A + 2(\beta^S + \beta^D), 0, 0, 0, i\omega \mathcal{B}^S, i\omega \beta^S, i\omega \beta_0^S, T^o).$$

The specific forms of β_0^S , β^S , \mathcal{B}^S , g_0 , g and G will depend on the underlying asset associated with the option. Various specific cases are detailed in Table 2.

Examples

In this section, two well-known examples which have closed form solutions to the exponential affine operator are discussed. The first is the Vasicek bond pricing model. The associated bond option pricing formula was first developed by Jamshidian(1989). The second model is the Heston stochastic volatility model for stock options. These simple examples will make more concrete the notation used here and the general pricing principles.

The Vasicek bond pricing model uses a Gaussian affine one factor model to describe the spot interest rate (x) under the risk neutral measure:

$$dx = (a + Ax)dt + C dW,$$

($b = 1$, $B = 0$), with $r_0 = 0$ and $r = 1$. The parameters for the value of a zero coupon bond can be verified to equal

$$\beta_0^Z(\tau) = \left(\frac{C^2}{2A^2} + \frac{a}{A} \right) (\tau + \beta^Z(\tau)) + \frac{\left(C\beta^Z(\tau) \right)^2}{4A}$$

and

$$\beta^Z(\tau) = \frac{1 - e^{-A\tau}}{A}.$$

Table 2. Summary of Pricing Formulas – Quadratic Model

Futures maturing at T^f on a spot asset:

$$[\mathcal{B}^F, \beta^F, \beta_0^F] = \text{expquad}(a, A, 0, 0, 0, H(T^f), h(T^f), h_0(T^f), T^f)$$

Bond maturing at T^z :

$$[\mathcal{B}^Z, \beta^Z, \beta_0^Z] = \text{expquad}(a, A, R, r, r_0, 0, 0, 0, T^z)$$

Futures maturing at T^f on a bond maturing at T^z :

$$[\mathcal{B}^{FZ}, \beta^{FZ}, \beta_0^{FZ}] = \text{expquad}(a, A, 0, 0, 0, \mathcal{B}^Z(T^z - T^f), \beta^Z(T^z - T^f), \beta_0^Z(T^z - T^f), T^f)$$

Option on a spot asset:

$$[\mathcal{B}^S(\tau), \beta^S(\tau), \beta_0^S(\tau)] = [G(T^o - \tau), g(T^o - \tau), g_0(T^o - \tau)]$$

$$[G, g, g_0] = [W, w, w_0]$$

Option on a futures on a spot asset:

$$[\mathcal{B}^S(\tau), \beta^S(\tau), \beta_0^S(\tau)] = [\mathcal{B}^F(\tau + T^f - T^o), \beta^F(\tau + T^f - T^o), \beta_0^F(\tau + T^f - T^o)]$$

$$[G, g, g_0] = [R, r, r_0]$$

Option on a bond:

$$[\mathcal{B}^S(\tau), \beta^S(\tau), \beta_0^S(\tau)] = [\mathcal{B}^Z(\tau + T^z - T^o), \beta^Z(\tau + T^z - T^o), \beta_0^Z(\tau + T^z - T^o)]$$

$$[G, g, g_0] = [0, 0, 0]$$

Option on a bond futures:

$$[\mathcal{B}^S(\tau), \beta^S(\tau), \beta_0^S(\tau)] = [\mathcal{B}^{FZ}(\tau + T^f - T^o), \beta^{FZ}(\tau + T^f - T^o), \beta_0^{FZ}(\tau + T^f - T^o)]$$

$$[G, g, g_0] = [R, r, r_0]$$

(Note that stationarity requires that $A < 0$ and hence $\beta^Z < 0$.) For an option on a bond that matures Δ periods after the option expires, $\beta_0^S(\tau) = \beta_0^Z(\tau + \Delta)$ and $\beta^S(\tau) = \beta^Z(\tau + \Delta)$.

The probabilities needed have the form

$$Prob\left(\beta_0^S(\Delta) + \beta^S(\Delta)x(T) \leq \ln(K)|x(t)\right).$$

In the Vasicek model $x(T)|x(t)$ is normally distributed. The probabilities are evaluated with the adjusted constants in the drifts

$$a^k(s) = a + \frac{C^2}{A}\left(1 - e^{A(s-t)}\right)$$

and

$$a^s(s) = a + \frac{C^2}{A}\left(1 - e^{A(s-t+\Delta)}\right).$$

Heston proposed a model a stock price behavior with stochastic volatility which can be written in terms of the affine diffusion (under the risk neutral distribution) as

$$dx = \left(\begin{bmatrix} r_0 \\ \alpha\kappa \end{bmatrix} + \begin{bmatrix} 0 & -\frac{1}{2} \\ 0 & -\kappa \end{bmatrix} x \right) dt + \begin{bmatrix} 1 & 0 \\ \rho\sigma & \sqrt{1-\rho^2}\sigma \end{bmatrix} \text{diag} \left(\sqrt{\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} x} \right) dW$$

(with $b = 0$). The first factor is interpreted as the log of the stock price, the second factor is the stock price volatility. The short interest rate is taken as the constant r_0 .

For pricing options, $g = [0 \ 0]$, $g_0 = r_0$, $h = [1 \ 0]$ and $h_0 = 0$. This implies that $S = \exp(x_1)$, $B = \exp(-r_0T)$ and $D = 1$. The first of the probabilities, P^k , uses the

risk-neutral parameters with no adjustments because $B_x = 0$, implying that $\Psi^k = 0$.

For P^s the drift is adjusted using

$$\Psi^s = \begin{bmatrix} 1 & 0 \\ \rho\sigma & 0 \end{bmatrix},$$

resulting in the adjustment

$$\hat{A} = \begin{bmatrix} 0 & \frac{1}{2} \\ 0 & \rho\sigma - \kappa \end{bmatrix}.$$

The characteristic functions associated with P^k and P^s can be solved in closed form. The Riccati equations for β , with initial conditions $\beta(0) = [i\omega \ 0]$, have the form

$$\beta'_1 = 0,$$

implying $\beta_1(\tau) = i$, and

$$\beta'_2 = c_0 + c_1\beta_2 + c_2\beta_2^2,$$

where, for P^k ,

$$c_0 = -\frac{\omega^2 + i\omega}{2}, \quad c_1 = -\kappa + i\omega\sigma\rho \quad \text{and} \quad c_2 = \frac{\sigma^2}{2},$$

and, for P^s ,

$$c_0 = -\frac{\omega^2 - i\omega}{2} \quad \text{and} \quad c_1 = \rho\sigma - \kappa + i\omega\sigma\rho$$

(with c_2 unchanged). Letting $\theta = \sqrt{4c_0c_2 - c_1^2}$, it is easily verified that

$$\beta_2(\tau) = \frac{1}{\sigma^2} \left(\theta \tan \left(\frac{\theta\tau}{2} + \tan^{-1} \left(\frac{c_1}{\theta} \right) \right) - c_1 \right).$$

The constant term, β_0 , solves $\beta_0' = i\omega r_0 + \alpha\kappa\beta_2$ with $\beta_0(0) = 0$, the solution to which is

$$\beta_0(\tau) = \left(i\omega r_0 - \frac{\alpha\kappa c_1}{\sigma^2} \right) \tau + \frac{\alpha\kappa}{\sigma^2} \left(\ln \left(1 + \tan \left(\frac{\theta\tau}{2} + \tan^{-1} \left(\frac{c_1}{\theta} \right) \right) \right)^2 \right) - \ln \left(\frac{4c_0c_2}{4c_0c_2 - c_1^2} \right).$$

Conclusions

The increased use of multi-factor models to represent asset price dynamics gives rise to the need for multi-factor option pricing models. A general formula for pricing European options has been presented that expresses the value of the option in terms of two bond prices, one actual and one artificial, and two terms that can be interpreted as probabilities that the option expires in-the-money under two alternative artificial diffusion processes.

The difficulty in using such a formula is in evaluating the required probabilities, especially in multi-factor models. In general, these probabilities are solutions to multi-state partial differential equations. For the affine and quadratic asset pricing models, however, the computational difficulties can be reduced to solving a system of ordinary differential equations a number of times and performing a numerical integration using

the resulting values. These computational tasks can be carried out efficiently in a reasonable time.

The approach is illustrated with specific formulas for options on spot assets, futures on spot assets, zero-coupon bonds and futures on zero-coupon bonds. Applications are not limited to these assets, however. Extensions to caps, floors and swaps are straightforward. Options on other kinds of assets, such as foreign currency based assets, could also be priced in this framework.

The desirability of the affine and quadratic models is further enhanced by the fact that many of the currently used multi-factor models for financial assets are of these forms. Recent econometric developments have made estimation of model parameters quite feasible as well (e.g. Singleton, Ahn et al.). Further work is required to determine whether these classes are sufficiently flexible to capture important characteristics of real prices and therefore to serve as a practical base for asset pricing.

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