ECG790C - Midterm Exam 1 - Spring 2009

This exam is due Thursday, Feb.12 by 10:15 AM. You should not discuss this exam with anyone except me. All work should be your own.

1. Consider the process given by

$$dS = \alpha(\mu - S)dt + \sqrt{2\alpha\sigma S(1-S)}dW$$

defined for $S \in [0,1]$, with $0 < \mu < 1$. This process is mean reverting in that values of $S > \mu$ imply negative drift, causing the process to tend to decline and, similarly, values of $S < \mu$ cause a tendency for the process to increase. Note that this process might describe a variable representing a fraction, share or event probability.

The ergodic (or long-run) distribution of S is Beta, i.e. has the form

$$B(S,a,b) = \frac{S^{a-1}(1-S)^{b-1}}{B(a,b)}$$

where $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$. Specifically, the density is $B(S, \mu/\sigma, (1-\mu)/\sigma)$. This implies that samples of values of S_T should be well described by this distribution for large enough T. Note that the long-run mean of the process is μ.

Write a MATLAB script that generates random sample paths and validates that I have in fact provided the correct ergodic distribution. I have provided MEX files to compute the PDF and CDF of the Beta distribution. Use the parameter values $\alpha = 0.5$, $\mu = 0.45$ and $\sigma = 0.1$. Be sure to handle simulated values that fall outside of the $[0,1]$ interval in a reasonable way.

2. Suppose we are interested in the joint process

$$dZ = \eta(P - Z)dt + \sigma_Z dW_1$$

$$dP = \delta(\mu - P)dt + \sigma_P dW_2$$

where W_1 and W_2 are independent standard Brownian processes. In this model the drift in Z depends on whether Z is above or below P, while the drift in P

depends on whether it is above or below μ. Intuitively μ is the long-run mean of both P and Z but P can be interpreted as an intermediate mean level for Z. Generally we would suppose that P moves slowly towards its mean and that Z moves more quickly towards P; this is true when $\delta < \eta$.

It can be shown that

$$E[Z_t|P_0 = P, Z_0 = Z] = \mu + \frac{\eta}{\eta - \delta}\left(e^{-\delta t} - e^{-\eta t}\right)(P - \mu) + e^{-\eta t}(Z - \mu)$$

Notice that this implies that the future deviation of Z from its long run mean depends on the deviations of the current values of both Z and P from μ.

Verify this formula by simulating 25000 time paths over the interval $t \in [0, 100]$ for the (Z, P) process and taking averages over the paths. Use the parameters $\delta = 0.02$, $\eta = 0.1$, $\mu = 1$, $\sigma_Z = 0.1$ and $\sigma_P = 0.05$. Also use the starting values of $(Z_0, P_0) = (1.2, 1, 1)$ and $(Z_0, P_0) = (1.2, 0.9)$. Create a plot that shows the deviations of the simulated time path of $E[Z_t|Z_0, P_0]$ from the formula values for both sets of starting values. Your answer should include a script file generating the time paths. Also estimate how many time paths would be needed to obtain errors of less than 0.001 (explain your reasoning).

3. Consider an option written on S, where

$$dS = rSdt + \sigma\sqrt{S}dW$$

The option pays pays $B - K$ the first time S hits B prior to T or it pays $\max(S_T - K, 0)$ at time T if S did not exceed B during the period $[0, T]$. The discount rate is r. Use Monte Carlo with 5000 paths to determine the time 0 value of this option with $r = 0.05$, $\sigma = 0.2$, $B = 1.4$, $K = 1$, $T = 1$ and $S_0 = 1.1$.

The price of a standard call option assuming that

$$dS = rSdt + \sigma SdW$$

where W is the same Brownian process as above can be determined using the Black-Scholes formula. Use this as a control variate. How much does this increase the accuracy of the Monte Carlo estimate?

Your answer should provide a script that generates the sample paths and computes the ordinary and CV Monte Carlo estimates as when an interpretation of the results.

4. Consider the process

$$dS = \alpha S(\mu - S)dt + \sigma S dW$$

Determine the transform $y = f(S)$ such that y is a constant volatility process. Simulate S using the original form and the transformation. Determine which form yields more accurate results and quantify how much more accuracy is obtained. Your answer should include a MATLAB script that performs the accuracy analysis and an interpretation of the results.

5. Consider the system of equations

$$f(x) = \exp(Ax) - b = 0$$

where A is $n \times n$ and b is $n \times 1$ (here the exponential operator is applied element-wise). Clearly the solution $x = A^{-1} \ln(b)$ but I would like you to solve this function using Newton's and Broyden's method on the system as written. To use Newton's method you need to know that Jacobian matrix, which is $f'(x) = \text{diag}(\exp(Ax))A$.

To solve this, write a function with the syntax `[fx,Jx]=testfunc(x,A,b)` that returns the function value and the Jacobian matrix. Be sure to test whether the Jacobian is needed and only compute it if necessary (use `nargout`).

Generate random values of A and b for different values of n and assess which of the two solution methods is more efficient at solving this problem (and whether it depends on n). Your answer should include a MATLAB script that performs these comparisons and an interpretation of the results.