ECG790C - Midterm Exam 3 - Spring 2009

This exam is due Thursday, April 16 by 10:15 AM. You should not discuss this exam with anyone except me. All work should be your own.

1. Suppose that

$$dX = \mu(X)dt + \sigma(X)dW$$

where $X \in [a, b]$ and we would like to know

$$Prob(X_t \leq y | X_0 = x) \equiv F(t, y; x)$$

It can be shown that

$$\frac{dF(t,y)}{dt} = (\sigma(y)\sigma'(y) - \mu(y))\frac{dF(t,y)}{dy} + \frac{\sigma(y)^2}{2}\frac{d^2F(t,y)}{dy^2}$$

The boundary conditions are that

$$F(0, y; x) = \begin{cases} 0 & \text{if } y < x \\ 1 & \text{if } y \geq x \end{cases}$$

and that $F(t, a; x) = 0$ and $F(t, b; x) = 1$.

Write a function that produces an approximation to F. The approximation should use a user-specified family of approximating functions and an implicit approximation (for the time dimension) with m time steps. Thus the function should be passed x, t, m, a function or functions that compute μ, σ and σ' (along with any parameters needed to compute these), and a function definition structure (i.e., an `fspace` variable) and should return the coefficients of the approximating function.

I suggest you write your code ignoring the boundary conditions at the endpoints of the X domain; the code should produce reasonable approximations even without these constraints if the approximation interval is wide enough. Once it is running, modify the code to include the boundary conditions (ideally making their imposition optional).

To demonstrate your function assume that

$$dx = \lambda(\mu - x)dt + \sigma\sqrt{x}dW$$

with $\lambda = 0.35$, $\mu = 0.07$, $\sigma = 0.3$, $x = 0.06$, $t = 1/12$. Use linear splines on the interval $[0, 0.3]$. Plot F (the conditional CDF) and F_y (the conditional PDF).

2. Consider a chooser option that, at time T, can be exercised to obtain either a put option or a call option expiring at time $T+h$. Thus the value of the option at T is the maximum of the value of the put and the call. Suppose the put and call options both have underlying S and strike price K, i.e., the put is worth $\max(0, K-S)$ at time $T+h$ and the call is worth $\max(0, S-K)$ at time $T+h$. Suppose that S is described (under the risk-neutral measure) by

$$dS = rSdt + \sigma S^{\gamma} dW$$

where r is the risk free rate of return.

One way to solve this problem is to determine the time T price of a put and a call expiring at time $T+h$ and use the maximum of these two values as the terminal payout of an the chooser option. Another way to approach valuing the chooser option is to note that the payoff function at time T is

$$\max(C(S,K,h), P(S,K,h)) = C(S,K,h) + \max(0, e^{-rh}K - S)$$

The latter term is a payout on a put option with strike price e^{-rh} that expires at time T. Thus the chooser option is worth $C(S,K,T+h) + P(S,e^{-rh}K,T)$. We can use the same approach to see that it is also equal to $P(S,K,T+h) + C(S,e^{-rh}K,T)$. Note that this approach relies on the existence of the put-call parity relationship.

Write a Matlab program to determine the value of the option using the PDE approach with finite difference derivatives (a piece-wise linear approximating function) using both of these approaches. You may use `finsolve` is you want. Use the following parameter values: $r = 0.05$, $\sigma = 0.2$, $\gamma = 1.25$, $K = 1$, $T = 1$ and $h = 0.5$. Note that with $\gamma = 1$ we could use the Black-Scholes formula to obtain a closed form expression for the value of the chooser option, a fact that can be used to test your code.

Plot the value of the option for $S \in [0.8, 1.2]$.

Create a table of values of the option for $S = \{0.8, 0.9, 1, 1.1, 1.2\}$.

3. Consider the two factor model in which the short rate r is described by

$$dr = \alpha(x - r)dt + \sigma\sqrt{r}dW_1$$

and the second factor X is described by

$$dx = \lambda(\mu - x)dt + \eta\sqrt{x}dW_2$$

(where W_1 and W_2 are uncorrelated).

Use `finsolve` to price a 5 year bond with this model with the parameter values $\alpha = 0.5$, $\sigma = 2$, $\lambda = 0.1$, $\mu = 0.06$ and $\eta = .25$. Note that the range of both r and x is $[0, \infty]$ but approximating them on $[0, 0.3]$ should be sufficient to ensure accuracy in the usual range of interest rates.

Evaluate and report the bond price when $r = 0.07$ and $x = 0.065$.

4. Consider a process described by

$$dS = rSdt + \sigma S^\gamma dW$$

where r is the risk free rate of interest. Price a European put option with strike price K and T periods until maturity using the implicit finite difference method and piecewise linear basis functions. Use the parameter values $r = 0.05$, $\sigma = 0.3$, $\gamma = 1.25$, $K = 1$ and $T = 1$.

Now suppose that the option can be exercised either at maturity or earlier at times $T/3$ or $2T/3$. Compute the value of this option and plot the value of both options for $S \in [0.75, 1.25]$ (on a single plot). What are the values of the at-the-money options (i.e., $S = K$)?