Technical Appendix to the paper "On the Implications of Portfolio Heterogeneity for Money Demand and the Great Moderation"

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This is the technical appendix to the paper entitled "Money Demand Heterogeneity and the Great Moderation". The first section provides a proof that equation (2) in the paper have a unique deterministic solution and admits a first order approximation. Section 2 derives the log-linearized money demand reported in the paper. Finally, the last section provides additional details on the GMM strategy pursued in the paper.

1 Existence of Solution to Equation (2) in Main Text

This section heavily draws on Lang (1993), Woodford (1998, and 2003). The interested reader is referred to those papers for additional details on the proofs and definitions. Recall that the first order condition for money holdings, Q, implies:

$$\begin{aligned} 0 &= \left[-R_t + 1 + \eta'(V_{i,t}) \left(\frac{p_t c_{i,t}}{Q_{i,t}}\right)^2\right] + \qquad (1) \\ & \widetilde{E}_t \sum_{l=1}^{\infty} (\xi\beta\pi g_c)^l \frac{\lambda_{t+l+1}}{\lambda_t} \left[(-R_{t+l} + 1) + \left(\eta_{i,t+l}\right)' \left(V_{i,t+l}^t\right)^2\right], \end{aligned}$$

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where $\eta_{i,t+l} \equiv \eta(V_{i,t+l}^t)$, $V_{i,t+l}^t \equiv \frac{p_{t+l}C_{i,t+l}}{(\pi g_c)^l Q_{i,t}}$, and $\widetilde{E}_t$ is the expectation operator upon the event that household j does not re-optimize her money balances after period t.

Proposition 1 *Suppose η is twice continuously differentiable and $\xi\beta\pi < 1$. Then the stochastic difference equation (1) has a locally unique deterministic solution given by $R = 1 + \eta' V^2$ and admits a first-order Taylor approximation around this steady state.*

Proof. The proof essentially consist of showing that the conditions for the inverse and implicit mapping theorems hold. For simplicity, I consider a perfect forseight equilibirum, i.e. the entire sequences $\{V_{i,t}\}, \{R_t\}, \{\lambda_t\}$, and $\{\pi_t\}$ for $t \geq 0$ are known at time $t = 0$. Re-write (1) as follows

$$\left[-R_t + 1 + \eta'(V_{i,t})V_{i,t}^2\right] + \sum_{l=1}^{\infty} (\xi\beta)^l \frac{\lambda_{t+l+1}}{\lambda_t} (\pi g_c)^l \left[(-R_{t+l} + 1) + \left(\eta_{i,t+l}\right)' \left(V_{i,t+l}^t\right)^2\right] = 0, \quad (2)$$

where $\eta_{i,t+j}$ and $V_{i,t+j}^t$ are defined above. Next, consider a deterministic steady state in which $R_t = R$, $\lambda_t = \lambda$, $g_{c,t} = g_c$ and $\pi_t = \pi$, then it is straightforward to show that (2) has a solution, in which $V_{i,t} = V_{i,t+1}^t = V$ and $R = 1 + \eta' V^2$. Since $\eta \in \mathbb{C}^2$ and $\eta' \neq 0$ the implicit function theorem implies there exist a continuous function $\aleph : A \to B \subset \mathbb{R}$, with A an open ball around R, such that $V = \aleph(R)$ and $R = 1 + \eta'(\aleph(R))\aleph(R)^2$. Moreover, this solution is locally unique since the inverse theorem holds. Note that the sequence of equilibrium conditions (2) can be written in the form

$$\Gamma(V_i, X) = 0,$$

where V_i, X refer to the sequences of values for the variables $V_{i,t}, X_t = \{R_t, \lambda_t, \pi_t\}$ in periods $t = 0, 1, 2...$, and Γ maps a four-tuple of infinite sequences into an infinite sequence. The t element of Γ, Γ_t, is given by the left-hand side of (2). Define the derivative mapping $D\Gamma(V_i, X)$ as the linear operator that maps the sequences V_i, X into the sequence

$$\widehat{\Gamma}_t = \sum_{k=0}^{\infty} [\Gamma_{R,k} \widehat{R}_{t+k} + \Gamma_{V,k} \widehat{V}_{i,t+k}], \quad (3)$$

where $\Gamma_{R,k} = -R\left(\xi\beta\right)^k$, and $\Gamma_{V,k} = \left(\xi\beta\right)^k\left(\eta''V + 2\eta'\right)V^2$.[1]
Since $\xi\beta < 1$, then $\sum_{k=0}^{\infty}[|\Gamma_{R,k}| + |\Gamma_{V,k}|] < \infty$ and

$$\widehat{R}_t = \left(1 - \xi\beta L^{-1}\right)\left[\widehat{\Gamma}_t - \sum_{k=0}^{\infty}\Gamma_{V,k}\widehat{V}_{i,t+k}\right], \tag{4}$$

where L is the lag operator. Under the sup norm topology[2] on the linear space of sequences, the last two conditions imply D maps bounded sequences into bounded sequences and has a continuous inverse. Therefore, the inverse and implicit mapping theorems hold and (4) is a first-order Taylor approximation to (2). This approximate solution is accurate up to an error term of order $O\left(||V||^2\right)$ .Moreover, the solution given by (4) is locally unique. ■

The next section uses the results from proposition 1 to derive the aggregate implications of the time-dependent portfolio adjustment assumption.

2 Derivation of the Basic Money Demand Equation

I derived the equilibrium dynamics of velocity, V_t, in the case of small enough disturbances around the steady state defined in proposition 1. Log-linearizing the FONC for money balances (1) we obtain:

$$\begin{aligned}\widetilde{R}_t &= (2\eta'V^2 + \eta''V^3)\widehat{V}_t + \\ &+\widetilde{E}_t\sum_{j=1}^{\infty}(\xi\beta)^j\left(\frac{1}{\pi g_c}\right)^j\left\{-(g_c\pi)^j R\widehat{R}_{t+j} + (g_c\pi)^j(2\eta'V^2 + \eta''V^3)\widehat{V}_{t+j}^t\right\},\end{aligned} \tag{5}$$

where g_c is the growth rate of consumption and $\widetilde{R}_t = R_t - R$. In (5), I have suppressed the household subindex to make the derivation clearer. However, we must remember velocity refers to household i. The problem with the last expression is that V_{t+j}^t depends on consumption at time $t + j$, conditional on not having re-optimized for the last j periods. Moreover, the expectation operator, E_t^i, must be taken only over those histories in which household i does not re-optimize. As discussed in Christiano (2004) and Woodford (2005),

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[1] The term $\Gamma_\lambda\lambda_{t+k}$ is omitted because $\Gamma_\lambda = 0$ in the deterministic steady state.

[2] Definitions of the sup norm and proofs of the Implicit and Inverse Mapping theorems are provided in Lang (1993). Woodford (1998 and 2003) applies those theorems to the existance and uniqueness of a dynamic model characterized by equations of the type $f(p_{t-1}, p_t, p_{t+1}; u_t) = 0$ for some vector p and disturbance u.

a direct evaluation of the last expression is troublesome. Following, Woodford's insight, I guess and verify that individual velocity obeys the following relation

$$\widehat{V}_{i,t} = \widehat{V}_t^* + \Psi \widehat{\frac{Q_{i,t}}{Q_t^*}}, \tag{6}$$

where V_t^* is economy-wide velocity and Ψ is a coefficient to be determined. Individual velocity can be expressed in terms of relative variables and aggregate velocity:

$$\widehat{V}_{i,t} = \widehat{V}_t^* + \widehat{\widetilde{c}}_{i,t} - \widehat{\frac{Q_{i,t}}{Q_t^*}},$$

then use this last expression and (6) to show that $\widehat{\widetilde{c}}_{i,t} \equiv \widehat{\left(\frac{c_{i,t}}{c_t}\right)} = (\Psi + 1)\widehat{\frac{Q_{i,t}}{Q_t^*}}$. To recover Ψ use the fact that the multiplier is the same for everybody. That is, take a log-linear approximation to the complete market condition (equation 3 in the paper) to show:

$$-\rho(\widehat{\widetilde{c}}_t(i) - \widehat{\widetilde{c}}_t(l)) = \tau(\widehat{V}_{i,t} - \widehat{V}_{l,t}).$$

Combining the last two equations we obtain:

$$-\rho((\Psi + 1)\widehat{\frac{Q_{i,t}}{Q_t^*}} - (\Psi + 1)\widehat{\frac{Q_{l,t}}{Q_t^*}}) = \tau(\widehat{V}_t^* + \Psi\widehat{\frac{Q_{i,t}}{Q_t^*}} - \widehat{V}_t^* - \Psi\widehat{\frac{Q_{l,t}}{Q_t^*}}).$$

The last condition must be satisfied for households i and l regardless of when they re-optimized their money balances. There are two possible cases. The first and trivial solution indicates that the last equation is satisfied for any pair of households if their hold the same money balances, i.e., $Q_i = Q_l$. But this condition is not true in the presence of sluggish portfolio adjustment. The second solution requires the equality of the individual terms. That is,

$$-\rho(\Psi + 1)\widehat{\frac{Q_{i,t}}{Q_t^*}} = \tau\Psi\widehat{\frac{Q_{i,t}}{Q_t^*}} \text{ and } -\rho(\Psi + 1)\widehat{\frac{Q_{l,t}}{Q_t^*}} = \tau\Psi\widehat{\frac{Q_{l,t}}{Q_t^*}}.$$

But these conditions are satisfied if the unknown coefficient is given by:

$$\Psi = -\frac{\rho}{\rho + \tau},$$

where ρ is the coefficient of risk aversion and $\tau = \frac{(2\eta' V + \eta'' V^2)}{(1+\eta+\eta' V)}$. For the log-utility case used in the main text $\rho = 1$. Furthermore, the positive slope, non-negativity, and convexity of the transaction function η imply that τ is positive, which in turn involves $\Psi < 0$ as claimed in the paper. Furthermore, note that Ψ is independent of individual variables as desired. The evaluation of (5) requires the knowledge of $\widehat{V}_{t+j}^t$. This variable can be readily expressed in terms of aggregate terms and money holdings at time t:

$$\widehat{V}_{t+j}^t = \widehat{V}_{t+j}^* + \Psi \left(\frac{\widehat{Q_{i,t}}}{Q_t^*} - \sum_{k=1}^{j} \widehat{\mu}_{q,t+j} \right)$$

where μ_q is the growth rate of money balances. With an indexation rule of the type $Q_{i,t} = \pi Q_{i,t-1}$, there would be a term $\left(\frac{\pi}{\mu_q}\right)^j$ in front of $\sum_{k=1}^{j} \widehat{\mu}_{q,t+j}$. As a consequence, the model would lack of a stationary steady state in which all households choose the same portfolio. A related phenomenon is present in Altig et at. (2004). Now combine the last results with (5) to show

$$\begin{aligned} & -R\widehat{R}_t + \tau_1(\widehat{V}_t^* + \Psi \frac{\widehat{Q_{i,t}}}{Q_t^*}) + \\ & \widetilde{E}_t \sum_{j=1}^{\infty} (\xi\beta)^j \left\{ -R\widehat{R}_{t+j} + \tau_1 \left[\widehat{V}_{t+j}^* + \Psi \left(\frac{\widehat{Q_{i,t}}}{Q_t^*} - \sum_{k=1}^{j} \widehat{\mu}_{q,t+j} \right) \right] \right\} = 0, \end{aligned} \qquad (7)$$

where $\tau_1 = (2\eta' V + \eta'' V^2)$. In equation (7), the expectation operator is taken over aggregate variables.

Let L^{-1} denote the forward operator. Then (7) implies

$$\begin{aligned} 0 \;=\; & E_t \Bigg[(-\widetilde{R}_t + \tau_1 \widehat{V}_t^* + \omega \frac{\widehat{Q_{i,t}}}{Q_t^*})(1 - \xi\beta L^{-1}) + \\ & \xi\beta \left(-\widetilde{R}_{t+1} + \tau_1 \widehat{V}_{t+1}^* - \frac{\Psi \tau_1}{1-\xi\beta} \widehat{\mu}_{q,t+1} \right) \Bigg], \end{aligned} \qquad (8)$$

where $\omega = \tau_1 \Psi \frac{1}{1-\xi\beta} < 0$. In (8) the expectation operator $\widetilde{E}$ has been replaced by the regular expectational term since with aggregate variables those operators are the same. I am interested in a symmetric equilibrium in which active households choose the same levels of consumption and nominal balances. Moreover, individual and aggregate money holdings are related by $Q_t^* = \int_A Q_{i,t}da + \int_I Q_{i,t}di$ where A and I are the sets of active and inactive households, respectively. Then we can show that relative money balances for active households obey $\widehat{\frac{Q_{i,t}}{Q_t^*}} = \frac{\xi}{1-\xi}\widehat{\mu}_{q,t}$. Combining this equation and (8) and collecting terms we can obtain an expression similar in structure to the new Phillips curve equation,

$$-\widetilde{R}_t + \tau_1 \widehat{V}_t^* - \omega\frac{\xi}{1-\xi}\left(\widehat{\mu}_{q,t} - \xi\beta E_t\widehat{q}_{t+1}\right) - \frac{\Psi\tau_1}{1-\xi\beta}\xi\beta E_t\widehat{\mu}_{q,t+1} = 0.$$

Finally, the growth rate of money holding can be rewritten in terms of aggregate velocity and inflation using

$$\widehat{\mu}_{q,t} = \widehat{V}_{t-1}^* - \widehat{V}_t^* + \widehat{\pi}_t + \widehat{g}_{c,t}.$$

Combine the last two equations to arrive to $\widetilde{R}_t = \phi_1\widehat{V}_t^* + \phi_3(\widehat{\pi}_t + \widehat{g}_{c,t}^* + \widehat{V}_{t-1}^*) + E_t[\phi_2(\widehat{V}_{t+1}^* - \widehat{g}_{c,t+1}^* - \widehat{\pi}_{t+1})]$,with the reduced parameters given by

$$\begin{aligned}
\phi_1 &= \tau_1 - \frac{\xi}{1-\xi}\omega - \frac{\xi\omega}{1-\xi}\xi\beta - \frac{\Psi\tau_1}{1-\xi\beta}\xi\beta > 0, \\
\phi_2 &= \frac{\xi\omega}{1-\xi}\xi\beta + \frac{\Psi\tau_1}{1-\xi\beta}\xi\beta < 0, \\
\phi_3 &= \frac{\xi}{1-\xi}\omega < 0.
\end{aligned}$$

It follows that $\phi_1 + \phi_2 + \phi_3 = \tau_1 > 0$ and $\phi_1 > \tau_1$ because $\omega < 0$ and $\Psi < 0$.

Next, I show that the roots in the polynomial $\frac{\phi_3}{\phi_2}z^2 + \frac{\phi_1}{\phi_2}z + 1 = 0$ are such that $\lambda_i = z_i^{-1}$ satisfy $-1 < \lambda_1 < 1 < \lambda_2$. The solution to the polynomial is given by:

$$z_1 = \frac{-\frac{\phi_1}{\phi_2} + \sqrt{\left(\frac{\phi_1}{\phi_2}\right)^2 - 4\frac{\phi 3}{\phi_2}}}{2\frac{\phi_3}{\phi_2}}, z_2 = \frac{-\frac{\phi_1}{\phi_2} - \sqrt{\left(\frac{\phi_1}{\phi_2}\right)^2 - 4\frac{\phi 3}{\phi_2}}}{2\frac{\phi_3}{\phi_2}}.$$

$z_1 > 1$ is true if and only if $\sqrt{\left(\frac{\phi_1}{\phi_2}\right)^2 - 4\frac{\phi 3}{\phi_2}} > 2\frac{\phi_3}{\phi_2} + \frac{\phi_1}{\phi_2} > 0$. Squaring both sides of the inequality and canceling out common terms the last inequality holds if and only if $\phi_1 + \phi_2 + \phi_3 > 0$; but this condition holds for the relevant parameters in this study. Therefore, the root with positive sign in front of the square root is larger than one or $0 < z_1^{-1} < 1$. For the negative root, we have $-\sqrt{\left(\frac{\phi_1}{\phi_2}\right)^2 - 4\frac{\phi 3}{\phi_2}} < 0 < 2\frac{\phi_3}{\phi_2} + \frac{\phi_1}{\phi_2}$ which implies that $0 < z_2 < 1$ since $\frac{\phi_1}{\phi_2} > \sqrt{\left(\frac{\phi_1}{\phi_2}\right)^2 - 4\frac{\phi 3}{\phi_2}}$. Once again, this implies $1 < z_2^{-1}$. In all these derivations, I have assumed that the term inside the square root is positive. I could not find enough conditions on the structural parameters to guarantee this assumption. However, the estimated coefficients reported in the paper satisfy that assumption.

3 Derivation of Elasticities

I define the interest semi-elasticity of money demand as the percentage change in real balances stemming from an increase of a hundred basis points in the annualized interest rate. Operationally, this definition implies that the semi-elasticity, ζ, is given by:

$$\zeta = -\frac{1}{4}\frac{\partial(\log(Q/P)_t)}{\partial R_t}. \tag{9}$$

Equation (9) accounts for the quarterly time period of the model. For this definition to be meaningful, however, we must find a representation in which real balances are a function of interest rates. To that end, I proceed in two steps. First, I solve (8) for velocity as a function of interest rates and the other variables. Using standard techniques (see Hamilton, 1994), it is straightforward to show that velocity is governed by the stationary solution to the difference equation (8):

$$\widehat{V}_t^* = \lambda_1 \widehat{V}_{t-1}^* + \frac{1}{c_2\lambda_2} E_t \sum_{j=0}^{\infty} (\lambda_2)^{-j} \left[-\widetilde{R}_{t+j} + c_3(\widehat{\pi}_{t+j} + \widehat{g}_{c,t+j}^*) - c_2(\widehat{g}_{c,t+j+1}^* + \widehat{\pi}_{t+j+1}) \right], \tag{10}$$

where λ_1 and λ_2 are the inverse of the roots in the polynomial $\frac{c_3}{c_2}z^2 + \frac{c_1}{c_2}z + 1 = 0$.[3] The estimated coefficients satisfy $c_1 > 0, c_2, c_3 < 0$ and are such that $0 < \lambda_1 < 1 < \lambda_2$ as required for (10) to be a valid solution. The previous section showed that the structural parameters satisfy that relation between the characteristic roots. Second, because velocity and real balances are linked through the equation $V = \frac{PC}{Q}$, I combine this condition and the stationary solution for velocity and arrive at a condition involving real balances as a function of interest rates as desired.

The idea behind (10) is straightforward. First, the term in front of c_3 is a consequence of the relation $\widehat{V}_t^* = \widehat{V}_{t-1}^* + \widehat{\pi}_t + \widehat{g}_{c,t}^* - \widehat{q}_t^*$ used in the derivation of the equation for velocity. According to this condition, today's velocity differs from yesterday's because innovations in inflation, consumption growth, and money growth. The higher the first two factors are,

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[3] I have imposed a boundedness condition to eliminate terms of the form $a_1\lambda_1^{-t}$ and $a_2\lambda_2^{t}$.

the higher velocity is today. Second, the terms $\widehat{g}^*_{c,t+j+1}+\widehat{\pi}_{t+j+1}$ reflect the forward looking behavior of velocity stemming from sluggish money balances and transaction costs. If households expect higher consumption rates in the future, they will forecast an increase in velocity that will drive up transaction costs. To counterbalance the rise in costs and afford the additional consumption, households must increase their money balances in the future. As the Calvo friction implies, however, this will be impossible for some households. So these households raise their money balances today, when they can re-optimize. Moreover, higher inflation in the future implies that consumption will be more expensive requiring higher money balances to afford the same level of consumption. In either case, current velocity declines thanks to higher current money balances.

Equation (10) also indicates the positive relation between velocity and present and future interest rates. If households expect high interest rates in the future, holding money balances, Q, is more expensive than sending a dollar to the bank. Therefore, households decrease current money balances, sparking an increase in velocity. Furthermore, the increase in velocity, and the decline in real balances, depend on the persistence of interest rates. Consider, for example, the effect of a temporary increase in interest rates at time t. The immediate effect on velocity, keeping $\widehat{\pi}$ and $\widehat{g}^*_c$ constant, is measured by the term $-\frac{1}{\lambda_2 c_2}$. This result indicates that, other things equal, a 1-percent increase in interest rates raises velocity by $-\frac{1}{\lambda_2 c_2}$ percent. Considering this outcome, we can define an elasticity measuring the effect of a temporary increase in interest rates:

$$\zeta^T = -\frac{1}{4c_2\lambda_2}.$$

Here, the index T stands for "temporary" to reflect this definition's correspondence to a temporary increase in interest rates. Unlike temporary increases, permanent rises in interest rates have short- and long-term consequences on real balances. After the increase in interest rates, velocity declines because households forecast a permanent change in interest rates. Moreover, when the economy reaches its new steady state, velocity settles at a value different from its initial value. This latter effect happens because of the backward looking term in (10). Using the same logic as before, we can define the following semi-elasticites:

$$\begin{aligned}
\zeta^P_{short} &= -\frac{1}{4c_2\lambda_2(1-\lambda_2^{-1})},\\
\zeta^P_{long} &= -\frac{1}{4c_2\lambda_2(1-\lambda_1)(1-\lambda_2^{-1})}.
\end{aligned}$$

The index P stands for "permanent." A direct comparison between the temporary and permanent definitions reveals that the elasticities from a permanent increase in interest rates are larger than those from a temporary increase, $\zeta^P > \zeta^T$. The reason is that a more persistent change in interest rates generates a larger change in real balances because transaction costs, η, and the Calvo adjustment in money balances induce agents to be more forward looking than in the absence of time-dependent adjustment.

4 Estimation of the Demand for Money

A direct and testable implication of the money demand equation and rational expectations is

$$E[(-\widetilde{R}_t + \phi_1 \widehat{V}_t^* + \phi_3(\widehat{\pi}_t + \widehat{g}^*_{c,t} + \widehat{V}^*_{t-1}) + \phi_2(\widehat{V}^*_{t+1} - \widehat{g}^*_{c,t+1} - \widehat{\pi}_{t+1}))X_{t-1}] = 0, \tag{11}$$

where X_{t-1} is any variable on the household's information set at time $t-1$. I use the GMM estimation as outlined in Hansen (1982) to estimate the reduced parameters ϕ_i. Burnside and Eichenbaum (1996) show that imposing restrictions implied by the underlying model greatly improves the estimate of the weighting matrix. Furthermore, the moment condition (11) implies that the term in square brackets has an MA(1) representation. Therefore, I choose the weighting matrix for GMM to yield a consistent estimate of

$$\Omega = \sum_{k=-\tau}^{\tau} E[\psi_{t+1+k} X_{t+k-\tau}][\psi_{t+1+k} X_{t+k-\tau}]', \tag{12}$$

where $\psi_{t+1} \equiv -\widetilde{R}_t + \phi_1 \widehat{V}_t^* + \phi_3(\widehat{\pi}_t + \widehat{g}^*_{c,t} + \widehat{V}^*_{t-1}) + \phi_2(\widehat{V}^*_{t+1} - \widehat{g}^*_{c,t+1} - \widehat{\pi}_{t+1})$ and $\tau = 1$. The weighting matrix is computed using the Newey-West estimator.

5 Replication Code

The replication code is available at: http://www4.ncsu.edu/~paguerro/research.htm

References

[1] Burnside, C., and M. Eichenbaum, 1996. *Small Sample Properties of GMM based Wald Tests,* Journal of Business and Economics Statistics, Vol. 14, pp. 294-308.

[2] Christiano, L., 2004. *Firm-Specific Capital and Aggregate Inflation Dynamics in Woodford's Model*, mimeo Northwestern University.

[3] Guerron-Quintana, P., 2008. *Money Demand Heterogeneity and the Great Moderation*, mimeo Department of Economics, North Carolina State University.

[4] Hamilton, J., 1994. *Time Series Analysis,* Princeton University Press.

[5] Hansen, L., 1982. *Large Sample Properties of Generalized Method of Moments Estimators,* Econometrica, Vol. 50, pp. 1029-1060.

[6] Lang, S., 1993. *Real and Functional Analysis,* Third Edition, Springer-Verlag Ed.

[7] Woodford, M., 2003. *Interest and Prices*, Princeton University Press.

[8] , 2005. *Firm-Specific Capital and the New-Keynesian Phillips Curve*, working paper Columbia University.