ECG790C - Midterm Exam 2 - Spring 2009

This exam is due Thursday, March 26 by 10:15 AM. You should not discuss this exam with anyone except me. All work should be your own.

1. Solve numerically, using function approximation and collocation, as well as fixed step size Runge-Kutta (`rk4`) and variable step size Runge-Kutta (`ode45`)

$$x'(t) = \frac{x(t)}{t(1-\ln(t))}$$

   on the interval [0,1], with the boundary condition that $x(1) = 1$. Compare the behavior of the different methods and discuss.
   Hint: the behavior at t=0 is problematic; either solve backwards from 1 or approximate on $[\epsilon, 1]$. The closed form solution is $x(t) = 1/(1-\ln(t))$.

2. Suppose that

$$dS = \lambda(\alpha - S)dt + \sigma\sqrt{S}dW$$

   It can be shown that

$$V(S) = E\left[\int_0^\infty e^{-rt}S_t^\gamma dt | S_o = S\right]$$

   satisfies

$$rV(S) = S^\gamma + \lambda(\alpha - S)V'(S) + \frac{\sigma^2}{2}SV''(S)$$

   Approximate and plot V as a function of S. Be sure to discuss the choices made in solving this problem and provide evidence that the choices lead to an adequate approximation. The domain of S is $[0, \infty)$ so you will either need to map S into a bounded domain or truncate the domain at a suitably large value. Use the following parameter values: $r = 0.05$, $\lambda = 0.5$, $\alpha = 3$, $\sigma = 0.04$ and $\gamma = 1.5$.

3. Given a function $F(z,x)$ you would like to find a function $\pi(x)$ such that

$$\pi(z) = \int_D \pi(x)F(z,x)dx \; \forall z \in D$$

subject to the side condition that $\int_D \pi(x)dx = 1$. As an example, consider the function

$$F(z,x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{z-(\alpha+\beta x)}{\sigma}\right)^2\right)$$

Approximate π using a cubic spline approximating function. Use Gauss-Legendre quadrature to approximate the integral. Use the parameter values $\alpha = 1$, $\beta = 0.5$ and $\sigma = 0.2$. For this problem the natural domain is the whole real line ($D = (-\infty, \infty)$) but this is impractical for approximating. Therefore use instead a truncated domain $[a,b]$. Some experimentation may be required to find a suitable approximation interval.

The context for this problem is from probability theory. Specifically F is the conditional distribution of Z given X and we would like to know the common unconditional distribution of X and Z. In this example it can be shown π is the normal PDF with a certain mean and variance (I leave you to see if you can figure out the values of these parameters). Thus your result should look like a normal distribution.

4. Consider the process

$$dS = \eta(\mu - S)dt + \sigma S dW$$

The conditional mean of this process is

$$M(S,\Delta) = E[S_{t+\Delta}|S_t = S] = \mu + e^{-\eta\Delta}(S-\mu)$$

and its conditional variance is

$$V(S,\Delta) = Var[S_{t+\Delta}|S_t = S] = e^{c\Delta}S^2 + (e^{c\Delta}-1)\frac{2\eta\mu^2}{c} + (e^{c\Delta}-e^{-\eta\Delta})\frac{2\eta\mu(S-\mu)}{c+\eta} - M(S,\Delta)^2$$

where $c = \sigma^2 - 2\eta$.

Suppose that we have n observations on S, observed with a common time step of Δ. One way to estimate the parameters of this process is to assume that the conditional density function is Gaussian and to solve

$$\max_{\theta} -\frac{1}{2}\sum_{i=2}^{n}\left[\frac{(S_i - M(S_{i-1},\Delta;\theta))^2}{V(S_{i-1},\Delta;\theta)} + \ln(V(S_{i-1},\Delta;\theta)\right]$$

where $\theta = [\eta;\mu;\sigma]$. This is known as a quasi-maximum likelihood method (quasi because the true conditional density is not actually Gaussian).

Use the weekly Federal Funds data for the period 1963-1998 (provided), estimate the parameters of this model. Notice that a simpler method is to regress $S_{t+\Delta}$ on S_t and use these estimates to obtain η and μ. The errors of this regression divided by S_t have variance that is approximately $\sigma^2\Delta$ which can be used to obtain an estimate of σ. These estimates can be used as starting values for the quasi-maximum likelihood method.