Prosthaphaeresis

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The word *prosthaphaeresis* may sound like a complicated medical procedure, but is really a method of calculation that was used in the astronomical observatories of Europe during the sixteenth and early seventeen century, prior to the development of logarithms. The word is a combination of the Greek words for addition and subtraction. Prosthaphaeresis is a method which reduces multiplication and division of numbers to addition, subtraction and trigonometric table lookup using any of the prosthaphaeretic formulae:

$$2\cos\alpha\cos\beta = \cos(\alpha+\beta) + \cos(\alpha-\beta) \tag{1}$$

$$2\sin\alpha\sin\beta = \cos(\alpha+\beta) - \cos(\alpha-\beta) \tag{2}$$

$$2\sin\alpha\cos\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta) \tag{3}$$

$$2\cos\alpha\sin\beta = \sin(\alpha+\beta) - \sin(\alpha-\beta) \tag{4}$$

For example to multiply 0.925473 by 0.689745, using equation(1), we find angles α and β such that $0.925473 = \cos\alpha$ and $0.689745 = \cos\beta$. From a cosine table we find $\alpha = .388514$ and $\beta = .80966$. Thus $\alpha + \beta = 1.198174$, $\alpha - \beta = -.0421146$ and $(\cos(1.198174) + \cos(-.0421146))/2 = .638344$, the product correct to 6 figures.Numbers greater than one can be handled by adjusting the decimal point and division can be performed eith the aid of tables of secant or cosecatn. Since 10 to 15 place trigonometric tables were available at the time, arithmetic calculations could be handled with great accuracy and with a considerable saving of time and effort over hand calculations.

It is likely that Napier knew of the method of prosthaphaeresis and it is possible that this knowledge was a factor in his development of logarithms. Of course once logarithmic tables became available, prosthaphaeretic calculations quickly faded from use.

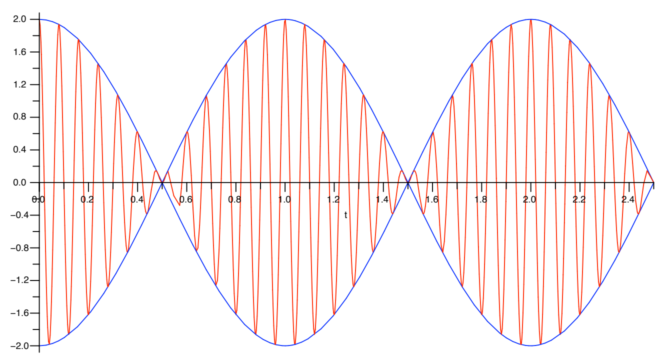
The trigonometric identities of prosthaphaeresis are helpful in understanding the phenomenon of beats in the theory of vibrations. Suppose two sinusoids of the same amplitude and almost the same frequency are added say

$$y = \cos(2\pi f_1 t) + \cos(2\pi f_2 t) \tag{5}$$

where f_1 and f_2 are the two frequencies in cycles per second. What does the graph of y versus t look like? Plotting each term and adding is very tedious and, unless done very carefully, would not show the essential nature of the graph. A better approach is to use Equation(1) with $\alpha - \beta = 2\pi f_1$ and $\alpha + \beta = 2\pi f_2$ to obtain

$$y = 2\cos(\pi(f_1 - f_2)t) \cdot \cos(\pi(f_1 + f_2)t) \tag{6}$$

Since $f_1 - f_2$ is small, therefore the above equations may be considered as a sinusoid with frequency $(f_1 + f_2)/2$ (which is approximately f_1 or f_2) with slowly varying amplitude $2\cos(\pi(f_1 - f_2)t)$. In the figure below we have sketched Equation (5) (or (6)) with $f_1 = 13$ and $f_2 = 12$. A maximum of amplitude, or a *beat*, will occur whenver the amplitude is $+1$ or -1. Thus the number of beats per second is twice the frequency $(f_1 - f_2)/2$, or the number of beats per second equals the difference in frequencies. In the figure the beat frequency is 1 beat per second.



$$y = \cos(2\pi \cdot 13t) + \cos(2\pi \cdot 12t) = 2\cos(\pi t) \cdot \cos(25\pi t)$$

The phenomenon of beats may be illustrated on a piano. The frequency of A above middle C is 440 cycles per second and the frequency of the adjacent note B is about 497 cycles per second (depending on the temperment of the scale). If these two notes are struck simultaneously and held down one hears an increase and decrease in loudness at the beat frequency (the difference in frequencies) of 57 cycles per second.One can tune two strings to the same tone by tightening one, while striking both, until the beats disappear. This is the principle used in tuning a piano or guitar.

A somewhat surprising visual illustration of the beat phenomenon occurs in the Wilberforce spring. This is simply a spiral spring which holds a mass with an appropriate moment of inertia. If one displaces the mass vertically, it begins to oscillate up and down with little rotation. Then the vertical motion decreases and the rotation increases. This continues until the mass is almost purely rotating with little vertical motion. The rotation then decreases and the vertical motion increases until it returns to vertical motion with practically no rotation. It can be shown that their are two normal modes of vibration, one of almost pure rotation, and one of almost pure translation. If these frequencies are close together, the phenomenon of beats occurs.

As a final application of the phosthaphaeretic formula we shall describe one cannot receive high fidelity sound on A-M radio. An A-M radio receives a high frequency electromagnetic wave called the carrier frequency whose amplitude is modulated in accordance with the desired sound or music. The carrier frequency in in the range of about 600 to 1500 kilocycles per second. The audible spectrum is about 0 to 15 kilocycles per second. For high fidelity sound we would need to transmit this range of frequencies. Suppose for simplicity we desire to transmit a single sinusoidal wave with a frequency f_a in the audible spectrum. The modulated wave has the form

$$(1 + 2a\cos(2\pi f_a t)) \cdot \cos(2\pi f_c t) \tag{7}$$

where f_c is the carrier frequency, and a is between 0 and 1/2. Using the first prosthaphaeretic formula (1) we may write equation (7) as

$$\cos(2\pi f_c t) + a\cos(2\pi(f_c + f_a)t) + a\cos(2\pi(f_c - f_a)t) \tag{8}$$

The result of the modulation is to introduce the "sideband frequencies" $f_c + f_a$ and $f_c - f_a$ in addition to the carrier frequency f_c. In order to have high fidelity sound it would be necessary to use a frequency spectrum of 15 kilocyles per second on both sides of the carrier frequency. However, in order to accomodate more stations, each station is restricted to frequencies of 5 kilocycles on both sides of the carrier frequency. Thus only a portion of the audible spectrum can be heard on A-M radios.