INTRODUCTION
TO
LINEAR ANALYSIS

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REVISED EDITION
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CHAPTER I
DIFFERENCE EQUATIONS

1.1–Introduction

Much of this book is devoted to the analysis of dynamical systems, that is, systems that change with time. The motion of a body under known forces, the flow of current in a circuit and the decay of a radioactive substance are examples of dynamical systems. If the quantity of interest in a dynamical system is considered to vary continuously with time, the system is called a continuous dynamical system. The physical laws that govern how continuous dynamical systems evolve in time are often given by equations involving time derivatives of the desired quantities; such equations are called differential equations. For example, the decay of a radioactive substance is governed by the differential equation

$$\frac{dm(t)}{dt} = -km(t), \quad t \geq 0,$$

where \(m(t)\) is the mass of the substance at time \(t\) and parameter \(k\) is a positive constant which depends on the particular substance involved. If the initial mass \(m(0)\) is known, \(m(t)\) can be found by solving the differential equation. Methods for solving differential equations will be discussed in Chapter 2.

In this chapter we shall discuss discrete dynamical systems where the quantity of interest is defined, or desired, only at discrete points in time. Often these discrete time points are uniformly spaced. Economic data, for instance, is usually obtained by periodic reports – daily, weekly, monthly, or yearly. A firm takes inventory perhaps monthly or quarterly. The quantity of interest in a discrete dynamical system is sequence of values denoted by \(\{x_k\}\), or completely written out

$$x_0, \ x_1, \ x_2, \ldots, \ x_k, \ldots,$$

where \(x_0\) represents the value at the beginning of the initial time period, \(x_1\), the value at the end of the first time period, and \(x_k\), the value at the end of the \(k^{th}\) time period.

The laws which govern the evolution of a discrete dynamical system are often expressed by equations which involve two or more general terms of the sequence of values of the desired quantity. Some examples of such equations are

\[
\begin{align*}
x_{k+1} &= -2x_k, \quad k = 0, 1, 2, \ldots \\
x_{k+2} - 2x_{k+1} + x_k &= 0, \quad k = 0, 1, 2, \ldots \\
x_k &= x_{k-1} + 100, \quad k = 1, 2, 3, \ldots
\end{align*}
\]

Equations of this type which indicate how the terms of a sequence recur are called recurrence relations, or more commonly, difference equations. The methods of solution of certain types of difference equations will be studied in this chapter.

Example 1. A population of fish is observed to double every month. If there are 100 fish to start with, what is the population of fish after \(k\) months?

Solution. Let \(x_k\) represent the number of fish after \(k\) months, then \(x_{k+1}\) is the number of fish after \(k + 1\) months. Since the number of fish doubles each month, we must have

$$x_{k+1} = 2x_k, \quad k = 0, 1, 2, \ldots$$

(i)
This is the equation governing the growth of the fish population.

Since the initial population of fish is 100, we have \( x_0 = 100 \). Successively substituting \( k = 0, 1, 2 \), into the difference equation we obtain

\[
\begin{align*}
x_1 &= 2x_0, \\
x_2 &= 2x_1 = 2(2x_0) = 2^2x_0, \\
x_3 &= 2x_2 = 2(2^2x_0) = 2^3x_0.
\end{align*}
\]

By continuing in this manner the number of fish at the end of any particular month could be found. However, what is really desired is an explicit formula for \( x_k \) as a function of \( k \). Looking at the expressions for \( x_1, x_2, \) and \( x_3 \) given above, a good guess for \( x_k \) is

\[ x_k = 2^k x_0 = 2^k(100). \]

To verify this, we first put \( k = 0 \) into the above formula to find \( x_0 = 100 \). Next we note that \( x_{k+1} = 2^{k+1}(100) \). Substituting \( x_k \) and \( x_{k+1} \) into the difference equation (i) we find

\[ x_{k+1} - 2x_k = 2^{k+1}(100) - 2 \cdot 2^k(100) = 0 \text{ for all } k. \]

Thus \( x_k = 2^k \cdot 100 \) is the desired solution.

**Example 2.** Suppose that every year the growth of a fish population is twice the growth in the preceding year. Write a difference equation for the number fish after \( k \) years.

**Solution** Let \( x_k \) denote the number of fish after \( k \) years. The growth of fish during the \( k \)th year is \( x_k - x_{k-1} \), and the growth of fish during the \((k + 1)\)st year is \( x_{k+1} - x_k \). According to the law of growth stated we must have

\[ x_{k+1} - x_k = 2(x_k - x_{k-1}), \quad k = 1, 2, \ldots \]

or

\[ x_{k+1} - 3x_k + 2x_{k-1} = 0, \quad k = 1, 2, \ldots \]

Suppose that the number of fish is initially 100 and that the number at the end of one year is 120, this means that \( x_0 = 100 \) and \( x_1 = 120 \). To find \( x_2 \), the number of fish at the end of the 2nd year, we put \( k = 1 \) in the difference equation to get

\[ x_2 = 3x_1 - 2x_0 = 3(120) - 2(100) = 160 \]

In a similar manner we could compute \( x_3, x_4, \ldots \) However a formula for \( x_k \) as an explicit function of \( k \) is not as easy to come by as in the previous example. We shall find out how to do this later in this chapter.

**Exercises 1.1**

1. For each of the following difference equations find \( x_1, x_2, \) and \( x_3 \) in terms of \( x_0 \).
   a. \( x_{n+1} = -2x_n, \quad n = 0, 1, 2, \ldots \)
   b. \( x_k = 3x_{k-1} + 2k, \quad k = 1, 2, 3, \ldots \)
   c. \( x_{k+1} = (x_k)^2 - 2k, \quad k = 0, 1, 2, \ldots \)

2. For the following difference equations find \( x_2 \) and \( x_3 \) in terms of \( x_0 \) and \( x_1 \).
   a. \( x_{k+2} + 3x_{k+1} + 2x_k = 2k, \quad k = 0, 1, 2, \ldots \)
   b. \( x_{n+1} - 2x_n + x_{n-1} = 0, \quad n = 1, 2, 3, \ldots \)

3. For each of the following find \( x_k \) as a function of \( k \) and \( x_0 \) and verify that the difference equation is satisfied.
   a. \( x_{k+1} = -2x_k, \quad k = 0, 1, 2, \ldots \)
   b. \( x_{k+1} = x_k, \quad k = 0, 1, 2, \ldots \)
   c. \( 2x_k = 3x_{k-1}, \quad k = 1, 2, 3, \ldots \)
4. Suppose that you get a job with a starting salary of 20,000 dollars and you receive a raise of 10% each year. Write a difference equation for your salary during the $k$th year.

5. Redo problem 4 if the raise you receive each year is 1000 dollars plus 10%.

6. Initially a population of fish is 1000 and grows to 1200 at the end of one year. If the growth of fish in any year is proportional to the number of fish at the end of the previous year, set up a difference equation for the number of fish at the end of the $n$th year. Be sure to evaluate the constant of proportionality.

7. The first two terms in a sequence are $x_1 = 1$ and $x_2 = 1$. Thereafter, each term is the sum of the preceding two terms. Write out the first 5 terms of the sequence and set up a difference equation, with proper initial conditions, for the $n$th term.

1.2–Sequences and Difference Equations

In the previous section the terms ‘sequence’, ‘difference equation’, and ‘solution of a difference equation’ were introduced. Before proceeding, it is important to have a clear understanding of what these terms mean.

Definition 1. A sequence is a function whose domain of definition is the set of nonnegative integers $0, 1, 2, \ldots$. If the function is $f$, the terms of the sequence are

$$x_0 = f(0), \ x_1 = f(1), \ x_2 = f(2), \ldots$$

The sequence may be denoted by \{x_k\} or by writing out the terms of the sequence in order

$$x_0, \ x_1, \ x_2, \ldots, \ x_k, \ldots$$

For example, the sequence

$$1, \ 1, \ 1, \ 1, \ldots, \ 1, \ldots$$

is a constant sequence all of whose terms are equal to 1, that is, $x_k = 1$ for all $k$. The sequence defined by $x_k = k$ for $k = 0, 1, 2, \ldots$, when written out is

$$0, \ 1, \ 2, \ 3, \ 4, \ldots$$

This is an arithmetic sequence since the difference between successive terms is a constant; in this case the difference is 1. This sequence could also be defined implicitly by the equation

$$x_k = x_{k-1} + 1, \ k = 1, 2, 3, \ldots$$

provided we impose the initial condition $x_0 = 0$. The sequence

$$a, \ a + d, \ a + 2d, \ldots, \ a + kd, \ldots$$

is a general arithmetic sequence with an initial term of $a$ and a difference of $d$. The $k$th term of the sequence is $x_k = a + kd$. This sequence can be defined by the equation.

$$x_k = x_{k-1} + d, \ k = 1, 2, 3, \ldots$$

where $x_0 = a$.

A geometric sequence is one where each term is obtained by multiplying the preceding term by the same factor. If the initial term is $a$ and the factor is $r$, the terms are

$$q, \ qr, \ qr^2, \ qr^3, \ldots, \ qr^k, \ldots$$
The general term of the sequence is \( x_k = qr^k, k = 0, 1, 2, \ldots \). The sequence can also be defined implicitly by the equation
\[
x_k = r x_{k-1}, \quad k = 1, 2, \ldots
\]  
(7)
together with the initial condition \( x_0 = q \).

Finally, let us consider the sequence whose first few terms are
\[
1, 1, 2, 3, 5, 8, 13, 21, 34.
\]  
(8)

What is the next term of the sequence? This is a common type of problem on so-called “intelligence” tests. There is no one correct answer. The next number could be any number at all, since no general rule or function is given to determine the next term. What is expected on these “intelligence” tests is to try to determine some pattern or rule that the given terms satisfy and then to use this rule to get additional terms of the sequence. You may notice by looking at the terms in (8) that each term, after the first two terms, is the sum of the preceding two terms. If we assume that this is the pattern that the terms satisfy, we get the so-called Fibonacci numbers. That is, the Fibonacci numbers are defined implicitly by
\[
x_k = x_{k-1} + x_{k-2}, \quad k = 3, 4, 5, \ldots
\]  
(9)
together with the initial conditions \( x_1 = 1 \) and \( x_2 = 1 \). It is now easy to write down additional terms of the sequence—the next three terms are 55, 89, 144. It is not so easy to write an explicit formula for the \( k \)th term; this will be done later in this chapter. Notice also that we have chosen to call the initial term of the sequence \( x_1 \) instead of \( x_0 \).

It is often useful to sketch the graph of a given sequence. If the horizontal axis is the \( k \)-axis and the vertical axis is the \( x_k \)-axis, the graph consists of the points whose coordinates are \((0, x_0), (1, x_1), \) and so on. The sequences \( x_k = 3 - 1/2^k \) and \( y_k = 2 \cos(k \pi/4) \) are sketched below.

![Graphs of sequences](image)

These graphs provide a picture of what happens to the terms of the sequence as \( k \) increases. For example, the graphs illustrate that \( \lim_{k \to \infty} x_k = 3 \) while \( \lim_{k \to \infty} y_k \) does not exist. We have drawn curves through the points of the graphs of the sequences so that the changes in the terms of the sequence can be more easily seen; these curves, however, are not part of the graphs.

The equation defining an arithmetic sequence, \( x_k = x_{k-1} + d \), expresses the \( k \)th term of a sequence in terms of the one immediately preceding term; this equation is called a difference equation of the first order. The equation defining a geometric sequence, \( x_k = r x_{k-1} \) is also a first order difference equation.
Section 1.2–Sequences and Difference Equations

The equation which defines the Fibonacci sequence, \( x_k = x_{k-1} + x_{k-2} \) expresses the \( k \)th term in terms of the two immediately preceding terms; it is called a difference equation of the second order.

**Definition 2.** A difference equation of the \( p \)th order is an equation expressing the \( k \)th term of an unknown sequence in terms of the immediately preceding \( p \) terms, that is, a \( p \)th order equation is an equation of the form

\[
x_k = F(k, x_{k-1}, x_{k-2}, \ldots, x_{k-p}), \quad k = p, \ p + 1, \ p + 2, \ldots
\]

where \( F \) is some well defined function of its several arguments. The order of the equation is therefore equal to the difference between the largest and the smallest subscripts that appear in the equation.

A difference equation expresses the value of the \( k \)th term of a sequence as a function of a certain number of preceding terms. It does not directly give the value of the \( k \)th term. In other words the difference equation gives the law of formation of terms of a sequence, not the terms themselves. In fact there are usually many different sequences that have the same law of formation. For example, let us look at the equation \( x_k = x_{k-1} + 2, \ k = 1, 2, 3, \ldots \) Any sequence whose successive terms differ by 2 satisfies this equation, for instance 0, 2, 4, 6, 8, ..., or 3, 5, 7, 9, ... Each of these sequences is called a solution of the difference equation.

**Definition 3.** Let \( x_k \) be the unknown sequence in a difference equation. By a solution of the difference equation we mean a sequence given as an explicit function of \( k \), \( x_k = f(k) \), which satisfies the difference equation for all values of \( k \) under consideration. By the general solution of a difference equation we mean the set of all sequences that satisfy the equation.

**Example 1.** Is \( x_k = 2^k \) a solution of \( x_k = 2x_{k-1}, \ k = 1, 2, 3, \ldots \)?

Since \( x_k = 2^k \), we have

\[
x_k - 2x_{k-1} = 2^k - 2(2^{k-1}) = 2^k - 2^k = 0, \text{ for all } k.
\]

Thus the given sequence is a solution. Similarly it can be shown that the sequence \( x_k = a \cdot 2^k \) satisfies the difference equation for all values of the parameter \( a \). The difference equation therefore has infinitely many solutions. We shall see shortly that every solution of the difference equation can be represented in the form \( x_k = a \cdot 2^k \) for some value of \( a \); this is the general solution of the difference equation.

**Example 2.** Show that \( x_k = 1 + 2k \) satisfies the difference equation

\[
x_k = 2x_{k-1} - 2k + 3, \ k = 1, 2, 3, \ldots
\]

We find that \( x_{k-1} = 1 + 2(k - 1) = 2k - 1 \), therefore

\[
x_k - 2x_{k-1} + 2k - 3 = 1 + 2k - 2(2k - 1) + 2k - 3 = 0, \text{ for all } k.
\]

**Example 3.** Is \( x_k = 2^k \) a solution of the difference equation

\[
x_k = x_{k-1} + x_{k-2}, \ k = 2, 3, 4, \ldots
\]

We have \( x_{k-1} = 2^{k-1} \) and \( x_{k-2} = 2^{k-2} \), so that

\[
x_k - x_{k-1} - x_{k-2} = 2^k - 2^{k-1} - 2^{k-2}.
\]

This is certainly not zero for all \( k \), therefore, the given sequence is not a solution.

**Example 4.** Show that \( x_k = \cos(k\pi/2) \) is a solution of \( x_{k+2} + x_k = 0, \ k = 0, 1, 2, \ldots \).
6 Chapter I–DIFFERENCE EQUATIONS

Solution Since \( x_{k+2} = \cos((k + 2)\pi/2) = \cos(\pi + k\pi/2) = -\cos(k\pi/2) \), we have
\[
x_{k+2} + x_k = -\cos(k\pi/2) + \cos(k\pi/2) = 0, \quad k = 0, 1, 2, \ldots
\]

At the risk of beating a simple example to death, let us look again at the simple first order equation \( x_k = 2x_{k-1}, \) \( k = 1, 2, 3, \ldots \). We have seen in Example 1 that this equation has infinitely many solutions. In order to obtain a unique solution the value of some one term, usually \( x_0 \), must be prescribed. This is called an initial value problem which we write in the form.
\[
\Delta E : \quad x_k = 2x_{k-1} \quad k = 1, 2, 3, \ldots
\]
\[
IC : \quad x_0 = -3
\]
where \( \Delta E \) stands for difference equation and \( IC \) for initial condition. It is clear that there is only one sequence which satisfies the difference equation and the initial condition. For, if \( x_0 \) is known, the difference equation uniquely determines \( x_1 \), and once any value \( x_{k-1} \) is known, the difference equation uniquely determines \( x_k \). By the principle of mathematical induction, \( x_k \) is uniquely determined for all \( k \). It is easy to show that this unique solution is \( x_k = -3 \cdot 2^k \), but the point is that we know that there is one and only one solution beforehand.

In order to determine a unique solution for a second order difference equation, two successive values, say \( x_0 \) and \( x_1 \), need to be known. For a \( p \)th order equation we have the following theorem.

**Theorem 1.** The initial value problem
\[
\Delta E : \quad x_k = F(k, x_{k-1}, \ldots, x_{k-p}), \quad k = p, \ p + 1, \ldots
\]
\[
IC : \quad x_0 = a_0, \ x_1 = a_1, \ldots, x_{p-1} = a_{p-1}
\]
has a unique solution \( x_k \) for each choice of the initial values \( a_0, \ldots, a_{p-1} \).

**Proof** Substituting \( k = p \) into the difference equation we find that \( x_p \) is uniquely determined by the initial conditions. Once the values of \( x_0 \) up to \( x_{k-1} \) are known, the value \( x_k \) is uniquely determined by the equation. By the principle of mathematical induction, \( x_k \) is uniquely determined for all \( k \).

Finally we mention that a difference equation can be written in many different but equivalent forms. For example, consider
\[
x_k = 2x_{k-1} + 2, \quad k = 1, 2, 3, \ldots
\]
(11)

By replacing \( k \) everywhere it appears with \( k + 1 \), we obtain
\[
x_{k+1} = 2x_k + 2, \quad k + 1 = 1, 2, 3, \ldots
\]
or
\[
x_{k+1} = 2x_k + 2, \quad k = 0, 1, 2, \ldots
\]
(12)

Equations (11) and (12) are just two different ways of saying the same thing; they differ only in notation and not in content. The three equations below are also equivalent.
\[
x_k = x_{k-1} + x_{k-2}, \quad k = 2, 3, 4, \ldots
\]
(13)
\[
x_{k+1} = x_k + x_{k-1}, \quad k = 1, 2, 3, \ldots
\]
(14)
\[
x_{k+2} = x_{k+1} + x_k, \quad k = 0, 1, 2, \ldots
\]
(15)

Equation (14) is obtained from (13) by replacing \( k \) by \( k + 1 \), and (15) is obtained from (14) by again replacing \( k \) by \( k + 1 \). Note that it is necessary to change the range of \( k \) for which each equation holds.

**Exercises 1.2**

1. Determine whether or not the given sequence is a solution of the indicated difference equation:
   a. \( x_k = 5(-2)^k, \ x_k = -2x_{k-1}, \ k = 1, 2, \ldots \)
   b. \( x_k = 3 \cdot 2^k + 1, \ x_{k+1} = 2x_k + 1, \ k \geq 0 \)
   c. \( x_k = 2^k + 3^k - 2^k, \ x_{k+1} = 2x_k + 3^k, \ k = 0, 1, \ldots \)
   d. \( x_k = \sin(k\pi/2), \ x_{k+2} + 3x_{k+1} + x_k = 0, \ k = 0, 1, \ldots \)
2. Find the values of the constant \( A \), if any, such that the constant sequence \( x_k = A \) is a solution of the difference equations:

- a. \( x_k = 3x_{k-1} - 1, \quad k = 1, 2, \ldots \)
- b. \( x_{k+1} = x_k, \quad k = 0, 1, 2, \ldots \)
- c. \( x_{k+1} = 2x_k + k, \quad k = 0, 1, 2, \ldots \)

3. Find the values of the constant \( \lambda \), if any, so that \( x_k = \lambda^k \) is a solution of

- a. \( x_{k+1} = 3x_k, \quad k = 0, 1, 2, \ldots \)
- b. \( x_{k+1} = kx_k, \quad k = 0, 1, 2, \ldots \)
- c. \( x_{k+1} = 5x_k + 6x_{k-1}, \quad k = 1, 2, \ldots \)
- d. \( x_{k+2} + 6x_{k+1} + 5x_k = 0, \quad k = 0, 1, 2, \ldots \)
- e. \( x_{k+1} = 3x_k + 5^k, \quad k = 0, 1, 2, \ldots \)
- f. \( x_{k+2} - x_k = 1, \quad k = 0, 1, 2, \ldots \)

4. Rewrite each of the following difference equations in terms of \( x_k, \ x_{k+1}, \) and \( x_{k+2} \). Also indicate the appropriate range of values of \( k \).

- a. \( x_k = 2x_{k-1}, \quad k = 2, 3, \ldots \)
- b. \( x_{k+1} - 2x_k + 3x_{k-1} = 2^{k-3}, \quad k = 1, 2, \ldots \)

### 1.3–Compound Interest

An understanding of the cumulative growth of money under compound interest is a necessity for financial survival in today’s world. Fortunately, compound interest is not difficult to understand and it provides a simple but useful example of difference equations. Here is the central problem.

> **Suppose an initial deposit of \( A \) dollars is placed in a bank which pays interest at the rate of \( r_p \) per conversion period. How much money is accumulated after \( n \) periods?**

The conversion period is commonly a year, a quarter, a month or a day. Interest rates are usually given as **nominal annual rates**. A monthly interest rate of one-half of one percent or \( r_p = .005 \), is equivalent to an annual rate of \( .005 \times 12 = .06 \) or 6%. In general, if \( r \) is the nominal annual rate, there are \( m \) conversion periods in a year, then \( r_p = r/m \).

Let \( x_n \) denote the amount in the bank at the end of the \( n \)-th period. The interest earned during the \( (n+1) \)-st period is \( r_px_n \), thus the amount of money accumulated at the end of the \( (n+1) \)-st period is given by

\[
x_{n+1} = x_n + r_px_n = (1 + r_p)x_n
\]

Since \( x_0 = A \), the amount accumulated must satisfy the initial value problem

\[
\Delta E: \quad x_{n+1} = (1 + r_p)x_n, \quad n = 0, 1, 2, \ldots
\]

\[
IC: \quad x_0 = A
\]

There is only one sequence which is a solution of this problem. It is rather easy to find this solution. Successively substituting \( n = 0, 1, 2 \), into the difference equation we find

\[
x_1 = (1 + r_p)x_0 = (1 + r_p)A
\]

\[
x_2 = (1 + r_p)x_1 = (1 + r_p)^2A
\]

\[
x_3 = (1 + r_p)x_2 = (1 + r_p)^3A
\]

The obvious guess for \( x_n \) is

\[
x_n = (1 + r_p)^nA, \quad n = 0, 1, 2, \ldots
\]

It is easily verified that this is indeed the solution, for

\[
x_{n+1} - (1 + r_p)x_n = (1 + r_p)^{n+1}A - (1 + r_p)^{n+1}A = 0
\]

Equation (3) is the fundamental formula for compound interest calculations. Tables of \( (1 + r_p)^n \) for various values of \( r_p \) and \( n \) are available. However, it is very easy to perform the necessary calculations on a modern scientific calculator.
Example 1. If $1000 is invested in a bank that pays 7.5% interest compounded quarterly, how much money will be accumulated after 5 years?

Solution  We have \( r_p = 0.075/4 \), \( A = 1000 \) and, since there are 4 conversion periods in each year, \( n = 5 \times 4 = 20 \). Thus, using Equation (3) yields

\[
x_{20} = (1 + 0.075/4)^{20}1000 = 1449.95, \text{ to the nearest cent.}
\]

It is instructive to compare compound interest with simple interest. In simple interest the amount of interest earned in each period is based on the initial amount deposited. Thus for simple interest we have the difference equation

\[
x_{n+1} = x_n + rA, \quad n = 0, 1, 2, \ldots \tag{4}
\]

with \( x_0 = A \). The solution of this difference equation is

\[
x_n = A + nrA, \quad n = 0, 1, 2, \ldots \tag{5}
\]

Therefore for simple interest the difference \( x_{n+1} - x_n \) is constant while for compound interest the ratio \( x_{n+1}/x_n \) is a constant; simple interest yields an arithmetic sequence and compound interest yields a geometric sequence. The graphs of these sequences are shown in Figure 1.

![Graphs of Simple and Compound Interest](image)

**Figure 1**

Modern banks offer a variety of annual interest rates and frequencies of compounding. Without some analysis it is difficult to tell, for instance, whether 6% compounded daily is a better deal than 6.25% compounded annually. We shall analyze the general situation using equation (3). Suppose the annual nominal interest rate is \( r \) and that there are \( m \) compounding periods in a year. The interest per period is \( r_p = r/m \), and the number of periods in \( k \) years is \( n = km \). If we let \( y_k \) denote the amount accumulated after \( k \) years we have

\[
y_k = (1 + r/m)^km y_0, \quad k = 0, 1, 2, \ldots \tag{6}
\]

where \( y_0 = x_0 \) is the initial amount deposited. To compare various interest policies, it is convenient to define the effective annual interest rate, \( r_E \), to be

\[
r_E = (1 + r/m)^m - 1, \tag{7}
\]

that is, \( r_E \) represents the increase in a one dollar investment in one year. Equation (6) can now be written as

\[
y_k = (1 + r_E)^k y_0, \quad k = 0, 1, 2, \ldots \tag{8}
\]
Table 1 shows the effective annual interest rate for various compounding frequencies, assuming an annual interest rate of \( r = .06 \).

A glance at this table shows that the effective annual rate does not show a great increase as the number of compounding periods increases. We see that the effective annual rate under daily compounding is \( .0618 \). Thus you would be better off with 6.25% compounded annually than with 6% compounded daily.

**Example 2.** A bank offers compound interest and advertises that it will triple your money in 15 years. (a) What is the effective annual interest rate? (b) What is the nominal annual rate if interest is compounded monthly? (c) What is the nominal annual rate if interest is compounded daily?

**Solution**  (a) Putting \( k = 15 \), and \( y_{15} = 3y_0 \) into equation (8) we find that

\[
y_0 = (1 + r_E)^{15} y_0, \text{ thus } (1 + r_E)^{15} = 3 \text{ or } r_E = 3^{1/15} - 1 = .07599.
\]

(b) Since \( m = 12 \) for monthly compounding we have

\[
1 + r_E = (1 + r/12)^{12} = 3^{1/15}, \text{ or } r = 12 \left( 3^{(1/(12))-(15)} \right) - 1 = .07346
\]

(c) For daily compounding \( r = 365 \left( 3^{(1/(15))-(365)} \right) - 1 = .07325 \).

Finally we discuss **continuous compound interest**, which is not often offered by banks, probably because of the difficulty of explaining it to the general public; also there is little difference in return over daily compounding for usual interest rates. For continuous compounding we let \( m \), the number of compounding periods in a year, approach infinity in equation (6)

\[
y_k = \lim_{m \to \infty} (1 + r/m)^mk = \left\{ \lim_{m \to \infty} (1 + r/m)^m \right\}^k y_0. \tag{9}
\]

In order to evaluate this limit we recall the definition of \( e \), the base of natural logarithms:

\[
e = \lim_{h \to \infty} (1 + 1/h)^h = 2.7182818\ldots \tag{10}
\]

Now let \( r/m = 1/h \) or \( h = m/r \) in equation (9). This yields

\[
\lim_{m \to \infty} (1 + r/m)^m = \lim_{h \to \infty} (1 + 1/h)^{hr} = \lim_{h \to \infty} \left\{ (1 + 1/h)^h \right\}^r = e^r.
\]

Using the above fact in equation (9) gives us

\[
y_k = e^{rk} y_0. \tag{11}
\]
Chapter I–DIFFERENCE EQUATIONS

This is the amount accumulated at the end of \( k \) years with interest compounded continuously. The effective interest rate for continuous compounding, that is, the increase in a one dollar investment in one year is

\[
 r_E = e^r - 1. \quad (12)
\]

For \( r = .06 \), we get \( r_E = .0618 \), which, to four decimal places, is the same as the effective interest rate for daily compounding shown in table 1.

In formula (9) there is no reason to restrict \( k \) to integer values. We replace \( k \) by \( t \), allowing \( t \) to be any nonnegative real number, and replace \( y_k \) by \( y(t) \) to obtain

\[
y(t) = e^{rt} y(0). \quad (13)
\]

as the amount accumulated after \( t \) years. Differentiating (11) we find that \( \frac{dy}{dt} = r e^{rt} y(0) \) so that \( y(t) \) satisfies the differential equation

\[
\frac{dy}{dt} = ry(t). \quad (14)
\]

This is one of the differential equations we will study later.

Exercises 1.3

1. If $1000 is invested at 8% compounded quarterly (a) What is the accumulated amount after 10 years? (b) What is the effective annual interest rate? (c) How many conversion periods are needed to double the original amount?

2. If money, at compound interest, will double in ten years, answer the following questions doing all calculations mentally (a) by what factor will the money increase in 30 years? (b) in 5 years? (c) how many years will it take for the money to quadruple?

3. If money, at compound interest, doubles in 10 years, by what factor will it increase in 13 years? How long will it take for the money to triple?

4. If $500 is invested at 9% compounded continuously (a) how much is accumulated after 5.2 years? (b) what is the effective annual interest rate? (c) how long will it take for your money to double?

5. The Taylor series for \( e^r \) about \( r = 0 \) is

\[
e^r = 1 + r + r^2/2! + r^3/3! + \ldots.
\]

For small values of \( r \), \( e^r = 1 + r + r^2/2 \) is a good approximation. Use this to determine whether 5% compounded continuously is better than 5.15% compounded annually.

6. a. If the nominal annual interest rate is \( r \) and interest is compounded \( m \) times a year, show that the doubling time in years is

\[
k = \frac{\ln 2}{m \cdot \ln(1 + r/m)}.
\]

b. If interest is compounded continuously, show the doubling time in years is

\[
t = \frac{\ln 2}{r}.
\]

c. If \( r \) is “small” show that the result in (a) is approximately \( (\ln 2)/r \). (Since \( \ln 2 \) is about .7, a rough formula for the doubling time is .7/r; for example, 7 years at 10% or 12 years at 6%).
7. Show that the amount accumulated after \( k \) years using continuous compounding is

\[
y_k = (1 + r_E)^k y_0.
\]

where \( r_E \) is given by (12). (Thus formula (8) can be used for any frequency of compounding.)

### 1.4–Amortization of a Mortgage

To **amortize** a debt is to pay it off in periodic payments, often equal in size. Monthly payments on a house mortgage or an automobile loan are familiar examples. The problem considered in this section is:

**Suppose** \( A \) **dollars is borrowed from a bank which charges interest at the rate of** \( r_p \) **per period on the unpaid balance. The debt is to be paid off in equal payment of** \( b \) **dollars at the end of each period. If the debt is to be completely paid off in** \( N \) **periods, what is the size of each payment.**

Let \( x_n \) be the amount owed to the bank at the end of the \( n \)th period, that is, just after the \( n \)th payment. Since the amount owed after the \((n + 1)\)th period is equal to the amount owed after the \( n \)th payment plus the interest charged during the \((n + 1)\)st period minus the payment made at the end of the \((n + 1)\)st period, we can write the following difference equation

\[
x_{n+1} = x_n + r_p x_n - b = (1 + r_p) x_n - b, \quad n = 0, 1, 2, \ldots
\]

If we let \( a = 1 + r_p \), the amount \( x_n \) at the end of the \( n \)th period satisfies

\[
\Delta E: \quad x_{n+1} = ax_n - b, \quad n = 0, 1, 2, \ldots
\]

\[
IC: \quad x_0 = A.
\]

We shall solve this problem for \( x_n \) and then choose \( b \) so that \( x_N = 0 \). By successively substituting \( n = 0, 1, 2, \) into the difference equation we find

\[
x_1 = ax_0 - b = aA - b,
\]

\[
x_2 = ax_1 - b = a(ax_0 - b) - b = a^2 A - (1 + a)b,
\]

\[
x_3 = ax_2 - b = a^3 A - (1 + a + a^2)b.
\]

It is easy to guess that the solution must be

\[
x_n = a^n A - (1 + a + a^2 + \ldots + a^{n-1})b
\]

Recall the formula for the sum of a geometric series

\[
1 + a + a^2 + \ldots + a^{n-1} = (1 - a^n)/(1 - a), \quad a \neq 1.
\]

Since \( a = 1 + r_p \) and \( r_p > 0 \), we have \( a > 1 \) so that (4) holds. Thus (3) becomes

\[
x_n = a^n A - \frac{1 - a^n}{1 - a} b.
\]

By direct substitution it can be verified that the difference equation and the initial condition are satisfied. To find the payment \( b \), we set \( x_N = 0 \) and solve for \( b \).

\[
0 = a^N A - \frac{1 - a^N}{1 - a} b \quad \text{or} \quad b = \frac{(a - 1)A}{1 - a^{-N}}.
\]
Since \( a = 1 + r_p \), we obtain the following result:

If \( A \) dollars is paid off in \( N \) periods with equal payments of \( b \) dollars per period with an interest rate of \( r_p \) per period then

\[
b = \frac{r_p A}{1 - (1 + r_p)^{-N}}.
\]

(*Example 1.*) A $50,000 mortgage is to be paid off in 30 years in equal monthly payments with an interest rate of 10%. What are the monthly payments.

*Solution* We have \( A = 50000 \), \( r_p = .10/12 \), and \( N = 30 \times 12 = 360 \), thus

\[
b = \frac{(.10/12)(50000)}{1 - (1 + .10/12)^{-360}} = 438.79.
\]

**Exercises 1.4**

1. Verify that (5) is the solution of (2).

2. What is the monthly payment necessary to pay off a debt of $20,000 in twenty years with an interest charge of 11.5%? What is the total amount paid?

3. If the monthly payment is $200 and the interest rate is 10%, how much money can be borrowed and be paid off in 20 years?

4. Find and check the solution of the following initial value problems.
   a. \( \Delta E: x_{n+1} = x_n + b \), IC: \( x_0 = c \)  
   b. \( \Delta E: x_{n+1} = ax_n + b \), \( a \neq 1 \), IC: \( x_0 = c \)

5. Using the results of problem 4, find and check the solution to:
   a. \( \Delta E: 2x_{n+1} = 2x_n - 1 \), IC: \( x_0 = 2 \)  
   b. \( \Delta E: 2x_{n+1} = -3x_n + 4 \), IC: \( x_0 = 0 \)

6. Suppose \( A \) dollars is deposited in a bank that pays interest at the rate of \( r_p \) per period and that at the end of each period additional deposits of \( b \) dollars are made. Write a difference equation for the amount in the bank at the end of the \( n \)th period, just after the \( n \)th deposit, and show that the solution is

\[
x_n = a^n A + \frac{1 - a^n}{1 - a} b, \text{ where } a = 1 + r_p.
\]

7. In problem 6, how much would be accumulated after 10 years if an initial deposit of $100 dollars is made and 10 dollars is deposited each month. Assume that the bank pays 8% interest compounded monthly.

8. How much money would need to be deposited initially in a bank which pays 6% interest compounded *quarterly*, in order to withdraw $500 a *month* for 20 years. The entire amount is to be consumed at the end of 20 years.

9. How much would you need to deposit quarterly over a period of 30 years in order to accumulate the fund needed for the subsequent 20 year withdrawals described in problem 8.
1.5–First Order Linear Difference Equations

A first order linear difference equation is one of the form

\[ a_k x_{k+1} = b_k x_k + c_k, \quad k = 0, 1, 2, \ldots \]  

(1)

where \( a_k, b_k \) and \( c_k \) are given sequences and \( x_k \) is the unknown sequence. If the sequence \( a_k \) is equal to zero for a certain value of \( k \), then, for this value of \( k \), \( x_{k+1} \) cannot be determined from knowledge of \( x_k \). To avoid this unpleasant situation, we assume that \( a_k \) is never zero, and divide through by \( a_k \) to obtain the normal form

\[ x_{k+1} = p_k x_k + q_k, \quad k = 0, 1, 2, \ldots \]  

(2)

where we have renamed the coefficients as indicated.

We shall first consider the special case of (2) when the sequence \( p_k \) is a constant sequence, \( p_k = a \) for all \( k \). Thus our problem is to solve

\[ \Delta \text{E: } x_{k+1} = a x_k + q_k, \quad k = 0, 1, 2, \ldots \]  

(3)

\[ \text{IC: } x_0 = c. \]

The procedure is the same as we have used previously, namely, write the first few terms, guess at the general term, and check. For \( k = 0, 1, 2 \) we find

\[ x_1 = a x_0 + q_0 = a c + q_0, \]  
\[ x_2 = a x_1 + q_1 = a(a c + q_0) + q_1 = a^2 c + a q_0 + q_1, \]  
\[ x_3 = a x_2 + q_2 = a^3 c + a^2 q_0 + a q_1 + q_2. \]

The pattern is clear. For \( x_k \) we expect

\[ x_k = a^k c + a^{k-1} q_0 + a^{k-2} q_1 + \ldots + a q_{k-2} + q_{k-1}. \]  

(4)

It can be easily checked that (4) satisfies the difference equation. Using summation notation equation (4) may be written in either of the forms

\[ x_k = a^k c + \sum_{i=0}^{k-1} q_i a^{k-1-i} = a^k c + \sum_{i=0}^{k-1} a^i q_{k-1-i} \]  

(5)

Since \( x_0 = c \), the summations in (5) should be interpreted as 0 when \( k = 0 \).

**Example 1.**

\[ \Delta \text{E: } x_{k+1} = 2 x_k + 3^k, \quad k = 0, 1, 2, \ldots \]  

\[ \text{IC: } x_0 = c. \]

Substituting \( a = 2 \) and \( q_k = 3^k \) into the first formula in (5) we get

\[ x_k = 2^k c + \sum_{i=0}^{k-1} 3^i 2^{k-1-i} = 2^k c + 2^{k-1} \sum_{i=0}^{k-1} (3/2)^i. \]

Summing the geometric series (see equation (4) of Section 1-4) gives the solution

\[ x_k = 2^k c + 2^{k-1} \frac{1 - (3/2)^k}{1 - (3/2)} = 2^k c + 3^k - 2^k. \]
Example 2.

\[ \Delta E: \quad x_{k+1} = 2x_k + 2^k, \quad k = 0, 1, 2, \ldots \]

IC: \quad x_0 = 0.

Using formula (5) we obtain

\[ x_k = \sum_{i=0}^{k-1} 2^i 2^{k-1-i} = \sum_{i=0}^{k-1} 2^{k-1} = k 2^{k-1}. \]

In the last step notice that each term in the sum is the same, namely, \(2^{k-1}\), and there are \(k\) terms.

We now turn to the general linear equation

\[ x_{k+1} = p_k x_k + q_k, \quad k \geq 0. \]

If \(q_k = 0\) for all \(k\), the equation is called homogeneous, otherwise it is called nonhomogeneous. Let us consider the solution of the initial value problem for the homogeneous equation

\[ \Delta E: \quad x_{k+1} = p_k x_k, \quad k \geq 0 \]

IC: \quad \(x_0 = c\). \tag{6}

Proceeding by successive substitutions we find

\[ x_1 = p_0 x_0 = p_0 c, \]
\[ x_2 = p_1 x_1 = (p_0 p_1) c, \]
\[ \vdots \]
\[ x_k = p_{k-1} x_{k-1} = (p_0 p_1 \cdots p_{k-1}) c, \quad k > 0. \tag{7} \]

It is convenient to introduce the product notation

\[ \prod_{i=0}^{n} a_i = a_0 a_1 a_2 \cdots a_n. \]

Using this notation the solution (7) of equation (6) can be written

\[ x_k = \left( \prod_{i=0}^{k-1} p_i \right) c, \quad k > 0. \tag{8} \]

Example 3.

\[ \Delta E: \quad x_{k+1} = (k + 1) x_k, \quad k \geq 0 \]

IC: \quad \(x_0 = 1\)

Using equation (8) above the solution is

\[ x_k = \prod_{i=0}^{k-1} (i + 1) \cdot 1, \]

or

\[ x_k = (1 \cdot 2 \cdots k) = k!. \]
Example 4.

\[ \Delta E: \quad y_{k+1} = ky_k, \quad k \geq 0 \]
\[ \text{IC: } \quad y_0 = 1. \]

Since \( y_1 = 0 \cdot y_0 = 0 \), this implies that \( y_2 = 1 \cdot y_1 = 0 \) or that \( y_k = 0 \) for \( k > 0 \) the solution is therefore

\[ y_k = \begin{cases} 1, & k = 0 \\ 0, & k > 0. \end{cases} \]

Finally we consider the nonhomogeneous problem

\[ \Delta E: \quad x_{k+1} = p_k x_k + q_k, \quad k \geq 0 \]
\[ \text{IC: } \quad x_0 = c. \]

We proceed by successive substitutions

\[ x_1 = p_0 x_0 + q_0, \]
\[ x_2 = p_1 x_1 + q_1 = (p_1 p_0) x_0 + p_1 q_0 + q_1, \]
\[ x_3 = p_2 x_2 + q_2 = (p_2 p_1 p_0) x_0 + p_2 p_1 q_0 + p_2 q_1 + q_2, \]

\[ \vdots \]
\[ x_k = p_{k-1} x_{k-1} + q_{k-1}, \]
\[ x_k = x_0 \prod_{i=0}^{k-1} p_i + q_0 \prod_{i=1}^{k-1} p_i + q_1 \prod_{i=2}^{k-1} p_i + \ldots + q_{k-2} p_{k-1} + q_{k-1}. \]

The solution can be written,

\[ x_k = c \prod_{i=0}^{k-1} p_i + \sum_{j=0}^{k-2} q_j \prod_{i=j+1}^{k-1} p_i + q_{k-1}, \quad k \geq 0 \]
\[ x_0 = c. \]

\[ (9) \]

Example 5.

\[ \Delta E: \quad x_{k+1} = (k + 1) x_k + (k + 1)!, \quad k \geq 0 \]
\[ \text{IC: } \quad x_0 = c. \]

From equation (8) we have

\[ x_k = c \prod_{i=0}^{k-1} (i + 1) + \sum_{j=0}^{k-2} (j + 1) 1 \prod_{i=j+1}^{k-1} (i + 1) + k! \]
\[ = c \cdot k! + \sum_{j=0}^{k-2} k! + k! \]
\[ = c \cdot k! + (k!) k \]
\[ = k!(c + k) \quad k \geq 0. \]

Exercises 1.5

1. \( x_{k+1} + 5x_k = 0, \quad k \geq 0; \quad x_0 = 5 \)
2. \( x_{k+1} = 3x_k + 2^k, \quad k \geq 0; \quad x_0 = 2 \)
3. \( x_{k+1} = 3x_k + 5 + k, \quad k \geq 0; \quad x_0 = 2 \)
4. \( (k + 1)x_{k+1} = (k + 2)x_k, \quad k \geq 0; \quad x_0 = 1 \)
5. \( y_{n+2} = n y_{n+1} + 1, \quad n \geq 0; \quad y_1 = 3 \)
1.6–The Method of Undetermined Coefficients

The following theorems give simple but important relationships between solutions of a nonhomogeneous linear difference equation and the corresponding homogeneous equation.

**Theorem 1.** If \( u_k \) and \( v_k \) are solutions of the nonhomogeneous equation

\[
x_{k+1} = p_k x_k + q_k \quad k \geq 0,
\]

then \( x_k = u_k - v_k \) is a solution of the corresponding homogeneous equation

\[
x_{k+1} = p_k x_k.
\]

**Proof** Since \( u_k \), \( v_k \) are solutions of (1) we know that \( u_{k+1} = p_k u_k + q_k \) and \( v_{k+1} = p_k v_k + q_k \). Therefore

\[
x_{k+1} = u_{k+1} - v_{k+1} = p_k u_k + q_k - (p_k v_k + q_k)
\]

\[
= p_k (u_k - v_k) = p_k x_k,
\]

so that \( x_k \) satisfies the homogeneous equation (2).

**Theorem 2.** The general solution of the nonhomogeneous equation \( x_{k+1} = p_k x_k + q_k \) can be written in the form

\[
x_k = x_k^{(h)} + x_k^{(p)},
\]

where \( x_k^{(h)} \) is the general solution of the homogeneous equation \( x_{k+1} = p_k x_k \) and \( x_k^{(p)} \) is any one (or particular) solution of the nonhomogeneous equation.

**Proof** Let \( x_k \) be any solution the nonhomogeneous equation (1) and \( x_k^{(p)} \) be a known particular solution, then by Theorem 1, \( x_k - x_k^{(p)} \) is a solution of the homogeneous equation (2). Thus \( x_k - x_k^{(p)} = x_k^{(h)} \) or \( x_k = x_k^{(h)} + x_k^{(p)} \).

The main purpose of this section is to describe a simple method for solving the nonhomogeneous equation

\[
x_{k+1} = a x_k + q_k, \quad k \geq 0,
\]

when \( q_k \) is a polynomial in \( k \), or an exponential. According to Theorem 2 we need to find the general solution of the homogeneous equation

\[
x_{k+1} = a x_k, \quad k \geq 0
\]

The general solution of equation (5) is simply

\[
x_k^{(h)} = a^k \cdot c,
\]

where \( c \) is an arbitrary constant (see problem 1). Now it is only necessary to find any one solution of equation (4). It is easy to guess the form of a solution when \( q_k \) is a polynomial in \( k \) or an exponential. The technique is best illustrated through examples.

**Example 1.** Solve

\[
\Delta E: \ x_{k+1} = 3 x_k - 4, \quad k \geq 0
\]

\[
IC: \ x_0 = 5.
\]

The general solution of the homogeneous equation, \( x_{k+1} = 3 x_k \), is \( x_k^{(h)} = 3^k \cdot c \). We need to find a particular solution of the nonhomogeneous equation. Because \( q_k = -4 \) is a constant sequence, we guess
that \( x^{(p)}_k \equiv A \), where \( A \) is a constant which must be determined to satisfy the equation. Since \( x^{(p)}_k \equiv A \), also \( x^{(p)}_{k+1} \equiv A \), thus substituting in the difference equation we find

\[
A = 3A - 4, \text{ or } A = 2.
\]

Thus \( x^{(p)}_k = 2 \) is a particular solution and the general solution is

\[
x_k = 3^k \cdot c + 2.
\]

All that remains is to determine \( c \) so that the initial condition is satisfied. Putting \( k = 0 \) in the solution yields

\[
x_0 = 5 = 3^0 \cdot c + 2 = c + 2,
\]

or \( c = 3 \). The solution is therefore

\[
x_k = 3^k \cdot 3 + 2 = 3^{k+1} + 2.
\]

which can easily be checked.

**Example 2.**

\[
\begin{align*}
\Delta E: & \quad x_{k+1} = 2x_k + 3^k, \quad k \geq 0 \\
IC: & \quad x_0 = \alpha.
\end{align*}
\]

The solution of the homogeneous equation is \( x^{(h)}_k = 2^k \cdot c \). A good guess for a particular solution is \( x^{(p)}_k = A \cdot 3^k \). Substituting in the difference equation we get

\[
A \cdot 3^{k+1} = 2A3^k + 3^k.
\]

Dividing by \( 3^k \) we find \( 3A = 2A + 1 \) or \( A = 1 \). Thus \( x^{(p)}_k = 3^k \) is a particular solution and

\[
x_k = 2^k \cdot c + 3^k
\]

is the general solution. To satisfy the initial condition set \( k = 0 \) and \( x_0 = \alpha \) to find that \( c = \alpha - 1 \). Thus the final solution is

\[
x_k = 2^k(\alpha - 1) + 3^k.
\]

**Example 3.**

\[
\begin{align*}
\Delta E: & \quad x_{k+1} = 2x_k + 2^k, \quad k \geq 0 \\
IC: & \quad x_0 = 0.
\end{align*}
\]

The homogeneous solution is \( x^{(h)}_k = 2^k \cdot c \). For a particular solution we might be tempted to try \( x^{(p)}_k = A \cdot 2^k \), however this cannot work because any constant times \( 2^k \) is a solution of the homogeneous equation \( x_{k+1} = 2x_k \) and therefore cannot also be a solution of the nonhomogeneous equation \( x_{k+1} = 2x_k + 2^k \). We must therefore modify our first attempt. It is clear that \( x_k \) must contain a factor of \( 2^k \), the simplest modification is to try \( x^{(p)}_k = Ak2^k \). Substituting in the difference equation we find

\[
A(k + 1)2^{k+1} = 2A2^k + 2^k.
\]

It is easy to solve for \( A \) to obtain \( A = 1/2 \). The particular solution is \( x^{(p)}_k = k2^k/2 = k2^{k-1} \) and the general solution is

\[
x_k = 2^k \cdot c + k2^{k-1}.
\]
Using the initial condition $x_0 = 0$ we get $c = 0$ and the solution is

$$x_k = k2^{k-1}.$$  

On the basis of these examples we can formulate a rule which gives the correct form for a particular solution of

$$x_{k+1} = ax_k + q_k,$$  

when $q_k$ is an exponential function.

**Rule 1.** If $q_k = c \cdot b^k$, a particular solution of equation (6) can be found in the form $x_k^{(p)} = Ab^k$ unless $b^k$ is a solution of the homogeneous equation (i.e., unless $b = a$) in which case a particular solution of the form $x_k^{(p)} = Akb^k$ can be found.

A similar rule exists in case $q_k$ is a polynomial in $k$.

**Rule 2.** If $q_k = c_0 + c_1k + \ldots + c_mk^m$, a particular solution of equation (6) can be found in the form $x_k^{(p)} = A_0 + A_1k + \ldots + A_mk^m$, unless $a = 1$, in which case a particular solution exists of the form

$$x_k = (A_0 + A_1k + \ldots + A_mk^m) \cdot k.$$  

**Example 4.** Find the general solution of $2x_{k+1} = 3x_k + 2k$

The homogeneous equation is $2x_{k+1} = 3x_k$ or $x_{k+1} = \frac{3}{2}x_k$ which has $x_k^{(h)} = \left(\frac{3}{2}\right)^k \cdot c$ for its general solution. For a particular solution we use (according to Rule 2)

$$x_k^{(p)} = A + Bk.$$  

Substituting into the difference equations produces

$$2(A + B(k + 1)) = 3(A + Bk) + 2k,$$

or

$$(-A + 2B) + (-B - 2)k = 0.$$  

Thus we must have $-B - 2 = 0$, or $B = -2$ and $-A + 2B = 0$ or $A = -4$. A particular solution is therefore

$$x_k^{(p)} = -4 - 2k,$$

and the general solution is

$$x_k = \left(\frac{3}{2}\right)^k c - 4 - 2k.$$  

**Example 5.** Find the general solution of $x_{k+1} = x_{k-1} + k$. The general solution of the homogeneous equation, $x_{k+1} = x_k$ is $x_k^{(h)} = c$. According to rule 2 the form $x_k^{(p)} = A + Bk$ will not work. The correct form is $x_k^{(p)} = (A + Bk)k = Ak + Bk^2$, we find

$$A(k + 1) + B(k + 1)^2 = Ak + Bk^2 - 1 + k.$$  

Simplifying we obtain

$$(A + B + 1) + (2B - 1)k = 0.$$  

Thus we must have $A + B + 1 = 0$ and $2B - 1 = 0$ this produces $B = 1/2$ and $A = -\frac{3}{2}$. A particular solution is $x_k^{(p)} = -\frac{3}{2}k + \frac{1}{2}k^2$ and the general solution is

$$x_k = c - \frac{3}{2}k + \frac{1}{2}k^2.$$
Example 6. Find a formula for the sum

\[ s_k = 1^2 + 2^2 + \ldots + k^2 \]

of the squares of the first \( k \) integers.

Solution We first write the problem in the form of difference equation. It is easy to see that \( s_k \) satisfies

\[
\begin{align*}
\Delta E: & \quad s_{k+1} = s_k + (k + 1)^2 = s_k + 1 + 2k + k^2 \\
\text{IC:} & \quad s_1 = 1.
\end{align*}
\]

The homogeneous solutions is \( s_k^{(h)} = c \). To find a particular solution, we assume

\[ s_k^{(p)} = Ak + Bk^2 + Ck^3. \]

Substituting into the \( \Delta E \): we obtain

\[ A(k + 1) + B(k + 1)^2 + C(k + 1)^3 = Ak + Bk^2 + Ck^3 + 1 + 2k + k^2. \]

Equating the constant terms, and the coefficients of \( k \) and \( k^2 \), leads to the values

\[ A = 1/6, \quad B = 1/2, \quad C = 1/3, \]

and hence to the general solution

\[ s_k = c + k/6 + k^2/2 + k^3/3. \]

Requiring \( s_1 = 1 \) leads to \( c = 0 \), so that

\[ s_k = \frac{1}{6}k + \frac{1}{2}k^2 + \frac{1}{3}k^3 = \frac{1}{6}k(k + 1)(2k + 1), \]

is the desired formula.

Exercises 1.6

1. Show that the general solution of \( x_{k+1} = ax_k \) is \( x_k = a^k c \). That is, first show \( x_k = a^k c \) is a solution and then let \( x_k \) be any solution and show it can be written in the form \( x_k = a^k c \) for some \( c \).

Solve each of the following initial value problems

2. \( \Delta E: 2x_{k+1} = 5x_k - 6, \ k \geq 0 \)
\[ \text{IC:} \quad x_0 = 0 \]

3. \( \Delta E: 2x_{k+1} = 5x_k + 3(-3)^{k-1}, \ k \geq 0 \)
\[ \text{IC:} \quad x_0 = 1 \]

4. \( \Delta E: x_{k+1} = 2x_k + 3 \cdot 2^{k-1}, \ k \geq 0 \)
\[ \text{IC:} \quad x_0 = 0 \]

5. \( \Delta E: x_{k+1} = 2x_k - 3 \cdot 2^{-k}, \ k \geq 1 \)
\[ \text{IC:} \quad x_1 = 0 \]

6. \( \Delta E: x_{k+1} = x_k + 2k, \ k \geq 1 \)
\[ \text{IC:} \quad x_1 = 0 \]

7. \( \Delta E: x_{k+1} + 2x_k = -3k, \ k \geq 0 \)
\[ \text{IC:} \quad x_0 = 1 \]

For each of the following write down the correct form for a particular solution. Do not evaluate the constants.

8. \( x_k - x_{k-1} = k - k^2 \)

9. \( x_{k+1} + 2x_k = 3(-2)^{k-5} \)

10. \( x_{k+1} - 2x_k = 3(-2)^{k-2} \)

11. \( 2x_{k+1} = x_k + 3 \cdot 2^{1-k} \)
12. a. Find a formula for the sum $s_k = 1^3 + 2^3 + \ldots + k^3$.
   b. Use the results of part (a) and of Example 6 to evaluate $\sum_{k=1}^{10}(k^3 - 6k^2 + 7)$.

13. Consider a community having $R$ residents. At time $n$, $x_n$ residents favor, and $R - x_n$ residents oppose, a local community issue. In each time period 100$b\%$ of those who previously favored the issue, and 100$a\%$ of those who previously opposed the issue, change their position.

   a. Write the difference equation describing this process. Then solve this equation, subject to the IC: $x_0 = A$.
   b. Describe the limiting behavior of $x_n$. That is, tell what happens as $n$ goes to infinity.

1.7–Complex Numbers

**Definition and Arithmetic of Complex Numbers.**

In many problems in engineering and science, it is necessary to use complex numbers. In this section we develop the background necessary to deal with complex numbers.

Complex numbers were invented so that all quadratic equations would have roots. The equation $x^2 = -1$ has no real roots, since, if $x$ is real, $x^2$ is non-negative. We therefore introduce a new number, symbolized by the letter $i$, having the property that

$$i^2 = -1, \quad \text{or } i = \sqrt{-1}. \quad (1)$$

The two roots of $x^2 = -1$ are now $x = \pm i$.

Let us now consider a general quadratic equation

$$ax^2 + bx + c = 0, \quad a, b, c \text{ real and } a \neq 0. \quad (2)$$

The quadratic formula yields the roots

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \quad (3)$$

If $b^2 - 4ac \geq 0$ the roots are real. However, if $b^2 - 4ac < 0$, (3) involves the square root of a negative number. In this case we write $\sqrt{b^2 - 4ac} = \sqrt{(-1)(4ac - b^2)} = \sqrt{-1}\sqrt{4ac - b^2} = i\sqrt{4ac - b^2}$. Therefore (3) becomes

$$x = \frac{-b \pm i\sqrt{4ac - b^2}}{2a}. \quad (4)$$

Are these numbers really roots of equation (2)? Let us find out by substituting (4) into (2), where we shall use the ordinary rules of algebra with the one additional rule that $i^2 = -1$

$$a \left(\frac{-b \pm i\sqrt{4ac - b^2}}{2a}\right)^2 + b \left(\frac{-b \pm i\sqrt{4ac - b^2}}{2a}\right) + c = \frac{b^2 \pm 2ib\sqrt{4ac - b^2} - (4ac - b^2)}{4a} + \frac{-b^2 \pm ib\sqrt{4ac - b^2}}{2a} + c = 0.$$ 

The equation is satisfied! We conclude that if we treat numbers of the form $x + iy$ with $x$ and $y$ real using the ordinary rules of algebra supplemented by the one additional rule that $i^2 = -1$, we are able to solve all quadratic equations.

With this background, we define a complex number to be a number of the form

$$z = a + ib, \ a, b \text{ real numbers.} \quad (5)$$
The number \(a\) is called the real part of \(z\) and we write \(a = \Re z\) and the number \(b\) is called the imaginary part of \(z\) and we write \(b = \Im z\); note that the imaginary part of \(z\) is a real number, the coefficient of \(i\) in (5). Numbers of the form \(a + 0i\) will be identified with the real numbers and we use the abbreviation \(a + 0i = a\). Numbers of the form \(0 + ib\) are called pure imaginary numbers and we use the abbreviation \(0 + ib = ib\). The number \(0 + 0i\) is the complex number zero, which we simply denote by 0.

Addition, subtraction and multiplication of complex numbers are defined below.

\[
(a + ib) + (c + id) = (a + c) + i(b + d) \tag{6},
\]
\[
(a + ib) - (c + id) = (a - c) + i(b - d) \tag{7},
\]
\[
(a + ib)(c + id) = (ac - bd) + i(bc + ad) \tag{8}.
\]

Note that these definitions are easy to remember since they are the usual rules of algebra with the additional rule that \(i^2 = -1\).

Before considering division of complex numbers, it is convenient to define the conjugate of a complex number \(z = a + ib\), denoted by \(\bar{z}\), to be \(\bar{z} = a - ib\). We find that

\[
z\bar{z} = (a + ib)(a - ib) = (a^2 + b^2) + i0 = a^2 + b^2. \tag{6}
\]

so that the product of a complex number and its conjugate is a real number. Now, to divide two complex numbers, we multiply numerator and denominator by the conjugate of the denominator so that the new denominator is real.

\[
\frac{a + ib}{c + id} = \frac{a + ib}{c + id} \cdot \frac{c - id}{c - id} = \frac{ac + bd}{c^2 + d^2} + i\frac{bc - ad}{c^2 + d^2}.
\]

provided \(c^2 + d^2 \neq 0\). However \(c^2 + d^2 = 0\) implies that \(c = d = 0\) making the denominator equal to zero; thus division of complex numbers (just as for real numbers) is defined except when the denominator is zero.

Example 1. \((2 + 3i)(1 - 2i) = 8 - i\).

Example 2. \(\frac{1 - 2i}{3 + 4i} = \frac{1 - 2i}{3 + 4i} \cdot \frac{3 - 4i}{3 - 4i} = \frac{-5 - 10i}{25} = -\frac{1}{5} - \frac{2}{5}i\).

Example 3. \(i^4 = 1, i^3 = -i, i^{-1} = \frac{1}{i} = \frac{i}{i^2} = -i, i^{-2} = -1, i^{-3} = i\).

Example 4. \(i^{75} = i^{(4\cdot18+3)} = (i^4)^{18} \cdot i^3 = -i\).

Addition, subtraction, multiplication, and division (except by zero) of complex numbers yield uniquely defined complex numbers. It can be verified that the same associative, commutative and distributive laws that hold for real numbers also hold for complex numbers. (In the language of modern algebra, the set of all complex numbers form a field).

Complex numbers were invented to allow us to solve quadratic equations. It might be expected that in order to solve cubic, quartic or higher degree equations it would be necessary to introduce more new ‘numbers’ like \(i\). However, this is not the case. Gauss proved what is called the Fundamental Theorem of Algebra: Every polynomial equation with complex coefficients has a complex root.

It is important to note that if two complex numbers are equal, \(a + ib = c + id\) then their real parts must be equal, \(a = c\), and their imaginary parts must be equal, \(b = d\). In other words, one equation between complex numbers is equivalent to two equations between real numbers.

Example 5. Find all possible square roots of \(3 + 4i\); that is, find all complex numbers \(z\) such that \(z^2 = 3 + 4i\).

Let \(z = x + iy\), then we must have \((x + iy)^2 = 3 + 4i\) or

\[(x^2 - y^2) + i(2xy) = 3 + 4i.\]
Setting real and imaginary parts of both sides equal we find that \( x^2 - y^2 = 3 \) and \( 2xy = 4 \). Thus \( y = 2/x \) and \( x^2 - 4/x^2 = 3 \), or \( x^4 - 3x^2 - 4 = (x^2 - 4)(x^2 + 1) = 0 \). We must have \( x^2 = 4 \) or \( x^2 = -1 \). Since \( x \) must be real, we find \( x = \pm 2 \). If \( x = 2 \), \( y = 1 \), while if \( x = -2 \), \( y = -1 \). Thus there are two square roots, \( z_1 = 2 + i \) and \( z_2 = -2 - i \). It is easily verified that \( z_1^2 = z_2^2 = 3 + 4i \).

**Geometric Interpretation — Polar Trigonometric Form**

A complex number \( z = x + iy \) is completely determined by knowing the ordered pair of real numbers \((x, y)\). Therefore we may interpret a complex number as a *point* in the \( x-y \) plane, or equally well, as a *vector* from the origin to the the point \((x, y)\) as shown in Figure 1. If complex numbers are interpreted as vectors then addition and subtraction of complex numbers follow the usual parallelogram law for vectors as shown in Figure 2.

![Figure 1](image1.png)  
![Figure 2](image2.png)

In order to see what happens geometrically when we multiply or divide complex numbers, it is convenient to use polar coordinates \((r, \theta)\) as shown in Figure 1. We have

\[
z = x + iy = r(\cos \theta + i \sin \theta).
\]

where \( r \) is the distance from the origin to the point \((x, y)\) and \( \theta \) is the angle between the positive \( x \)-axis and the vector \( z \). We call \( z = x + iy \) the *rectangular form* of \( z \) and \( z = r(\cos \theta + i \sin \theta) \) the *polar* or *trigonometric form* of \( z \). The distance \( r \) is called the *absolute value* or *modulus* of the complex number \( z \), also denoted by \(|z|\), and \( \theta \) is called the *angle* or *argument* of \( z \). From Figure 1 we derive the relations

\[
r = |z| = \sqrt{x^2 + y^2}, \quad \theta = \arctan \frac{y}{x}.
\]

The polar form allows a nice geometrical interpretation of the product of complex numbers. Let \( z = r(\cos \theta + i \sin \theta) \) and \( z' = r'(\cos \theta' + i \sin \theta') \) be two complex numbers. Then

\[
z z' = r(\cos \theta + i \sin \theta) r'(\cos \theta' + i \sin \theta')
\]
\[
= rr'\{(\cos \theta \cos \theta' - \sin \theta \sin \theta') + i(\cos \theta \sin \theta' + \sin \theta \cos \theta')\}
\]
\[
= rr'(\cos(\theta + \theta') + i \sin(\theta + \theta')).
\]

Therefore to multiply two complex numbers we multiply their absolute values and add their angles. In particular since \( i = 1(\cos \pi/2 + i \sin \pi/2) \), multiplication of a complex number \( z \) by the number \( i \) rotates \( z \) by \( 90^\circ \) in the positive (counterclockwise) direction.

Now dividing \( z \) by \( z' \) we find

\[
\frac{z}{z'} = \frac{r}{r'} \cdot \frac{\cos \theta + i \sin \theta}{\cos \theta' + i \sin \theta'}
\]
\[
= \frac{r}{r'} \cdot \frac{\cos \theta + i \sin \theta}{\cos \theta' + i \sin \theta'} \cdot \frac{\cos \theta' - i \sin \theta'}{\cos \theta' - i \sin \theta'}
\]
\[
= \frac{r}{r'} \cdot \frac{(\cos \theta \cos \theta' + \sin \theta \sin \theta') + (\sin \theta \cos \theta' - \cos \theta \sin \theta')}{\cos \theta'^2 + \sin \theta'^2}
\]
\[
= \frac{r}{r'}(\cos(\theta - \theta') + i \sin(\theta - \theta')).
\]
Thus to divide one complex number by another we divide the absolute values and subtract the angles. In particular, dividing a complex number \(z\) by \(i\), rotates \(z\) by 90° in the negative sense.

Formula (9) may be used to find powers of complex numbers. If \(z = r(\cos \theta + i \sin \theta)\), then
\[
z^2 = r^2(\cos 2\theta + i \sin 2\theta), \quad z^3 = r^3(\cos 3\theta + i \sin 3\theta).
\]
By induction we may prove, for \(n\) a nonnegative integer:
\[
z^n = (r(\cos \theta + i \sin \theta))^n = r^n(\cos n\theta + i \sin n\theta). \tag{11}
\]
For negative powers we define, for \(z \neq 0\), \(z^{-n} = 1/z^n\), where \(n\) is a positive integer. Using (10) and (11) we find
\[
z^{-n} = \frac{1}{z^n} = \frac{1}{r^n(\cos n\theta + i \sin n\theta)} = r^{-n}(\cos(-n\theta) + i \sin(-n\theta)) = r^{-n}(\cos n\theta - i \sin n\theta). \tag{12}
\]
Putting \(r = 1\) in (15) and (16) we have the famous formula of De Moivre:
\[
(\cos \theta + i \sin \theta)^n = (\cos n\theta + i \sin n\theta), \tag{13}
\]
which holds for all integers \(n\).

**Example 6.** To find \((1 + i)^4\), we first write \(1 + i\) in polar form (see figure below): \(1 + i = \sqrt{2}(\cos \pi/4 + i \sin \pi/4)\). Thus

\[
(1 + i)^4 = (\sqrt{2}(\cos \pi/4 + i \sin \pi/4))^4
= \sqrt{2}^4(\cos \pi + i \sin \pi)
= 4(-1 + i0) = -4.
\]

**The Polar Exponential Form and Euler’s Forms**

Consider the function \(E(\theta)\) defined by
\[
E(\theta) = \cos \theta + i \sin \theta, \quad \theta \text{ a real number.} \tag{14}
\]
For each \(\theta\), \(E(\theta)\) is a complex number of absolute value 1. As \(\theta\) varies, \(E(\theta)\) moves along the unit circle with center at the origin. Using equations (13)–(15) we deduce the following properties of this function:

\[
E(\theta) \cdot E(\theta') = E(\theta + \theta'),
\]
\[
\frac{E(\theta)}{E(\theta')} = E(\theta - \theta'),
\]
\[
(E(\theta))^n = E(n\theta), \quad (n \text{ an integer}),
\]
\[
\frac{d}{d\theta} E(\theta) = \frac{d}{d\theta}(\cos \theta + i \sin \theta)
= - \sin \theta + i \cos \theta = i(\cos \theta + i \sin \theta) = iE(\theta).
\]

The first three of these properties are shared by the real exponential function \(e^{a\theta}\). The last property suggests that \(a = i\). This motivates the definition
\[
e^{i\theta} = \cos \theta + i \sin \theta. \tag{15}
\]
This and the companion formula
\[
e^{-i\theta} = \cos \theta - i \sin \theta. \tag{16}
\]
are known as Euler’s forms.

Rewriting the formulas given above for \( E(\theta) \) in terms of \( e^{i\theta} \) we have

\[
e^{i\theta} e^{i\theta'} = e^{i(\theta + \theta')}, \quad e^{i\theta} / e^{i\theta'} = e^{i(\theta - \theta')}, \quad (e^{i\theta})^n = e^{in\theta}, \quad \frac{d}{d\theta} e^{i\theta} = ie^{i\theta}.
\]

These properties are easy to remember since they are the same as the properties of real exponentials. An important additional property of \( e^{i\theta} \) is that it is periodic of period 2\( \pi \) in \( \theta \), that is

\[
e^{i(\theta + 2k\pi)} = e^{i\theta},
\]

where \( k \) is an integer.

We now complete the definition of \( e^z \) for any complex number \( z \). If \( z = a + ib \), we define

\[
e^z = e^{a+ib} = e^a (\cos b + i \sin b).
\]

Using this definition it is straightforward to verify that the complex exponential obeys the following for all complex \( z_1 \) and \( z_2 \):

\[
e^{z_1} e^{z_2} = e^{z_1+z_2}, \quad e^{z_1} / e^{z_2} = e^{z_1-z_2}, \quad (e^{z_1})^n = e^{nz_1}
\]

In addition we have

\[
\frac{d}{dt} e^{(a+ib)t} = (a + ib) e^{(a+ib)t}.
\]

Euler’s forms allow us to write a complex number in the polar exponential form

\[
z = x + iy = r (\cos \theta + i \sin \theta) = re^{i\theta}.
\]

In computations involving multiplication, division and powers of complex numbers, the polar exponential form is usually the best form to use. The expression \( z = re^{i\theta} \) is compact and the rules for multiplication, division and powers are the usual laws of exponents; nothing new need be remembered.

**Example 7.** Evaluate \((-1 + i)^6\). From the simple diagram below we find that for the number \(-1 + i\), we have \( r = \sqrt{2} \), and \( \theta = 3\pi/4 \). Therefore

\[
(-1 + i)^6 = (\sqrt{2}e^{i3\pi/4})^6
\]

\[
= (\sqrt{2})^6 e^{i18\pi/4}
\]

\[
= 8e^{i\pi/2} \quad \text{(Using equation (17))}
\]

\[
= 8i.
\]

It is helpful to think of the complex number \( e^{i\theta} \) as a unit vector at an angle \( \theta \). By visualizing the numbers 1, \( i \), \(-1 \), and \(-i\) as vectors one easily finds

\[
1 = e^{i0}, \quad i = e^{i\pi/2}, \quad -1 = e^{i\pi}, \quad -i = e^{i3\pi/2} = e^{-i\pi/2}.
\]

Likewise one should think of \( re^{i\theta} \) as a vector of length \( r \) in the direction \( \theta \).

Suppose \( z(t) \) is a complex–valued function of the real variable \( t \). If \( x(t) = \Re z(t) \) and \( y(t) = \Im z(t) \), then

\[
z(t) = x(t) + iy(t).
\]
If the functions have derivatives then
\[ \frac{d}{dt} z(t) = \frac{d}{dt} x(t) + i \frac{d}{dt} y(t). \]

In other words

If \( x(t) = \Re z(t) \) then \( \frac{d}{dt} x(t) = \Re \frac{d}{dt} z(t) \) and if \( y(t) = \Im z(t) \) then \( \frac{d}{dt} y(t) = \Im \frac{d}{dt} z(t) \).

We now make use of these simple observations to derive a compact formula for \( n^{th} \) derivative of \( \cos at \) (a is real). Since \( \cos at = \Re e^{iat} \) we have

\[
\frac{d^n}{dt^n} \cos at = \Re \left( \frac{d^n}{dt^n} e^{iat} \right)
= \Re \left( (ia)^n e^{iat} \right)
= \Re \left( i^n a^n e^{iat} \right)
= \Re \left( \left(e^{i\pi/2}\right)^n a^n e^{iat} \right)
= \Re \left( a^n e^{i(at+n\pi/2)} \right)
= a^n \cos \left( at + n\pi/2 \right).
\]

As another example we compute \( \int e^{ax} \sin bx \, dx \), where \( a \) and \( b \) are real. We have

\[
\int e^{ax} \sin bx \, dx = \int \Im \left( e^{ax} e^{ibx} \, dx \right)
= \Im \int e^{ax} e^{ibx} \, dx
= \Im \left( e^{(a+ib)x} \right)
= \Im \left( \frac{e^{(a+ib)x}}{a + ib} \right)
= \Im \left( \frac{e^{ax}(\cos bx + i \sin bx)}{a + ib} \right)
= \Im \left( \frac{e^{ax}(a \cos bx + b \sin bx) + i(a \sin bx - b \cos bx)}{a^2 + b^2} \right)
= e^{ax} \left( \frac{a \sin bx - b \cos bx}{a^2 + b^2} \right).
\]

**Roots of Complex Numbers**

Let \( w = a + ib \neq 0 \) be a given complex number. We seek the \( n^{th} \) roots of \( w \), that is, all numbers \( z \) such that \( z^n = w \). We write \( w \) in polar form: \( w = R e^{i\alpha} \) where \( R \) and \( \alpha \) are known. Let \( z = r e^{i\theta} \) where \( r \) and \( \theta \) must be found to satisfy \( z^n = w \). We have

\[ z^n = (re^{i\theta})^n = Re^{i\alpha}, \text{ or } r^n e^{in\theta} = Re^{in\alpha}. \]  \hspace{1cm} (22)

Therefore \( r^n = R \) and \( n\theta = \alpha \), or \( r = \sqrt[n]{R} \) (the positive \( n^{th} \) root of \( R \)) and \( \theta = \alpha/n \). This yields one root

\[ z_0 = \sqrt[n]{R} e^{i\alpha/n} = \sqrt[n]{R} \left( \cos \alpha/n + i \sin \alpha/n \right). \]
However there are more roots. Note that, for any integer \( k \), \( e^{i\alpha} = e^{i(\alpha+2k\pi)} \); in other words the angle \( \alpha \) is only determined up to a multiple of \( 2\pi \). Now (26) becomes:

\[
(r e^{i\theta})^n = R e^{i(n\alpha+2k\pi)}, \quad \text{or} \quad r e^{in\theta} = R e^{i(n\alpha+2k\pi)}.
\]  

(23)

We now have that \( r^n = R \) or \( r = \sqrt[n]{R} \) as before, but \( n\theta = \alpha + 2k\pi \) or \( \theta = (\alpha + 2k\pi)/n \). If we let \( k = 0, 1, 2 \ldots, n - 1 \), we obtain \( n \) values of \( \theta \) which yield \( n \) distinct \( n^{\text{th}} \) roots of \( w \):

\[
z_k = \sqrt[n]{R} e^{i\frac{\alpha + 2k\pi}{n}} = \sqrt[n]{R} (\cos \frac{\alpha + 2k\pi}{n} + i \sin \frac{\alpha + 2k\pi}{n}), \quad k = 0, 1, 2, \ldots, n - 1.
\]  

(24)

We see that every non-zero complex number has exactly \( n \) distinct \( n^{\text{th}} \) roots, furthermore, these roots divide the circle of radius \( \sqrt[n]{R} \) into \( n \) equal parts.

**Example 8.** Find the cube roots of 1.

**Solution**  \( 1^{1/3} = (1 + i0)^{1/3} = (1 \cdot e^{i(0+2k\pi)})^{1/3} = 1 \cdot e^{i2k\pi/3}, \quad k = 0, 1, 2 \)

For \( k = 0 \), we have \( z_0 = e^{i0} = 1 \),

for \( k = 1 \), we have \( z_1 = e^{i2\pi/3} = \cos 2\pi/3 + i \sin 2\pi/3, \quad = -1/2 + i\sqrt{3}/2 \)

for \( k = 2 \), we have \( z_2 = e^{i4\pi/3} = \cos 4\pi/3 + i \sin 4\pi/3 \quad = -1/2 - i\sqrt{3}/2 \).

These roots are represented geometrically in the figure above.

**Example 9.** Find the cube roots of \(-1\).

**Solution**  \( (-1)^{1/3} = (-1 + i0)^{1/3} = (1 \cdot e^{i(\pi+2k\pi)})^{1/3} = 1 \cdot e^{i(\pi/3+2k\pi/3)}, \quad k = 0, 1, 2 \)

For \( k = 0 \), we have \( z_0 = e^{i\pi/3} = \cos \pi/3 + i \sin \pi/3, \quad = 1/2 + i\sqrt{3}/2 \)

for \( k = 1 \), we have \( z_1 = e^{i\pi} = \cos \pi + i \sin \pi = -1, \quad = 1/2 - i\sqrt{3}/2 \)

for \( k = 2 \), we have \( z_2 = e^{i5\pi/3} = \cos 5\pi/3 + i \sin 5\pi/3 \quad = 1/2 - i\sqrt{3}/2 \).

**Exercises 1.7**

1. Evaluate using exact arithmetic; write all answers in the form \( a + ib \):

   a. \( \frac{1 - i}{i - 1} \)  \quad b. \( \frac{4 - 5i}{2 - 3i} \)  \quad c. \( \frac{1}{2i}(i^7 - i^{-7}) \)  \quad d. \( i^{757} \)  \quad e. \( (-\frac{1}{2} + i\frac{\sqrt{3}}{2})^3 \)

2. Proceeding as in example 5, find all square roots of \( i \), that is find all complex \( z = x + iy \) such that \( z^2 = i \).

3. Show that for complex \( z_1 \) and \( z_2 \)

   a. \( |z_1 z_2| = |z_1| \cdot |z_2| \)  \quad b. \( |z_1| = \sqrt{|z_1 z_1|} \)  \quad c. \( |z_1 + z_2| \leq |z_1| + |z_2| \).

4. Show that for complex \( z_1 \) and \( z_2 \)

   a. \( \frac{z_1}{z_2} = \frac{\overline{z_1}}{z_2} \)  \quad b. \( (z_1 + z_2) = \overline{z_1} + \overline{z_2} \).
5. Using De Moivre’s theorem for \( n = 2 \): 
\[
(\cos \theta + i \sin \theta)^2 = \cos 2\theta + i \sin 2\theta,
\]
find \( \sin 2\theta \) and \( \cos 2\theta \) in terms of \( \sin \theta \) and \( \cos \theta \).

6. Calculate exactly, using the polar exponential form, and put answers in the form \( a + ib \):

a. \((1 - i)^9\) 
b. \((-\frac{1}{2} + i\frac{\sqrt{3}}{2})^{37}\) 
c. \((-1)^n\) 
d. \((-i)^n\)

7. If it is known that \((a + ib)^k = 2 - 6i\), evaluate

a. \((a - ib)^k\) 
b. \((-a + ib)^k\).

8. Using a calculator evaluate \((-0.791 + 0.892i)^5\), rounding the real and imaginary parts of the answer to three significant digits.

9. Using complex exponentials find

a. \(\int e^x \cos 2x \, dx\) 
b. The \(n^{th}\) derivative of \(\sin 2x\) 
c. \(\frac{d^{15}}{dx^{15}} (e^{\frac{\sqrt{2}}{2}x} \sin \frac{\sqrt{2}}{2}x)\).

10. Evaluate all roots exactly in the form \(a + ib\)

a. \((-1)^{1/4}\) 
b. \(1^{1/4}\) 
c. \(i^{1/2}\) 
d. \((3 - 3i)^{2/3}\).

11. Find all roots of \((2 - 3i)^{1/3}\), rounding the real and imaginary parts of the answers to three significant digits.

1.8–Fibonacci Numbers

As our first example of a second order difference equation we consider a problem that dates back to an Italian mathematician, Leonardo of Pisa, nicknamed Fibonacci (son of good nature), who lived in the thirteenth century. He proposed the following problem

On the first day of a month we are given a newly born pair of rabbits, how many pairs will there be at the end of one year? It is assumed that no rabbits die, that rabbits begin to bear young when they are two months old and produce one pair of rabbits each month.

Let \(f_n\) = number of pairs of rabbits at end of \(n^{th}\) month, \(n = 1, 2, \ldots\). We know that \(f_1 = 1, f_2 = 1\). We have the obvious relation

\[ f_n = \text{number of pairs of rabbits at end of the } (n - 1)^{th} \text{ month + births during } n^{th} \text{ month} \]

The births during the \(n^{th}\) month must be produced by pairs of rabbits that are at least two months old and there are exactly \(f_{n-2}\) such pairs of rabbits. Therefore, we have to solve

\[
\text{ΔE: } f_n = f_{n-1} + f_{n-2}, \quad n = 2, 3, 4, \ldots
\]

\[
\text{IC: } f_1 = 1, \quad f_2 = 1.
\]

(1)

It is easy to see that \(f_n\) is uniquely determined for all relevant \(n\). It is also easy to write down the first few terms of the sequence \(f_n\)

\[ f_1 = 1, \quad f_2 = 1, \quad f_3 = 2, \quad f_4 = 3, \quad f_5 = 8. \]

Each term is the sum of the preceding two terms. These numbers are called Fibonacci numbers; the first twelve Fibonacci numbers are

\[ 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144. \]

The number of pairs of rabbits at the end of one year is therefore 144.
Suppose we wish to find \( f_n \) for arbitrary \( n \), then we must solve the difference equation. We try to guess at a solution. After some reflection, the type of sequence most likely to produce a solution is

\[
f_n = \lambda^n, \; n = 1, 2, \ldots
\]

Substituting into \( f_n = f_{n-1} + f_{n-2} \) we find

\[
\lambda^n = \lambda^{n-1} + \lambda^{n-2}, \; \text{or} \; \lambda^{n-2}(\lambda^2 - \lambda - 1) = 0. \quad (2)
\]

Thus any \( \lambda \) satisfying \( \lambda^2 - \lambda - 1 = 0 \) will yield a solution. The solutions of the quadratic equation are

\[
\lambda_1 = \frac{1 + \sqrt{5}}{2}, \quad \lambda_2 = \frac{1 - \sqrt{5}}{2}. \quad (3)
\]

Thus, as may be easily checked, \( \lambda_1^n \) and \( \lambda_2^n \) are two solutions of the difference equation. Furthermore, we find that if \( c_1, c_2 \) are any constants, then

\[
f_n = c_1\lambda_1^n + c_2\lambda_2^n, \quad n = 1, 2, \ldots \quad (4)
\]

is also a solution. We now try to find \( c_1, c_2 \) so that the initial conditions \( f_1 = 1, f_2 = 1 \) are satisfied. We find that \( c_1, c_2 \) must satisfy

\[
1 = c_1\lambda_1 + c_2\lambda_2 \\
1 = c_1\lambda_1^2 + c_2\lambda_2^2
\]

Solving these linear equations we find \( c_1 = 1\sqrt{5}, c_2 = -1\sqrt{5} \), therefore our solution is

\[
f_n = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^n, \quad n = 1, 2, \ldots \quad (5)
\]

This is a general formula for the \( n \)th Fibonacci number.

From the initial value problem (1), it is clear that \( f_n \) is always an integer; however this is not at all obvious from the general formula (5).

Curiously, a number of natural phenomena seem to follow the Fibonacci sequence, at least approximately. Consider the branching of a tree; in some trees the number of branches at each level are successive Fibonacci numbers as shown below

![A Fibonacci Tree](image)
This would happen exactly if the main trunk of the tree sent out a branch every year starting with the 2nd year and each branch sent out a branch every year starting in its second year.

Other examples of the Fibonacci sequence in nature are, the number of spiral florets in a sunflower, the spiraled scales on the surface of a pineapple, the position of leaves on certain trees and the petal formations on certain flowers.

One can calculate (see problem 1) that

\[
\lim_{n \to \infty} \frac{f_{n+1}}{f_n} = \frac{1 + \sqrt{5}}{2} = 1.618 \ldots
\]

This number is often called the golden mean. The golden mean was thought by the ancient Greeks to be the ratio of the sides of the rectangle that was most pleasing to the eye. The golden mean occurs in several places in geometry, for instance, the ratio of the diagonal of a regular pentagon to its edge is the golden mean. A regular icosahedron contains 20 faces, each an equilateral triangle; it can be constructed from three golden rectangles intersecting symmetrically at right angles. Connecting the vertices of the rectangles, one obtains the regular icosahedron as shown below. A regular dodecahedron has twelve regular pentagonal faces. The midpoints of the faces are the vertices of a regular icosahedron, as shown in the figure below.

Exercises 1.8

1. Show \( \lim_{n \to \infty} \frac{f_{n+1}}{f_n} = \frac{1 + \sqrt{5}}{2} \).

2. If a line segment of unit length is divided so that the whole is to the larger as the larger is to the smaller, show these ratios are the golden mean.

3. Solve and check the solution of

\[
\begin{align*}
\Delta E: & \quad x_{n+2} + 5x_{n+1} + 6x_n = 0, \ n = 0, 1, 2, \ldots \\
IC: & \quad x_0 = 0, \ x_1 = 1.
\end{align*}
\]
1.9–Second Order Linear Difference Equations

A second order linear difference equation is one of the form

$$a_nx_{n+2} + b_nx_{n+1} + c_nx_n = f_n, \quad n = 0, 1, 2, \ldots$$  \hspace{1cm} (1)

where \(\{a_n\}, \{b_n\}, \{c_n\}\) and \(\{f_n\}\) are given sequences of real or complex numbers and \(\{x_n\}\) is the sequence we wish to find. We shall also assume that \(a_n \neq 0\) for \(n \geq 0\). This assures that the initial value problem

$$\Delta E: \quad a_nx_{n+2} + b_nx_{n+1} + c_nx_n = f_n, \quad n \geq 0$$

$$\text{IC: } x_0 = \alpha_0, \quad x_1 = \alpha_1.$$  \hspace{1cm} (2)

has a unique solution for each choice of \(\alpha_0, \alpha_1\) (see Theorem 1 of Section 1–3).

It is convenient to introduce an abbreviation for the left hand side of (1).

$$L(x_n) = a_nx_{n+2} + b_nx_{n+1} + c_nx_n, \quad n = 0, 1, 2, \ldots$$  \hspace{1cm} (3)

For a given sequence \(\{x_n\}\), \(L(x_n)\) is another sequence; \(L\) is an operator which maps sequences into sequences. For example, suppose \(L\) is defined by \(L(x_n) = x_{n+2} + x_n\), then \(L(n^2) = (n+2)^2 + n^2 = 2n^2 + 4n + 4\) and \(L\) maps the sequence \(\{n^2\}\) into the sequence \(\{2n^2 + 4n + 4\}\). Also \(L(i^n) = L(e^{i\pi n/2} = e^{i(n+2)\pi/2} + e^{in\pi/2} = e^{i\pi/2} + e^{i\pi} = 0\). Thus \(L\) maps the sequence \(\{i^n\}\) into the zero sequence; this of course means that \(x_n = i^n\) is a solution of \(x_{n+2} + x_n = 0\).

Using the operator \(L\) defined by (3), the difference equation (1) can be written in the abbreviated form

$$L(x_n) = f_n, \quad n = 0, 1, 2, \ldots$$  \hspace{1cm} (4)

Or, if \(f_n = 0\) for all \(n\), we have

$$L(x_n) = 0.$$  \hspace{1cm} (5)

If \(f_n\) is not identically zero (4) is called a nonhomogeneous difference equation and (5) is called the associated homogeneous equation.

In the next two sections we shall show how to solve equation (1) in the special case when the coefficient sequences \(a_n, b_n, c_n\) are all constant. Methods for solving (2) when the coefficients are not constant are beyond the scope of this book. In this section we shall discuss properties of solutions which will be needed for the constant coefficient case but are just as easy to demonstrate for the general linear equation (2).

We start with a simple, but fundamental, property of the operator \(L\).

**Theorem 1.** The operator \(L\) is linear that is

$$L(\alpha x_n + \beta y_n) = \alpha L(x_n) + \beta L(y_n).$$  \hspace{1cm} (6)

for any constants \(\alpha, \beta\) and any sequences \(x_n, y_n\).

**Proof** By direct computation we find

$$L(\alpha x_n + \beta y_n) = a_n(\alpha x_{n+2} + \beta y_{n+2}) + b_n(\alpha x_{n+1} + \beta y_{n+1}) + c_n(\alpha x_n + \beta y_n)$$

$$= \alpha(a_nx_{n+2} + b_nx_{n+1} + c_nx_n) + \beta(a_ny_{n+2} + b_ny_{n+1} + c_ny_n)$$

$$= \alpha L(x_n) + \beta L(y_n).$$

An useful property of solutions of the homogeneous equation is given in the following theorem.

**Theorem 2.** If \(u_n\) and \(v_n\) are solutions of the homogeneous equation \(L(x_n) = 0\) then \(\alpha u_n + \beta v_n\) is also a solution for constant values of \(\alpha, \beta\).
Proof. By hypothesis we have \( L(u_n) = 0 \) and \( L(v_n) = 0 \). From Theorem 1 we have

\[
L(\alpha u_n + \beta v_n) = \alpha L(u_n) + \beta L(v_n) = \alpha \cdot 0 + \beta \cdot 0 = 0.
\]

Example 1. It can be verified that \( 2^n \) and \( 3^n \) are solutions of \( x_{n+2} - 5x_{n+1} + 6x_n = 0 \). Thus the theorem guarantees then \( x_n = \alpha 2^n + \beta 3^n \) is also a solution. The question arises, is this the general solution or are there other solutions? To answer this question we need the notion of linear independence of two sequences.

Definition 1. Two sequences \( \{u_n\} \), \( \{v_n\} \), \( n = 0, 1, \ldots \), are linearly dependent (LD) if it is possible to find two constants \( c_1 \) and \( c_2 \), not both zero such that \( \alpha u_n + \beta v_n \equiv 0 \), \( n = 0, 1, 2 \ldots \). The sequences are called linearly independent (LI) if they are not linearly dependent, i.e., if \( \alpha u_n + \beta v_n \equiv 0 \), \( n = 0, 1, 2 \ldots \) then \( \alpha = \beta = 0 \).

Saying this another way \( \{u_n\} \), \( \{v_n\} \), are LD if and only if one sequence is a multiple of (or depends on) the other and LI otherwise. One can usually tell by inspection whether two sequences are LI or LD. For example the sequences \( u_n = n, v_n = n^2 \) are LI while \( u_n = 2n, v_n = 5n \) are LD. Also if \( u_n \equiv 0 \), then \( u_n, v_n \) are LD no matter what the sequence \( v_n \) is. A useful test for LI of two solutions of \( L(x_n) = 0 \) is given in the following theorem.

Theorem 3. Two solutions of \( u_n, v_n \) of \( L(x_n) = 0 \), \( n \geq 0 \), are linearly independent if and only if

\[
\begin{vmatrix}
  u_0 & v_0 \\
  u_1 & v_1 \\
\end{vmatrix} = u_0v_1 - u_1v_0 \neq 0
\]

Proof. Assume that \( u_n, v_n \) are LI solutions of \( L(x_n) = 0 \). Suppose \( \begin{vmatrix} u_0 & v_0 \ \n \ u_1 & v_1 \end{vmatrix} = 0 \). Then the equations

\[
\begin{align*}
\alpha u_0 + \beta v_0 &= 0 \\
\alpha u_1 + \beta v_1 &= 0
\end{align*}
\]

have a nontrivial solution. Consider

\[
y_n = \alpha u_n + \beta v_n
\]

where \( \alpha \) and \( \beta \) are determined above. We have that \( y_n \) is a solution of \( L(x_n) = 0 \), with \( y_0 = 0 \) and \( y_1 = 0 \). Therefore by the uniqueness of solutions of the initial value problem, \( y_n \equiv 0 \). This means that \( u_n, v_n \) are LD, contrary to assumption.

Conversely, assume that \( \begin{vmatrix} u_0 & v_0 \ \n \ u_1 & v_1 \end{vmatrix} \neq 0 \). Suppose that \( u_n, v_n \) are LD. Then there exists constants \( \alpha \) and \( \beta \), not both zero, so that

\[
\alpha u_n + \beta v_n \equiv 0, \ n \geq 0
\]

In particular this must be true for \( n = 0 \) and \( n = 1 \). Thus the system of equations

\[
\begin{align*}
\alpha u_0 + \beta v_0 &= 0 \\
\alpha u_1 + \beta v_1 &= 0
\end{align*}
\]

has a nontrivial solution. This means that the determinant of the coefficients is zero \( \begin{vmatrix} u_0 & v_0 \ \n \ u_1 & v_1 \end{vmatrix} = 0 \), contrary to assumption.

We are now in a position to prove the following fundamental fact.

Theorem 4. If \( u_n \) and \( v_n \) are two LI solutions of the second order linear homogeneous difference equation, \( L(x_n) = 0 \), \( n = 0, 1, \ldots \), then \( x_n = \alpha u_n + \beta v_n \) is the general solution of \( L(x_n) = 0 \).
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Proof Let \( w_n \) be any solution of \( L(x_n) = 0 \). We must show that \( w_n = \alpha u_n + \beta v_n \) for suitable constants \( \alpha \) and \( \beta \). Now \( w_0 \) and \( w_1 \) are definite numbers, we determine \( \alpha, \beta \) so that

\[
\begin{align*}
    w_0 &= \alpha u_0 + \beta v_0 \\
    w_1 &= \alpha u_1 + \beta v_1.
\end{align*}
\]

Since the solutions are LI, the determinant of the coefficients is not zero, i.e.,

\[
\begin{vmatrix}
    u_0 & v_0 \\
    u_1 & v_1
\end{vmatrix} \neq 0.
\]

we can solve uniquely for \( \alpha \) and \( \beta \). With these values of \( \alpha, \beta \) define \( z_n = \alpha u_n + \beta v_n \). Note that \( L(z_n) = 0 \) and \( z_0 = w_0 \) and \( z_1 = w_1 \). Therefore \( z_n \) and \( w_n \) are solutions of the same difference equation and the same initial conditions. Since there is a unique solution to this initial value problem we must have \( z_n \equiv w_n \equiv \alpha u_n + \beta v_n \).

Example 2. In Example 1 we found that \( 2^n \) and \( 3^n \) were solutions of \( x_{n+2} - 5x_{n+1} + 6x_n = 0 \). Since \( 2^n, 3^n \) are LI we now know that \( x_n = \alpha 2^n + \beta 3^n \) is the general solution. In particular, we may find solutions satisfying any initial conditions. Suppose \( x_0 = 1, x_1 = 2 \) then we must solve

\[
\begin{align*}
    1 &= \alpha + \beta \\
    2 &= \alpha \cdot 2 + \beta \cdot 3
\end{align*}
\]

which yields the unique values \( \alpha = 1, \beta = 0 \) so the unique solution satisfying the initial conditions is \( x_n = 2^n \).

Theorem 4 reduces the problem of finding the general solution of \( L(x_n) = 0 \) to finding a LI set of solutions. In the next section we will show how this is done if \( L \) has constant coefficients.

We now turn to the general properties of solutions of the nonhomogeneous equation.

Theorem 5. If \( y_n \) and \( z_n \) are both solutions of the same nonhomogeneous equation \( L(x_n) = f_n \) then \( x_n = y_n - z_n \) is a solution of the homogeneous equation \( L(x_n) = 0 \).

Proof \( L(x_n) = L(y_n - z_n) = L(y_n) - L(z_n) - f_n - f_n = 0 \).

Theorem 6. If \( x_n^{(h)} \) is the general solution of \( L(x_n) = 0 \) and \( x_n^{(p)} \) is any one (or particular) solution of \( L(x_n) = f_n \) then \( x_n = x_n^{(h)} + x_n^{(p)} \) is the general solution of \( L(x_n) = f_n \).

Proof Let \( x_n \) be any solution of \( L(x_n) = f_n \) and let \( x_n^{(p)} \) be a particular solution of the same equation. By Theorem 5, \( x_n - x_n^{(p)} \) is a solution of \( L(x_n) = 0 \) therefore \( x_n - x_n^{(p)} = x_n^{(h)} \) as desired.

Example 3. Find the general solution of

\[ x_{n+2} - 5x_{n+1} + 6x_n = 4 \]

From Example 2 we know that \( x_n^{(h)} = \alpha 2^n + \beta 3^n \). According to Theorem 5 we need only find any one solution of the difference equation. We look for the simplest solution. The fact that the right hand side is a constant suggests that we try \( x_n^{(p)} \equiv A \) (a constant). Then \( x_n^{(p)} = A \) and \( x_{n+1}^{(p)} = A \) and substituting into the equation we find

\[
A - 5A + 6A = 4, \text{ or } A = 2
\]

Thus \( x_n^{(p)} = 2 \) and the general solution is \( x_n = \alpha 2^n + \beta 3^n + 2 \).

Example 4. Find the general solution of

\[ x_{n+2} - 5x_{n+1} + 6x_n = 2 \cdot 5^n \]
The homogeneous equation is the same as in example 3. The right hand side of the equation suggests that a particular solution of the form \( x_n^{(p)} = A \cdot 5^n \) may exist. Substituting into the difference equation we find

\[
A \cdot 5^{n+2} - 5(A \cdot 5^{n+1}) + 6(A \cdot 5^n) = 2 \cdot 5^n
\]

Note that \( 5^n \) is a common factor on both sides. Dividing both sides by this factor we find \( A = 1/3 \) or \( x_n^{(p)} = 5^n/3 \). Adding this to the homogeneous solution provides the general solution.

Our final theorem allows us to break up the solution of \( L(x_n) = \alpha f_n + \beta g_n \) into the solution of two simpler problems.

**Theorem 7.** If \( y_n \) is a solution of \( L(x_n) = f_n \) and \( z_n \) is a solution of \( L(x_n) = g_n \) then \( \alpha y_n + \beta z_n \) is a solution of \( L(x_n) = \alpha f_n + \beta g_n \).

**Proof** \( L(\alpha y_n + \beta z_n) = \alpha L(y_n) + \beta L(z_n) = \alpha f_n + \beta g_n \).

**Example 5.** Find a particular solution of

\[
x_{n+2} - 5x_{n+1} + 6x_n = 4 + 2 \cdot 5^n
\]

Using the results of examples 3 and 4 and Theorem 6 with \( \alpha = \beta = 1 \) we find

\[
x_n^{(p)} = -\frac{2}{5} + \frac{5^n}{3}
\]

**Example 6.** Find a particular solution of

\[
x_{n+2} - 5x_{n+1} + 6x_n = 1 - 5^{n-1}
\]

Comparing the right hand side of this equation with examples 3 and 4 we can find \( \alpha, \beta \) such that

\[
1 = \alpha \cdot 4 \quad \text{and} \quad -5^{n-1} = \beta \cdot 2 \cdot 5^n
\]

this yields \( \alpha = 1/4 \) and \( \beta = -1/10 \). Thus

\[
x_n^{(p)} = \frac{1}{4} \cdot \left(-\frac{2}{5}\right) + \left(-\frac{1}{10}\right) \cdot \frac{5^n}{3}
\]

**Exercises 1.9**

1. What is a solution of \( L(x_n) = 0 \), \( x_0 = 0 \), \( x_1 = 0 \). Is there more than one solution?
2. Let \( L(x_n) = x_{n+2} - 4x_n \). Compute \( L(x_n) \) in each of the following cases
   a. \( x_n = 3(-4)^{n-1} \)
   b. \( x_n = 2^n \)
3. If \( u_n \) is the solution of \( L(x_n) = 0 \), \( x_0 = 0 \), \( x_1 = 1 \) and \( v_n \) is the solution of \( L(x_n) = 0 \), \( x_0 = 1 \), \( x_1 = 0 \) what is the solution of \( L(x_n) = 0 \), \( x_0 = 5 \), \( x_1 = 6 \).
4. If \( 2^{n-1} \) is a solution of \( L(x_n) = 5 \cdot 2^{n+1} \) and \( 2 \cdot 3^n \) is a solution of \( L(x_n) = 4 \cdot 3^{n-2} \) what is the solution of \( L(x_n) = 2^n - 3^n \).
5. If \( 2^n - 3^n \), \( 2^{n+1} - 3^n \), \( 2^{n+1} - 3^n + 1 \) are each solutions of the same difference equation \( L(x_n) = f_n \)
   a. What is the general solution of \( L(x_n) = 0 \).
   b. What is the general solution of \( L(x_n) = f_n \).
1.10–Homogeneous Second Order Linear Difference Equations

We shall restrict ourselves to equations with constant coefficients. Consider the difference equation

\[ a x_{n+2} + b x_{n+1} + c x_n = 0, \quad n = 0, 1, 2, \ldots \]  

(1)

where \( a, b, c \) are real constants and both \( a \neq 0 \) and \( c \neq 0 \) (if \( a = 0 \) or \( c = 0 \), the equation is of first order).

Obviously, one solution is \( x_n = 0 \) for all \( n \); this is called the trivial solution. We look for non-trivial solutions of the form \( x_n = \lambda^n \) for a suitable value of \( \lambda \neq 0 \). Substituting this into (1) we find

\[ a\lambda^{n+2} + b\lambda^{n+1} + c\lambda^n = 0 \quad \text{or} \quad \lambda^n(a\lambda^2 + b\lambda + c) = 0. \]

If this is to hold for all \( n \), \( \lambda \) must satisfy

\[ a\lambda^2 + b\lambda + c = 0 \]  

(2)

which is called the characteristic equation associated with (1).

Solving this quadratic equation we obtain

\[ \lambda_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad \lambda_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \]  

(3)

and the corresponding solutions \( \lambda_1^n \) and \( \lambda_2^n \) of the difference equation (1). Note that if \( b^2 - 4ac > 0 \), \( \lambda_1 \) and \( \lambda_2 \) are real and distinct, if \( b^2 - 4ac = 0 \) then \( \lambda_1 = \lambda_2 \) and if \( b^2 - 4ac < 0 \) the roots \( \lambda_1 \) and \( \lambda_2 \) are conjugate complex numbers. We analyze each of these situations.

(a) Real distinct roots \( (b^2 - 4ac > 0) \).

We know from (3) that \( \lambda_1, \lambda_2 \) are real and \( \lambda_1 \neq \lambda_2 \), thus \( u_n = \lambda_1^n \) and \( v_n = \lambda_2^n \) are two solutions. Since these solutions are clearly LI, the general solution is

\[ x_n = \alpha \lambda_1^n + \beta \lambda_2^n. \]  

(4)

(b) Real equal roots \( (b^2 - 4ac = 0) \).

We have only one root \( \lambda_1 = -b/2a \) and a corresponding solution \( u_n = \lambda_1^n \). We must find a second solution. Of course \( \alpha \lambda_1^n \) is also a solution but the set \( \{\lambda_1^n, \alpha \lambda_1^n\} \) is not a LI set. It turns out that a second LI solution is

\[ v_n = n\lambda_1^n. \]

Let us verify this

\[ a v_{n+2} + b v_{n+1} + c v_n = a(n + 2)\lambda_1^{n+2} + b(n + 1)\lambda_1^{n+1} + c n\lambda_1^n \]

\[ = \lambda_1^n \{n(a\lambda_1^2 + b\lambda_1 + c) + 2a\lambda_1 + b\lambda_1\} \]

but \( a\lambda_1^2 + b\lambda_1 + c = 0 \) since \( \lambda_1 \) is a root of this equation; also \( 2a\lambda_1^2 + b\lambda_1 = 0 \) since \( \lambda_1 = -b/2a \). Therefore, the right-hand side of the above equation is 0 and \( v_n = n\lambda_1^n \) is a solution. Clearly \( \{\lambda_1^n, n\lambda_1^n\} \) is a LI set. Thus the general solution is

\[ x_n = \alpha \lambda_1^n + \beta n\lambda_1^n. \]  

(5)

(c) Complex roots \( (b^2 - 4ac < 0) \).

We can write the roots of the characteristic equation as

\[ \lambda_1 = \frac{-b + i\sqrt{4ac - b^2}}{2a}, \quad \lambda_2 = \frac{-b - i\sqrt{4ac - b^2}}{2a}. \]
Since \(a, b, c\) are real numbers, \(\lambda_1, \lambda_2\) are complex conjugate pairs;

\[
\lambda_1 = \xi + i\eta, \quad \lambda_2 = \xi - i\eta
\]

where \(\eta = -b/2aa, \, \eta = \sqrt{4ac - b^2}/2a\) and \(\eta \neq 0\).

Writing these roots in polar exponential form

\[
\lambda_1 = re^{i\theta}, \quad \lambda_2 = re^{-i\theta}, \quad \theta \neq m\pi \ (m \text{ an integer})
\]

we obtain the complex valued solutions

\[
z^{(1)}_n = \lambda_1^n = r^n e^{i \theta} \quad \text{and} \quad z^{(2)}_n = \lambda_2^n = r^n e^{-i \theta}.
\]

These form a LI set and the general complex valued solution is

\[
x_n = c_1 r^n e^{i \theta} + c_2 r^n e^{-i \theta}
\]

where \(c_1, c_2\) are any complex numbers.

We usually desire real solutions. To get these we take appropriate linear combinations of the complex solutions to get the real solutions \(x^{(1)}_n\) and \(x^{(2)}_n\)

\[
x^{(1)}_n = \frac{z^{(1)}_n + z^{(2)}_n}{2} = r^n \frac{e^{i \theta} + e^{-i \theta}}{2} = r^n \cos n\theta
\]

\[
x^{(2)}_n = \frac{z^{(1)}_n - z^{(2)}_n}{2i} = r^n \frac{e^{i \theta} - e^{-i \theta}}{2i} = r^n \sin n\theta
\]

These solutions are clearly LI, thus, the general real valued solution is

\[
x_n = r^n(\alpha \cos n\theta + \beta \sin n\theta)
\]

with \(\alpha, \beta\) arbitrary real constants.

**Example 1.** \(x_{n+2} + 5x_{n+1} + 6x_n = 0, \quad n = 0, 1, 2, \ldots\)

Assume \(x_n = \lambda^n\) to find \(\lambda^2 + 5\lambda + 6 = 0\) where \(\lambda = -3, -2\). The general solution is

\[
x_n = \alpha(-2)^n + \beta(-3)^n.
\]

**Example 2.** \(x_{n+2} + 2x_{n+1} + x_n = 0, \quad n = 0, 1, 2, \ldots\)

We have \(x_n = \lambda^n\), \(\lambda^2 + 2\lambda + 1 = 0\), \(\lambda = -1, -1\). Thus \((-1)^n\) is one solution. The other solution is \(n(-1)^n\) and the general solution is \(x_n = \alpha(-1)^n + \beta n(-1)^n\).

**Example 3.** \(x_{n+2} - 2x_{n+1} + 2x_n = 0, \quad n \geq 0\)

We have \(\lambda^2 - 2\lambda + 2 = 0\) or \(\lambda_1 = 1 + i = \sqrt{2}e^{i\pi/4}\) and \(\lambda_2 = 1 - i = \sqrt{2}e^{-i\pi/4}\). Therefore the general solution in real form is

\[
x_n = 2^{n/2} \left(\alpha \cos \frac{n\pi}{4} + \beta \sin \frac{n\pi}{4}\right).
\]

**Exercises 1.10**

In problems 1–8 find the general solution in real form and check them.

1. \(x_{n+2} - x_n = 0, \quad n \geq 0\)
2. \(x_{n+2} - 6x_{n+1} + 9x_n = 0, \quad n \geq 0\)
3. \(x_n + x_{n-2} = 0, \quad n \geq 2\)
4. \(4y_{n+1} - 4y_n + y_{n-1} = 0, \quad n \geq 1\)
5. \(y_n + 4y_{n-1} + 4y_{n-2} = 0, \quad n \geq 2\)
6. \(x_{n+2} + x_{n+1} + x_n = 0, \quad n \geq 0\)
7. \(x_{k+2} + 2x_{k+1} + 2x_k = 0, \quad k \geq 0\)
8. \(6x_{n+1} - 7x_n + 4x_{n-1} = 0, \quad n \geq 3764\)
9. Solve \[ \Delta E: x_{k+2} + 2x_{k+1} + 2x_k = 0, \ k \geq 0 \]
   IC: \[ x_0 = 0, \ x_1 = 1 \]
10. Solve \[ \Delta E: x_{n+2} + x_n = 0, \ n \geq 0 \]
    IC: \[ x_0 = 1, \ x_1 = 1 \]
11. Solve \[ \Delta E: x_{n+1} - 2x_n \cdot \cos(\pi/7) + x_{n-1} = 0, \ n \geq 1 \]
    IC: \[ x_0 = 1, \ x_1 = \cos(\pi/7) + \sin(\pi/7) \]
12. Find the second order homogeneous equation with constant coefficients that have the following as general solutions.
   a. \[ x_n = \alpha(-1)^n + \beta 3^n \]
   b. \[ x_n = (\alpha + \beta n)4^n \]
   c. \[ x_n = 2^{n/2}(\alpha \cos \frac{3n\pi}{4} + \beta \sin \frac{3n\pi}{4}) \]
   d. \[ x_n = \alpha - \beta n \]

1.11–The Method of Undetermined Coefficients

We recall that the general solution of the non-homogeneous equation

\[
ax_{n+2} + bx_{n+1} + cx_n = f_n, \quad n = 0, 1, 2, \ldots
\]

is \( x_n = x_n^{(h)} + x_n^{(p)} \) where \( x_n^{(h)} \) is the general solution of the homogeneous equation and \( x_n^{(p)} \) is a particular solution of the non-homogeneous equation. In section 8, we indicated how to find \( x_n^{(p)} \). We shall consider how to find a particular solution when \( f_n \) has one of the following forms

(a) \( f_n = k \), a constant  
(b) \( f_n = a \) polynomial in \( n \)  
(c) \( f_n = k\alpha^n \), \( k, \alpha \) constants 

or a sum of terms of these types.

**Example 1.** \( x_{n+2} + 5x_{n+1} + 6x_n = 3 \)

The homogeneous solution is \( x_n^{(h)} = c_1(-2)^n + c_2(-3)^n \). We assume \( x_n^{(p)} = A \), a constant, where \( A \) must be determined. Substituting into the differences equation we find

\[ A + 5A + 6A = 3 \quad \text{or} \quad A = 1/4 \]

Thus \( x_n^{(p)} = 1/4 \) and

\[ x_n = c_1(-2)^n + c_2(-3)^n + 1/4 \]

**Example 2.** \( x_{n+2} + 5x_{n+1} + 6x_n = 1 + 2n \). Assume \( x_n^{(p)} = A + Bn \). Upon substitution we get

\[ A + B(n + 2) + 5(A + B(n + 1)) + 6(A + Bn) = 1 + 2n \]

\[ 12B = 2, \quad 12A + 7B = 1 \quad \text{or} \quad B = 1/6, \ A = -1/72. \]

Thus

\[ x_n^{(p)} = -\frac{1}{72} + \frac{n}{6}. \]

**Example 3.** \( x_{n+2} + 5x_{n+1} + 6x_n = 2 \cdot 4^n \). Assume \( x_n^{(p)} = A4^n \). We find

\[ A4^{n+2} + 5A4^{n+1} + 6A4^n = 2 \cdot 4^n \]

\[ (16A + 20A + 6A)4^n = 2 \cdot 4^n \]

\[ 42A = 2, \ A = 1/21. \]
Thus \( x_n^{(p)} = \frac{4n}{21} \).

**Example 4.** \( x_{n+2} + 5x_{n+1} + 6x_n = 3(-2)^n \). Recall that \( x_n^{(h)} = c_1(-2)^n + c_2(-3)^n \). If we assume \( x_n^{(p)} = A(-2)^n \), this will not work. The reason it will not work is that the assumed form is a solution of the homogeneous equation; it cannot also be a solution of the non-homogeneous equation. We modify our assumption slightly to \( x_n^{(p)} = An(-2)^n \). We find

\[
A(n + 2)(-2)^{n+2} + 5A(n + 1)(-2)^{n+1} + 6An(-2)^n = 3(-2)^n
\]
\[
A(-2)^n\{(n + 2)(-2)^2 + 5(n + 1)(-2) + 6n\} = 3(-2)^n
\]
\[
A(-2)^n\{4n + 8 - 10n - 10 + 6n\} = 3(-2)^n
\]
\[
-2A(-2)^n = 3(-2)^n, \text{ or } A = -3/2.
\]

Thus \( x_n^{(p)} = -3n(-2)^n/2 \) and \( x_n = c_1(-2)^n + c_2(-3)^n - \frac{3}{2} n(-2)^n \).

**Example 5.** \( x_{n+2} - 2x_{n+1} + x_n = 1 \). We find \( x_n^{(h)} = c_1 + c_2 n \). We first try \( x_n^{(p)} = A \), but this is a solution of the homogeneous equation. We modify it to \( x_n^{(p)} = An \), but this is also a solution. We modify it again to \( x_n^{(p)} = An^2 \); this will work.

\[
A(n + 2)^2 - 2A(n + 1)^2 + An^2 = 1.
\]

We find that the ‘\( n^2 \)’ terms and the ‘\( n \)’ terms cancel out on the left and we have \( 2A = 1 \) or \( A = 1/2 \). Thus \( x_n^{(p)} = n^2/2 \) and

\[
x_n = c_1 + c_2 n + \frac{n^2}{2}.
\]

**Example 6.** \( x_{n+2} + 5x_{n+1} + 6x_n = 3(-2)^n \). Recall that \( x_n^{(h)} = c_1(-2)^n + c_2(-3)^n \). If we assume \( x_n^{(p)} = A(-2)^n \), this will not work. For this assumed form is a solution of the homogeneous equation; it cannot also be a solution of the non-homogeneous equation. We modify our assumption slightly to \( x_n^{(p)} = An(-2)^n \). We find

\[
A(n + 2)(-2)^{n+2} + 5A(n + 1)(-2)^{n+1} + 6An(-2)^n = 3(-2)^n
\]
\[
A(-2)^n\{(n + 2)(-2)^2 + 5(n + 1)(-2) + 6n\} = 3(-2)^n
\]
\[
A(-2)^n\{4n + 8 - 10n - 10 + 6n\} = 3(-2)^n
\]
\[
-2A(-2)^n = 3(-2)^n, \text{ or } A = -3/2.
\]

Thus \( x_n^{(p)} = -3n(-2)^n/2 \) and \( x_n = c_1(-2)^n + c_2(-3)^n - \frac{3}{2} n(-2)^n \).

**Example 7.** \( x_{n+2} - 2x_{n+1} + x_n = 1 \). We find \( x_n^{(h)} = c_1 + c_2 n \). We first try \( x_n^{(p)} = A \) but this is a solution of the homogeneous equation, we modify it to \( x_n^{(p)} = An \), but this is also a solution. We modify it again to \( x_n^{(p)} = An^2 \); this will work.

\[
A(n + 2)^2 - 2A(n + 1)^2 + An^2 = 1.
\]

We find that the terms involving \( n^2 \) and \( n \) cancel out and we are left with

\[
2A = 1 \quad \text{or } A = 1/2.
\]

Thus \( x_n^{(p)} = n^2/2 \) and

\[
x_n = c_1 + c_2 n + \frac{n^2}{2}.
\]
We summarize the procedure.

**Method.** To find a particular solution of

\[ ax_{n+2} + bx_{n+1} + cx_n = f_n \]

where \( f_n \) is a polynomial of degree \( d \), assume \( x^{(p)}_n \) is an arbitrary polynomial of degree \( d \). If any term is a solution of the homogeneous equation, multiply by \( n \); if any term is still a solution multiply by \( n^2 \).

If \( f_n = k\alpha^n \), assume \( x^{(p)}_n = A\alpha^n \). If this is a solution of the homogeneous equation use \( x^{(p)}_n = An\alpha^n \); if this is also a solution of the homogeneous equation use \( x^{(p)}_n = An^2\alpha^n \).

**Example 8.** We illustrate the above method with a few examples.

<table>
<thead>
<tr>
<th>Homogeneous solution</th>
<th>( f_n )</th>
<th>Proper Form for ( x^{(p)}_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c_12^n + c_33^n )</td>
<td>( 5n^2 )</td>
<td>( A + Bn + Cn^2 )</td>
</tr>
<tr>
<td>( c_1 + c_2 \cdot 3^n )</td>
<td>( 3 + 5n^2 )</td>
<td>( (A + Bn + Cn^2)n )</td>
</tr>
<tr>
<td>( c_1 + c_2n )</td>
<td>( 3 + 5n^2 )</td>
<td>( (A + Bn + Cn^2)n^2 )</td>
</tr>
<tr>
<td>( c_1 \cdot 2^n + c_2 \cdot 3^n )</td>
<td>( 2 \cdot 5^{n-1} )</td>
<td>( A5^n ) or ( A5^{n-1} )</td>
</tr>
<tr>
<td>( c_1 \cdot 2^n + c_2 \cdot 3^n )</td>
<td>( 2 \cdot 3^{n-1} )</td>
<td>( An3^n )</td>
</tr>
<tr>
<td>( c_1 \cdot 2^n + c_2n\cdot 2^n )</td>
<td>( 5 \cdot 2^n )</td>
<td>( An^22^n )</td>
</tr>
<tr>
<td>( c_1 \cdot 2^n + c_2 \cdot 3^n )</td>
<td>( 6 \cdot 2^{-n} )</td>
<td>( A2^{-n} )</td>
</tr>
</tbody>
</table>

**Exercises 1.11**

1. Find the general solutions of
   a. \( x_{n+2} + x_{n+1} - 2x_n = 3 \)
   b. \( x_{n+2} + x_{n+1} - 2x_n = 5 \cdot 4^{n-2} \)
   c. \( x_{n+2} + x_{n+1} - 2x_n = (-2)^{n+1} \).

2. Using the results of problem 1, find a particular solution of
   a. \( x_{n+2} + x_{n+1} - 2x_n = 3 + 5 \cdot 4^{n-2} + (-2)^{n+1} \)
   b. \( x_{n+2} + x_{n+1} - 2x_n = 1 - 5 \cdot 4^n + (-2)^{n+2} \).

3. Solve \( \Delta E: x_{n+1} + 5x_n + 6x_{n-1} = 12 \)
   IC: \( x_0 = 0, \ x_1 = 0 \).

4. Solve \( \Delta E: x_{n+1} - 2x_n + x_{n-1} = 2 - 3n \)
   IC: \( x_0 = 0, \ x_1 = 1 \).

5. Find a second order difference equation whose general solution is
   a. \( c_12^n + c_2(-3)^{n} + 5 \cdot 4^n \)
   b. \( c_1 + c_2(-3)^{n} + 1 + 2n \)
   c. \( 2\pi(c_1 \cos(\frac{3\pi}{2}n) + c_2 \sin(\frac{3\pi}{2}n)) + 3 \).

6. A sequence starts off with \( x_1 = 0, \ x_2 = 1 \) and thereafter each term is the average of the two preceding terms. Find a formula for the \( n^{th} \) term of the sequence.

7. Write down the proper form for a particular solution of
   a. \( x_{n+1} + 5x_n + 6x_{n-1} = n + 2(-3)^{n-2} \)
   b. \( x_{n+2} - 2x_{n+1} + x_n = 2 - n + 3^n \)
   c. \( x_{n+2} - 3x_{n+1} + 2x_n = n - n^2 + 3^n - 2^n \).
1.12–A Simple Model of National Income

We shall study a simple mathematical model of how national income changes with time. Let \( Y_n \) be the national income during the \( n^{th} \) period. We assume that \( Y_n \) is made up of three components:

1. \( C_n \) = consumer expenditures during the \( n^{th} \) period
2. \( I_n \) = induced private expenditures during the \( n^{th} \) period
3. \( G_n \) = governmental expenditures during the \( n^{th} \) period.

Since we are assuming that these are the only factors contributing to national income, we have the simple accounting equation

\[
Y_n = C_n + I_n + G_n. \tag{1}
\]

Following Samuelson, we make three additional assumptions

4. Consumer expenditures in any period is proportional to the national income of the preceding period.
5. Induced private investment in any period is proportional to the increase in consumption of that period over the preceding period (the so-called acceleration principle)
6. Government expenditure is the same in all periods.

We restate these assumptions in mathematical terms. If we denote the constant of proportionality in (4) by \( a \), we have

\[
C_n = a \ Y_{n-1} \tag{2}
\]

The positive constant \( a \) is called the marginal propensity to consume. The constant of proportionality in assumption (5), we denote by \( b \) and we have the equation

\[
I_n = b(C_n - C_{n-1}) \tag{3}
\]

The positive constant \( b \) is called the relation. If consumption is decreasing, then \( (C_n - C_{n-1}) < 0 \) and therefore \( I_n < 0 \). This may be interpreted to mean a withdrawal of funds committed for investment purposes, for example, by not replacing depreciated machinery. Finally, assumption (6) states that \( G_n \) is a constant, and we may as well assume that we have chosen our units so that the government expenditure is equal to 1 (these days 1 stands for about 1 trillion dollars). Thus

\[
G_n = 1, \text{ for all } n \tag{4}
\]

Substituting equations (2), (3), and (4) into (1) we obtain a single equation for the national income

\[
Y_n = a \ Y_{n-1} + b(C_n - C_{n-1}) + 1
= a \ Y_{n-1} + b(aY_{n-1} - aY_{n-2}) + 1 \tag{5}
\]

or finally

\[
Y_n - a(1+b)Y_{n-1} + abY_{n-2} = 1, \; n = 2, 3, \ldots \tag{6}
\]

Let us analyze a particular case when \( a = 1/2, \; b = 1 \) and assume that \( Y_0 = 2 \) and \( Y_1 = 3 \). Thus we have to solve the following initial value problem:

\[
Y_n - Y_{n-1} + Y_{n-2}/2 = 1, \; n = 2, 3, \ldots
Y_0 = 2, \; Y_1 = 3. \tag{7}
\]

The general solution of (7) is

\[
Y_n = (1/\sqrt{2})^n \{ A \cos(n \pi/4) + B \sin(n \pi/4) \} + 2. \tag{8}
\]
From the initial conditions we find that $A = 0$ and $B = 2$, thus the solution is

$$Y_n = 2(1/\sqrt{2})^n \sin (n\pi/4) + 2.$$  

The presence of the sine term in (9) makes $Y_n$ an oscillating function of the time period $n$. Since $1/\sqrt{2} < 1$, the amplitude of the oscillations decrease as $n$ increases and the first term on the right of (9) approaches zero. The sequence $Y_n$ therefore approaches the limit 2 as $n$ approaches infinity. A graph of $Y_n$ is shown in Figure 1.

We conclude that a constant level of government expenditures results (in this special case) in damped oscillatory movements of national income which gradually approach a fixed value.

We now consider the case when $a = 0.8$ and $b = 2$. The difference equation (6) now becomes

$$Y_n - 2.4Y_{n-1} + 1.6Y_{n-2} = 1.$$  

(9)

The general solution of this equation is

$$Y_n = (\sqrt{1.6})^n (c_1 \cos n\theta + c_2 \sin n\theta) + 5.$$  

(10)

where $\theta = \arctan(0.4/1.2)$. We note that $Y_n$ has an oscillatory character but since $\sqrt{1.6} > 1$, the factor $(\sqrt{1.6})^n$ causes the oscillations to increase in amplitude.

The two special cases we have considered show that we can get very different behavior of the national income for different values of the parameters $a$ and $b$. We analyze the situation in general. Consider the difference equation (6). Let $\lambda_1$ and $\lambda_2$ be roots of the characteristic equation

$$\lambda^2 - a(1 + b)\lambda + ab = 0.$$  

(11)

The general solution of the difference equation (6) has one of the following three forms.

$$Y_n = c_1(\lambda_1)^n + c_2(\lambda_2)^n + 1/(1 - a), \quad \lambda_1, \lambda_2 \text{ real and distinct}$$  

(12)

$$Y_n = c_1(\lambda_1)^n + c_2n(\lambda_1)^n + 1/(1 - a), \quad \lambda_1 = \lambda_2$$  

(13)

$$Y_n = r^n(c_1 \cos(n\theta) + c_2 \sin(n\theta)) + 1/(1 - a), \quad \lambda_1 = re^{i\theta}.$$  

(14)
In all cases, in order for $Y_n$ to have a limit as $n$ approaches infinity, for any possible choices of $c_1$ and $c_2$ it is necessary and sufficient that $|\lambda_1| < 1$ and $|\lambda_2| < 1$. It can be shown that this will happen if and only if the positive parameters $a$ and $b$ satisfy the two conditions

$$a < 1 \quad \text{and} \quad ab < 1.$$  

If $a$ and $b$ satisfy these conditions the national income will approach the limit $1/(1-a)$ as $n$ approaches infinity independent of the initial conditions.

**Exercises 1.12**

Find and sketch the solution to equation (6) under the following assumptions:

1. $a = \frac{3}{4}, \quad b = \frac{1}{4}, \quad Y_0 = 1, \quad Y_1 = 2.$
2. $a = \frac{5}{8}, \quad b = \frac{1}{5}, \quad Y_0 = 1, \quad Y_1 = 2.$

1.13–The Gamblers Ruin.

We enter into a game of chance with an initial holding of $c$ dollars, our adversary begins with $d$ dollars. At each game we will win one dollar with probability $p$ and lose one dollar with probability $q = 1 - p, 0 < p < 1$. The gamblers ruin problem is to determine the probability of our ultimate ruin, that is the probability that we end up with zero if we keep playing the game.

Let $P_n$ = the probability of ruin given that we now hold $n$ dollars. In particular we are interested in determining $P_c$, that is the probability of ruin if we start with $c$ dollars. Since if we start with zero dollars we are already ruined we have the condition

$$P_0 = 1.$$  

Also if we end up with all the money, namely $(c+d)$ dollars, the game is over and we have no possibility of being ruined. Thus we have the condition

$$P_{c+d} = 0.$$  

We can set up a difference equation for $P_n$ by noting the following:

*The probability of ruin, $P_n$, is equal to the probability of winning the next game followed by eventual ruin plus the probability of losing the next game followed by eventual ruin.*

Translating this into mathematical language we have

$$P_n = pP_{n+1} + qP_{n-1}, \quad n = 1, 2, \ldots, c + d$$  

(together with the boundary conditions

$$P_0 = 1 \quad \text{and} \quad P_{c+d} = 0.$$  

This is a homogeneous second order difference equation with constant coefficients. Assume $P_n = \lambda^n$ and substitute into (3) to find

$$\lambda^n = p\lambda^{n+1} + q\lambda^{n-1} \quad \text{or} \quad \lambda = p\lambda^2 + q \quad \text{or} \quad p\lambda^2 - \lambda + q = 0.$$  

Recalling that $q = 1 - p$ the roots of this quadratic equation are

$$\lambda_1 = 1 \quad \text{and} \quad \lambda_2 = q/p.$$
If \( p \neq 1/2 \), the roots are distinct and the general solution is

\[ P_n = A + B(q/p)^n. \]  \hfill (6)

Using the boundary conditions \( P_0 = 1 \) and \( P_{c+d} = 0 \), we get the following equations for the constants \( A \) and \( B \):

\[ 1 = A + B, \quad 0 = A + (q/p)^{c+d}B. \]  \hfill (7)

Solving for \( A \) and \( B \) we find

\[ A = \frac{- (q/p)^{c+d}}{1 - (q/p)^{c+d}}, \quad B = \frac{1}{1 - (q/p)^{c+d}}. \]  \hfill (8)

Thus we have

\[ P_n = \frac{(q/p)^n - (q/p)^{c+d}}{1 - (q/p)^{c+d}}. \]  \hfill (9)

In a typical Las Vegas slot machine we would have \( p = 0.4 \) and \( q = 0.6 \). If you had $100 (c = 100) and the house had $1000 (d = 1000) you would find that

\[ P_{100} = \frac{(1.5)^{100} - (1.5)^{1100}}{1 - (1.5)^{1100}} = 1.00000. \]

This result is correct to many decimal places. In other words, if you play long enough you will lose!!

**Exercises 1.13**

1. In a “fair” game of chance, \( p = q = 1/2 \).
   a. Find the solution of the gamblers ruin problem in this case, that is find \( P_c \).
   b. If \( c = d \), what is \( P_c \)?
   c. If \( c = 100 \) and \( d = 1000 \), what is \( P_c \)? What conclusion can you draw for such a fair game?
CHAPTER II

DIFFERENTIAL EQUATIONS AND THE LAPLACE TRANSFORM

2.1–Introduction

The main aim of this chapter is to develop the method of Laplace Transforms for solving certain linear differential equations. However, first we shall review some of the elementary differential equations studied in calculus.

A **differential equation** (DE) is an equation involving the derivative of an unknown function. The **order** of a DE is the order of the highest derivative appearing the equation. The following are examples of differential equations.

\[
\frac{dy}{dx} = x^2, \quad -\infty < x < \infty \quad (1)
\]

\[
\frac{dy}{dt} = ry, \quad t \geq 0 \quad (2)
\]

\[
\ddot{x} + 5\dot{x} + 6x = 0, \quad t \geq 0 \quad (3)
\]

The first two equations are first order, the third equation is of second order.

By a **solution** of a DE of the \(n^{th}\) order we mean a function, defined in some interval \(J\), which possesses \(n\) derivatives and which satisfies the differential equation identically in \(J\). The **general solution** of a DE is the set of all solutions.

Let us discuss each of the DE’s above. One solution of (1) is \(y = x^3/3\) since \(\frac{dy}{dx} = \frac{d}{dx}(x^3/3) \equiv x^2\). From calculus we know that \(y = x^3/3 + c\), where \(c\) is an arbitrary constant, represents all solutions of (1), therefore this is the general solution. Here the interval \(J\) is the entire real axis \(-\infty < x < \infty\).

For the DE (2), we may use the method of separation of variables.

\[
\frac{dy}{y} = rdt, \quad \text{or} \quad \ln|y| = rt + c, \quad |y| = e^{rt+c} = e^c e^{rt}, \quad y = ke^{rt}, \quad \text{where} \quad k = \pm e^c
\]

It is easy to verify that \(y = ke^{rt}\) is a solution for all \(t\), where \(k\) is any constant. In fact, this is the general solution.

For the DE (3), we note that since the equation is linear, homogeneous and has constant coefficients, we should look for solutions of the form \(e^{\lambda t}\) for an appropriate constant \(\lambda\). Substituting into the DE we find \(\ddot{x} + 5\dot{x} + 6x = (\lambda^2 + 5\lambda + 6)e^{\lambda t} = 0\). Thus \(\lambda^2 + 5\lambda + 6 = 0\). This yields the two values \(\lambda = -3\) and \(\lambda = -2\), and the two solutions \(e^{-3t}\) and \(e^{-2t}\). Since the equation is linear and homogeneous, it follows that \(x = c_1e^{-3t} + c_2e^{-2t}\) is also a solution for arbitrary constants \(c_1\) and \(c_2\); it can be shown that this is the general solution.

Generally speaking differential equations have infinitely many solutions as in the examples above. Notice that the general solution of the first order DE’s (1) and (2) contain one arbitrary constant and the general solution of the second order DE (3) contains two arbitrary constants. This is the usual situation. In order to obtain a unique solution it is necessary to require that the solution satisfy subsidiary initial or boundary conditions.

*Example 1.* Find the solution of the DE \(\frac{dy}{dx} = x^2\) that satisfies the initial condition (IC) \(y(0) = 5\).
We know that the general solution of the DE is \( y = y(x) = \frac{x^3}{3} + c \). Thus \( y(0) = 5 = 0 + c \), and \( c = 5 \). The unique solution to the initial value problem is \( y = \frac{x^3}{3} + 5 \).

**Example 2.** Solve \( \dot{x} + 5\dot{x} + 6x = 0 \)

IC: \( x(0) = 2, \ \dot{x}(0) = -5 \).

We have seen above that the general solution of the DE is \( x = c_1 e^{-3t} + c_2 e^{-2t} \). Thus \( \dot{x} = -3c_1 e^{-3t} - 2c_2 e^{-2t} \). Putting in the IC’s we obtain

\[
\begin{align*}
2 &= c_1 + c_2 \\
-5 &= -3c_1 - 2c_2
\end{align*}
\]

Solving these we obtain \( c_1 = c_2 = 1 \) and the solution \( x = e^{-3t} + e^{-2t} \).

**Exercises 2.1**

1. Show that \( u = e^{2t} \) is a solution of \( \ddot{u} - 4u = 0 \).
2. Is \( y = e^{-2x} \) a solution of \( y \frac{dy}{dx} + y^2 = -e^{-4x} \)?
3. Find the value of the constant \( a \), if any, so that \( y = ax^3 \) is a solution of
   - a. \( x^2 y'' + 6xy' + 5y = 0 \)
   - b. \( x^2 y'' + 6xy' + 5y = x^3 \)
   - c. \( x^2 y'' + 6xy' + 5y = 2x^2 \).
4. Find the values of the constant \( \lambda \), if any, so that \( e^{\lambda x} \) is a solution of
   - a. \( y'' - 4y = 0 \)
   - b. \( y' + 4y = 0 \)
5. a. Solve \( \text{DE: } y' = x \) IC: \( y(0) = 1 \)
   b. Solve \( \text{DE: } y'' - 4y = 0 \) IC: \( y(0) = 1, \ y'(0) = -2 \)

### 2.2–Separation of Variables

A first order differential equation is said to have its variables separated if it is in the form

\[ A(x) + B(y) \frac{dy}{dx} = 0 \quad (1) \]

or in the equivalent differential form

\[ A(x) \, dx + B(y) \, dy = 0 \quad (2). \]

Here \( A(x) \) and \( B(x) \) are assumed to be given continuous functions.

Suppose \( \Phi(x) \) is a solution of DE (1) for \( x \) in some interval \( J \). Then

\[ A(x) + B(\Phi(x))\Phi'(x) = 0 \quad \text{for all } x \in J. \]

This is an identity in \( x \) which may be integrated to yield

\[
\int A(x) \, dx + \int B(\Phi(x))\Phi'(x) \, dx = c
\]

where \( c \) is an arbitrary constant. In the second integral we use the substitution \( y = \Phi(x), \ dy = \Phi'(x) \, dx \) to obtain

\[
\int A(x) \, dx + \int B(y) \, dy = c. \quad (3)
\]
Thus any solution of the DE (1) must satisfy the implicit equation (3). Conversely, assuming that this equation determines \( y \) as a differentiable function of \( x \), we shall show that it is a solution of the DE. Differentiating (3) we find

\[
\frac{d}{dx} \left( \int A(x) \, dx \right) + \frac{d}{dx} \left( \int B(y) \, dy \right) = 0. \tag{4}
\]

However \( \frac{d}{dx} \left( \int A(x) \, dx \right) = A(x) \), and \( \frac{d}{dx} \left( \int B(y) \, dy \right) = \frac{d}{dy} \left( \int B(y) \, dy \right) \frac{dy}{dx} = B(y) \frac{dy}{dx} \).

Thus (4) becomes \( A(x) + B(y) \frac{dy}{dx} = 0 \), and any function determined by the implicit equation (3) represents a solution to the given DE.

**Example 1.** Find and check the general solution of \( \frac{dy}{dx} = \frac{y \cos x}{1 + 2y^2} \).

Separating variables we find

\[
\frac{1 + 2y^2}{y} \, dy = \cos x \, dx
\]

which yields

\( \ln |y| + y^2 = \sin x + c. \) \tag{i}

To verify that this is a solution we use implicit differentiation. If (i) determines \( y \) as a differentiable function of \( x \) in some interval \( J \), then we have

\( \ln |y(x)| + (y(x))^2 \equiv \sin x + c. \)

Differentiating both sides with respect to \( x \) we find

\[
\frac{1}{y} \frac{dy}{dx} + 2y \frac{dy}{dx} = \cos x
\]

which, when solved for \( \frac{dy}{dx} \), yields the original DE.

**Example 2.** Population growth—unrestricted growth.

Let \( x(t) \) be the number of individuals in a population at time \( t \). Assume that the population has a constant growth rate of \( r \) (net births per individual per year). The population satisfies

DE: \( \frac{dx}{dt} = rx, \ t \geq 0 \)

IC: \( x(0) = x_0 \)

We may solve the DE by separation of variables.

\[
\frac{dx}{x} = r \, dt, \ \ln x = rt + c, \ x = e^{rt+c} = e^r t e^c = e^r k.
\]

Since \( x(0) = x_0 \), we have \( k = x_0 \) and thus \( x = x_0 e^{rt} \). We see that the population increases exponentially as shown in the figure to the right. One measure of how fast the population is growing is the doubling time, that is the time it takes for the population to grow from \( x_0 \) to \( 2x_0 \). Solving \( 2x_0 = x_0 e^{rt} \) for \( t \) we obtain the doubling time of \( t_d = \frac{\ln 2}{r} \).

Finally we note that this is the same differential equation as the growth of money under continuous compound interest at the nominal interest rate of \( r \).
Example 3. Population growth—with limited food supply.

Usually, populations of a given species do not continue to grow exponentially but level off due to a limited food supply or other limiting factors. The simplest model that takes this into account is

\[ \frac{dx}{dt} = rx(c-x), \quad r > 0, \ c > 0 \]
\[ IC: \ x(0) = x_0. \]

If \( x \) is between 0 and \( c \) then \( \frac{dx}{dt} \) is positive so that \( x \) is increasing. However when \( x \) is close to \( c \), \( c-x \) is small so that the rate of increase of the population is small.

We solve the DE by separating variables

\[ \int \frac{dx}{x(c-x)} = \int r \ dt = rt + k \]

To integrate the left side, we use partial fractions to obtain

\[ \int \frac{dx}{x(c-x)} = \frac{1}{c} \int \left( \frac{1}{x} + \frac{1}{c-x} \right) dx = \frac{1}{c} \ln \frac{x}{c-x}. \]

Therefore

\[ \frac{1}{c} \ln \frac{x}{c-x} = rt + k \quad \text{or} \quad \frac{x}{c-x} = e^{rt} A, \quad \text{where we have set} \ A = e^{ck}. \]

Solving for \( x \) we find

\[ x = \frac{Ac}{e^{rt} + A}, \]

where \( A \) is an arbitrary constant. Using the initial condition \( x(0) = x_0 \) we find \( A = \frac{x_0}{c - x_0} \), and the final solution can be written

\[ x(t) = \frac{c}{1 + \left( \frac{c}{x_0} - 1 \right) e^{-rt}}. \quad (5) \]

A sketch of the solution for \( 0 < x_0 < c \) is shown on the right. Note that \( x \) continually increases, has a point of inflection when \( x \) reaches \( c/2 \) and approaches \( c \) as \( t \to \infty \). Thus \( c \) is the upper limit to the population. The solution looks somewhat like a \( S \)-shaped curve and is often called the \textit{logistic} curve.

There are three parameters in the solution (5), \( x_0, r, \) and \( c \). In order to evaluate these parameters we must know the population at three distinct times. Some years ago Pearl and Read used this model for the population of the United States using the data below.

<table>
<thead>
<tr>
<th>Year</th>
<th>Population in Millions</th>
<th>t</th>
</tr>
</thead>
<tbody>
<tr>
<td>1790</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>1850</td>
<td>23</td>
<td>60</td>
</tr>
<tr>
<td>1910</td>
<td>92</td>
<td>120</td>
</tr>
</tbody>
</table>
The final result, where \( x \) is measured in millions and 1790 represents \( t = 0 \), is

\[ x(t) = \frac{210}{1 + 15.15 e^{-0.063t}}. \] (6)

This formula predicts the limiting population to be 210 million, which is below the actual population today. However, the Pearl–Read formula gave quite accurate predictions until about 1970. This illustrates the fact that such a simple model, which ignores many important phenomena in population growth, should be used with skepticism. It may yield useful results over short time periods, but should not be expected to hold over long periods.

**Exercises 2.2**

In problems 1–6 find the general solution, put in as simple form as possible and check.

1. \( y \frac{dy}{dx} + x^3 = 0 \)
2. \( x^2 + \frac{1 + x}{y} \frac{dy}{dx} = 0 \)
3. \( x + ye^{-x} \frac{dy}{dx} = 0 \)
4. \( \frac{dx}{dy} = \frac{y^2 + y}{x + xy^2} \)
5. \( \frac{dv}{du} = \frac{4uv}{u^2 + 1} \)
6. \( \frac{dy}{dx} = xy^2 \)

For the next two problems, find the solution, put into as simple a form as possible and check.

7. \( x^2(1 + y^2)dx + 2y dy = 0, \ y(0) = 1 \)
8. \( x^2(1 + y) dx + 3 dy = 0, \ y(0) = 1 \)

9. Suppose a population grows according to the law \( \frac{dx}{dt} = rx \). If the doubling time is 60 years, what is the birth rate (to 4 decimals). What is the population after 30 years, if the initial population is 1000.

10. The mass \( m(t) \) of a radioactive substance decreases according to the law \( \frac{dm}{dt} = -km \). Find the mass at any time if the initial mass is \( m_0 \). Find the time for the initial mass to be reduced to half the initial amount (the half–life).

11. An island in the Pacific is contaminated by radioactive fallout. If the amount of radioactive material is 100 times that considered safe, and if the half–life of the material is 1620 years, how long will it be before the island is safe?

### 2.3–Linear Differential Equations

**First Order Linear DE**

A first order linear DE is one of the form

\[ \frac{dx}{dt} + p(t)x = q(t), \] (1)

where \( p(t) \) and \( q(t) \) are given functions which we assume are continuous. To solve this equation we proceed as follows

1. Multiply the DE by the integrating factor \( e^{\int p(t) dt} \) to obtain

\[ e^{\int p(t) dt} \left( \frac{dx}{dt} + p(t)x \right) = e^{\int p(t) dt} q(t). \]

2. Rewrite the left hand side as the derivative of the product of \( x \) times the integrating factor

\[ \frac{d}{dt} \left( xe^{\int p(t) dt} \right) = e^{\int p(t) dt} q(t). \]
(We leave to the reader to verify that the left hand sides of items 2 and 3 are the same).

3. Integrate both sides and solve for the solution $x(t)$.

**Example 1.** Solve $\frac{dx}{dt} - 2tx = t$. The integrating factor is $e^{\int -2t \, dt} = e^{-t^2}$. Multiplying through by the integrating factor we find

$$
e^{-t^2} \left( \frac{dx}{dt} - 2tx \right) = te^{-t^2}, \text{ or } \frac{d}{dt} \left( xe^{-t^2} \right) = te^{-t^2}.
$$

Thus

$$xe^{-t^2} = \int te^{-t^2} \, dt + c = -\frac{e^{-t^2}}{2} + c,$$

and the final solution is

$$x = \frac{1}{2} + ce^{t^2}.$$

**Example 2.** Solve $\frac{dx}{dt} = ax$. Rewriting in the form (1) we have $\frac{dx}{dt} - ax = 0$. The integrating factor is $e^{-at}$. Proceeding as above we see that the general solution is $x = ce^{at}$. This is worthwhile remembering—it is easy to see that one function whose derivative is $a$ times itself is $e^{at}$, and since the equation is linear and homogeneous, any constant times a solution is a solution.

**Second Order Linear DE’s**

Consider a second order linear homogeneous DE with constant coefficients, that is, one of the form

$$a\ddot{x} + b\dot{x} + cx = 0. \tag{2}$$

where $a$, $b$ and $c$ are real constants and $a \neq 0$.

Recall that if $x_1(t)$ and $x_2(t)$ are two solutions then $c_1x_1(t) + c_2x_2(t)$ are also solutions for arbitrary $c_1$ and $c_2$. Furthermore if the two solutions are linearly independent (LI) (one is not identically equal to a constant times the other) then $c_1x_1(t) + c_2x_2(t)$ represents the general solution.

To solve the DE (2) we look for solutions of the form $x = e^{\lambda t}$. Substituting into the differential equation we find $(a\lambda^2 + b\lambda + c)e^{\lambda t} \equiv 0$, thus $\lambda$ must satisfy the characteristic equation $a\lambda^2 + b\lambda + c = 0$. There are three cases:

Case 1—Real unequal roots. If the roots of the characteristic equation are $\lambda_1 \neq \lambda_2$, then $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ are two LI solutions and the general solution is

$$x = c_1e^{\lambda_1 t} + c_2e^{\lambda_2 t}.$$

Case 2—Real equal roots. If the two roots are equal, $\lambda_1 = \lambda_2$, then $e^{\lambda_1 t}$ is the only solution of the assumed form. A second LI solution is $te^{\lambda_1 t}$ and the general solution is

$$x = c_1e^{\lambda_1 t} + c_2te^{\lambda_1 t}.$$

Case 3—Complex conjugate roots. Let the roots be $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$, then $z_1 = e^{(\alpha + i\beta)t} = e^{\alpha t}e^{i\beta t}$ and $z_2 = e^{(\alpha - i\beta)t} = e^{\alpha t}e^{-i\beta t}$ are two complex valued solutions. We would like real solutions. Recall Euler’s forms

$$e^{\pm i\theta} = \cos \theta \pm i \sin \theta$$

or equivalently

$$\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \text{and} \quad \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$
Then \( x_1 = \frac{z_1 + z_2}{2} = e^{at} \cos \beta t \) and \( x_2 = \frac{z_1 - z_2}{2i} = e^{at} \sin \beta t \) are two LI independent real solutions and the general solution is

\[
x = c_1 e^{at} \cos \beta t + c_2 e^{at} \sin \beta t.
\]

**Example 3.** Solve (a) \( \ddot{x} - 4\dot{x} = 0 \) (b) \( 4\ddot{x} + 4\dot{x} + x = 0 \) (c) \( \ddot{x} + 4\dot{x} + 5x = 0 \).

(a) Let \( x = e^{\lambda t} \) to find \( \lambda^2 - 4\lambda = 0, \lambda = 0, 4 \), the general solution is \( x = c_1 e^{0t} + c_2 e^{4t} = c_1 + c_2 e^{4t}. \)

(b) Let \( x = e^{\lambda t} \) to find \( 4\lambda^2 + 4\lambda + 1 = 0 = (2\lambda + 1)^2 \). Thus \( \lambda = -1/2, -1/2 \) and the general solution is \( x = c_1 e^{-t/2} + c_2 e^{-t/2}. \)

(c) Let \( x = e^{\lambda t} \) to find \( \lambda^2 + 4\lambda + 5 = 0 \). Thus \( \lambda = -2 \pm i \) and the general solution is \( x = e^{-2t}(c_1 \cos t + c_2 \sin t). \)

Finally we consider the nonhomogeneous DE

\[
a\ddot{x} + b\dot{x} + cx = f(t).
\]  

Recall that the general solution of 3 is given by

\[
x = x_h(t) + x_p(t),
\]

where \( x_h(t) \) is the general solution of the associated homogeneous equation \( a\ddot{x} + b\dot{x} + cx = 0 \), and \( x_p(t) \) is any one (or particular) solution of 3. When \( f(t) \) is an exponential, a sinusoid, a polynomial or a product of these, one may use undetermined coefficients to find a particular solution. The following examples illustrate this.

**Example 4.** Solve \( \ddot{x} + 5\dot{x} + 6x = 4e^{-4t} \). We first solve the homogeneous equation to obtain

\[
x_h = c_1 e^{-2t} + c_2 e^{-3t}.
\]

Since the right hand side is an exponential we expect a particular solution of the form \( x_p = A e^{-4t} \).

Substituting into the DE we find

\[
(-4)^2 A + 5(-4A) + 6A \ e^{-4t} = 4e^{-4t}.
\]

Thus \( 2A = 4, \ A = 2, \ x_p = 2e^{-4t} \), and the general solution is

\[
x = c_1 e^{-2t} + c_2 e^{-3t} + 2e^{-4t}.
\]

**Example 5.** Solve \( \ddot{x} + 5\dot{x} + 6x = \cos 2t \). For a particular solution we look for a solution of the form

\[
x_p = A \cos 2t + B \sin 2t.
\]

Substituting into the DE we find

\[
\ddot{x} + 5\dot{x} + 6x = (-4A \cos 2t - 4B \sin 2t) + 5(-2A \sin 2t + 2B \cos 2t) + 6(A \cos 2t + B \sin 2t) = \cos 2t
\]

This simplifies to

\[
(2A + 10B) \cos 2t + (2B - 10A) \sin 2t = \cos 2t.
\]

Thus \( 2A + 10B = 1 \) and \( 2B - 10A = 0 \). This yields \( A = 1/52, \ B = 5/52 \), and the general solution

\[
x = c_1 e^{-2t} + c_2 e^{-3t} + \frac{1}{52} \cos 2t + \frac{5}{52} \sin 2t.
\]

**Exercises 2.3**

1. \( \frac{dx}{dt} - 2t = e^{2t} \)
2. \( \frac{dx}{dt} - \frac{x}{t} = t^2 \)
3. \( 2\frac{dx}{dt} + 3x = e^{t/2} \)
4. \( \frac{dy}{dx} - \frac{y}{x} = x^2 + 2 \)
5. \( \frac{dx}{dy} + 2y = e^{x/2} \)
6. \( (2 - x)\frac{dy}{dx} + y = 2x - x^2 \)
7. \( \ddot{x} + 4\dot{x} = 0 \)
8. \( \ddot{x} - 4\dot{x} + 4x = 0 \)
9. \( \ddot{x} - \dot{x} = 0 \)
10. \( 6\ddot{x} - 5\dot{x} + x = 0 \)
11. \( \dddot{x} - 2\dot{x} + 5x = 0 \)
12. \( \ddot{x} - \dot{x} + x = 0 \)
13. \( \dddot{x} + 4\dot{x} + 4x = e^{3t} \)
14. \( 6\ddot{x} - 5\dot{x} + x = 2\sin t \)
15. \( \ddot{x} + 4x = \cos 2t \)
2.4–The Laplace Transform

The Laplace transform of a function \( f(t) \), \( 0 \leq t < \infty \), is defined to be the improper integral

\[
\mathcal{L} \{ f(t) \} = \hat{f}(s) = \int_0^\infty e^{-st} f(t) \, dt
\]

provided this integral converges for at least one value of \( s \). We see that functions \( f(t) \) are transformed into new functions of \( s \), denoted by \( \hat{f}(s) = \mathcal{L}(f(t)) \). We can consider that equation (1) defines an operator or transformation \( \mathcal{L} \) which transforms object functions \( f(t) \) into image functions or transforms \( \hat{f}(s) \).

**Example 1.** Find the Laplace transform of \( e^{at} \).

\[
\mathcal{L} \{ e^{at} \} = \int_0^\infty e^{-st} e^{at} \, dt = \lim_{R \to \infty} \int_0^R e^{-(s-a)t} \, dt.
\]

If \( s > a \), \( \lim_{R \to \infty} e^{-(s-a)R} = 0 \), therefore

\[
\mathcal{L} \{ e^{at} \} = \frac{1}{s-a}, \quad s > a.
\]

Thus the transform of the transcendental function \( e^{at} \) is the “simpler” rational function \( \frac{1}{s-a} \). Putting \( a = 0 \) we find

\[
\mathcal{L} \{ 1 \} = \frac{1}{s}, \quad s > 0.
\]

The Operator \( \mathcal{L} \) is a linear operator, that it,

\[
\mathcal{L} \{ af(t) + bg(t) \} = a\mathcal{L} \{ f(t) \} + b\mathcal{L} \{ g(t) \} = a\hat{f}(s) + b\hat{g}(s)
\]

for those values of \( s \) for which both \( \hat{f}(s) \) and \( \hat{g}(s) \) exist. For example \( \mathcal{L} \{ 1 + 3e^{-2t} \} = \mathcal{L} \{ 1 \} + 3\mathcal{L} \{ e^{-2t} \} = \frac{1}{s} + \frac{3}{s+2} \) using equation (2).

**Example 2.** \( \mathcal{L} \{ \cos bt \} = \mathcal{L} \left\{ \frac{e^{ibt} + e^{-ibt}}{2} \right\} = \frac{1}{2} \frac{1}{s+ib} + \frac{1}{2} \frac{1}{s-ib} = \frac{s}{s^2 + b^2}, \)

\( \mathcal{L} \{ \sin bt \} = \mathcal{L} \left\{ \frac{e^{ibt} - e^{-ibt}}{2i} \right\} = \frac{1}{2i} \frac{1}{s+ib} + \frac{1}{2i} \frac{1}{s-ib} = \frac{b}{s^2 + b^2}. \)

To each object function \( f(t) \) there exists a unique transform \( \hat{f}(s) \), assuming the transform exists. In order to guarantee that to each transform there exists a unique object function, we shall restrict ourselves to continuous object functions. This is the content of the following theorem.

**Theorem 1.** If \( \mathcal{L} \{ f(t) \} \equiv \mathcal{L} \{ g(t) \} \) and \( f(t) \) and \( g(t) \) are continuous for \( 0 \leq t < \infty \), then \( f(t) \equiv g(t) \).

If \( F(s) \) is a given transform and \( f(t) \) is a continuous function such that \( \mathcal{L} \{ f(t) \} = F(s) \), then no other continuous function has \( F(s) \) for its transform; in this case we shall write

\[
f(t) = \mathcal{L}^{-1} \{ F(s) \}
\]
and call \( f(t) \) the inverse Laplace transform of \( F(s) \). For example if \( F(s) = 1/(s - a) \) then

\[
f(t) = \mathcal{L}^{-1} \left\{ \frac{1}{s - a} \right\} = e^{at}.
\]

This formula (or equivalently Equation 2) is worth remembering.

Corresponding to the linearity property of \( \mathcal{L} \) we have the linearity property of \( \mathcal{L}^{-1} \)

\[
\mathcal{L}^{-1} \{ aF(s) + bG(s) \} = a\mathcal{L}^{-1} \{ F(s) \} + b\mathcal{L}^{-1} \{ G(s) \}
\]

The Laplace Transform of a function will not exist if the function grows too rapidly as \( t \to \infty \), for example \( \mathcal{L}\{e^{bt}\} \) does not exist. To describe a useful set of functions for which the transform exists, we need the following definition.

**Definition 1.** \( f(t) \) is of exponential order \( \alpha \) if \( |f(t)| \leq Me^{\alpha t}, \ 0 \leq t < \infty \), where \( M \) and \( \alpha \) are constants.

In other words a continuous function is of exponential order if it does not grow more rapidly than an exponential as \( t \to \infty \).

**Example 3.** \( \sin bt \) is of exponential order 0, since \(|\sin bt| < 1 = 1 \cdot e^{0t}\).

**Example 4.** Clearly \( e^{at} \) is of exponential order \( a \) (take \( M = 1, \ \alpha = a \) in the definition).

**Example 5.** \( t^2 \) is of exponential order 1. Since, from \( e^t = 1 + t + t^2/2! + \cdots \), we have \( t^2 \leq 2!e^t \).

Similarly \( t^n, n \) a positive integer, is of exponential order 1.

We now show that continuous functions of exponential order have Laplace transforms.

**Theorem 2.** If \( f(t) \) is continuous and of exponential order \( \alpha \), then \( \mathcal{L}\{f(t)\} \) exists for \( s > \alpha \).

**Proof** \( \int_0^\infty e^{-st} f(t) \, dt = \int_0^\infty e^{-st}|f(t)| \, dt \leq M \int_0^\infty e^{-st}e^{\alpha t} \, dt \leq M \int_0^\infty e^{-(s-\alpha) t} \, dt \). If \( s > \alpha \) the last integral converges, therefore the original integral converges absolutely.

It can be shown that if \( f(s) = \mathcal{L}\{f(t)\} \) exists for some value of \( s \), say \( s = a \), then it must also exist for \( s > a \). The exact interval for which the transform exists is not important. The only important thing is that we know that the transform of a given function exists. Of course, to show that the transform of a given function exists it is necessary to find at least one value of \( s \) for which it exists. However, after this has been done, the values of \( s \) for which the transform exists can be ignored.

Before proceeding with the development of the properties of Laplace Transforms, we indicate how they are used to solve differential equations by means of a simple example.

**Example 6.** Solve DE: \( \frac{dy}{dt} - 4y = e^t \)

**IC:** \( y(0) = 1 \)

We take the Laplace transform of both sides of the equation, assuming \( y \) and \( dy/dt \) possess transforms

\[
\mathcal{L} \left\{ \frac{dy}{dt} - 4y \right\} = \mathcal{L} \{ e^t \} \quad \text{or} \quad \mathcal{L} \left\{ \frac{dy}{dt} \right\} - 4\mathcal{L} \{ y \} = \frac{1}{s - 4}
\]

where we have used Equation (2) for the right hand side. Consider \( \mathcal{L} \left\{ \frac{dy}{dt} \right\} = \int_0^\infty e^{-st} \frac{dy}{dt} \, dt \). Integrating by parts we find

\[
\mathcal{L} \left\{ \frac{dy}{dt} \right\} = \int_0^\infty e^{-st}\frac{dy}{dt} \, dt
\]

\[
= e^{-st}y(t)|_0^\infty + s \int_0^\infty e^{-st}y(t) \, dt
\]

\[
= \lim_{R \to \infty} e^{-st}y(R) - y(0) + s\mathcal{L} \{ y(t) \}.
\]
In Problem 4 below we show that \( \lim_{R \to \infty} e^{-sR}y(R) = 0 \), therefore
\[
\mathcal{L} \left\{ \frac{dy}{dt} \right\} = s\mathcal{L}\{y(t)\} - y(0) = s\hat{y}(s) - y(0) = s\hat{y}(s) - 1.
\]
Thus the transformed differential equation now becomes
\[
s\hat{y}(s) - 1 - 4\hat{y}(s) = \frac{1}{s - 1}.
\]
or
\[
\hat{y}(s) = \frac{1}{s - 1} + \frac{1}{(s - 1)(s - 4)}
\]
In order to find the solution \( y(t) \) which is the inverse transform of \( \hat{y}(s) \), we replace the second term on the right above with its partial fraction expansion
\[
\frac{1}{(s - 1)(s - 4)} = \frac{1}{3} \frac{1}{s - 4} - \frac{1}{3} \frac{1}{s - 1}.
\]
Therefore
\[
\hat{y}(s) = \frac{4}{3} \frac{1}{s - 4} - \frac{1}{3} \frac{1}{s - 1}.
\]
Using Equation (4) and the linearity of the inverse transform we find
\[
y(t) = \mathcal{L}^{-1}\{\hat{y}(s)\} = \frac{4}{3} e^{4t} - \frac{1}{3} e^{t}.
\]
By direct substitution into the \( \text{DE} \), it can be verified that the above is the solution.

**Exercises 2.4**

1. Starting from the definition find \( \mathcal{L}\{te^{at}\} \).
2. If \( f(t) = 0, \ 0 \leq t \leq 1 \), and \( f(t) = 1, \ t > 1 \), find \( \mathcal{L}\{f(t)\} \).
3. Using the method of Example 2, find \( \mathcal{L}\{e^{at}\cos bt\} \).
4. If \( f(t) \) is of exponential order (\( |f(t)| \leq Me^{\alpha t} \)), show that \( \lim_{R \to \infty} e^{-sR}f(R) = 0, \ s > \alpha \).
5. If \( f(t) \) is of exponential order \( \alpha \), show that \( F(t) = \int_0^t f(\tau) d\tau \) is also of exponential order \( \alpha \).
6. If \( f \) and \( g \) are of exponential order \( \alpha \), show that
   a. \( af(t) + bg(t) \) is also of exponential order \( \alpha \)
   b. \( f(t)g(t) \) is of exponential order \( 2\alpha \).
7. Prove that if \( f(t) \) is continuous of exponential order and \( y \) is a solution of
   \[
y' + ay = f(t)
   \]
then \( y \) and \( y' \) are also continuous and of exponential order and therefore possess Laplace transforms.
8. Prove that if \( f(t) \) is continuous of exponential order and \( y \) is a solution of
   \[
a y'' + by' + cy = f(t)
   \]
then \( y, \ y', \ y'' \) are also continuous and of exponential order and therefore possess Laplace transforms.
9. Using Laplace transforms, solve
   \[
   \text{DE: } \dot{x} - 5x = e^{3t} + 4
   \]
   \[
   \text{IC: } x(0) = 0
   \]
2.5–Properties of the Laplace Transform

We shall develop those properties of Laplace transforms which will permit us to find the transforms and inverse transforms of many functions and to solve linear differential equations with constant coefficients. The first theorem concerns the transform of a derivative; this is the key in the use of transforms to solve differential equations.

**Theorem 1.** If \( f(t) \) and \( f'(t) \) are continuous and \( f(t) \) is of exponential order, then \( \mathcal{L}\{f'(t)\} \) exists for \( s > \alpha \) and

\[
\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0).
\]

**Proof.** \( \mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t) \, dt \). Integrating by parts, we obtain

\[
\mathcal{L}\{f'(t)\} = e^{-st} f(t)|_0^\infty + s \int_0^\infty e^{-st} f(t) \, dt.
\]

Since \( f(t) \) is of exponential order, it follows (Problem 3 of Section 2.4) that \( \lim_{R \to \infty} \{e^{-st} f(R)\} = 0 \), therefore

\[
\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0).
\]

By repeated applications of Theorem 1, we may find the transforms of derivatives of any order.

**Theorem 2.** If \( f, f', \ldots, f^{(n-1)} \) are continuous and of exponential order and \( f^{(n)} \) is continuous, then \( \mathcal{L}\{f^{(n)}(t)\} \) exists and

\[
\mathcal{L}\{f^{(n)}(t)\} = s^n \hat{f}(s) - s^{n-1} f(0) - s^{n-2} f'(0) \cdots - f^{(n-1)}(0).
\]

In particular for \( n = 2 \) we have

\[
\mathcal{L}\{f''(t)\} = s^2 \hat{f}(s) - sf(0) - f'(0).
\]

**Example 1.** Solve

DE: \( \ddot{x} + 5\dot{x} + 6x = 0 \)

IC: \( x(0) = 1, \ \dot{x}(0) = 0 \)

From Problem 8 of Section 2.4 we know \( x, \dot{x}, \ddot{x} \) all possess Laplace transforms. Taking the transform of both sides of the DE, we find

\[
s^2 \hat{x} - s \hat{x} + 5(s \hat{x} - 1) + 6\hat{x} = 0.
\]

Solving for \( \hat{x} \), and using partial fractions, we obtain

\[
\hat{x} = \frac{s + 5}{s^2 + 5s + 6} = \frac{s + 5}{(s + 3)(s + 2)} = \frac{3}{s + 2} + \frac{-2}{s + 3}.
\]

Taking inverse Laplace transforms we have

\[
x(t) = 3e^{-2t} - 2e^{-3t}.
\]

Next we consider the transform of an integral.

**Theorem 3.** If \( f \) is continuous and of exponential order and \( F(t) = \int_0^t f(\tau) \, d\tau \), then \( F(t) \) is of exponential order, \( \mathcal{L}\{F(t)\} \) exists, and

\[
\mathcal{L}\{ \int_0^t f(\tau) \, d\tau \} = \frac{1}{s} \hat{f}(s), \ s > \alpha.
\]
Proof From Problem 5 of Section 2.4 we know that $F(t)$ is of exponential order. Since $F'(t) = f(t)$ and $F(0) = 0$, we have from (2), $\mathcal{L}\{F'(t)\} = \mathcal{L}\{f(t)\} = s\mathcal{L}\{F(t)\}$, from which the desired result follows.

Example 2. We have seen that $\mathcal{L}\{1\} = 1/s$, thus

$$\mathcal{L}(t) = \mathcal{L}\left\{ \int_0^t 1 \cdot dt \right\} = \frac{1}{s} \mathcal{L}\{1\} = \frac{1}{s^2}.$$ 

The next theorem tells us what happens to the transform if we multiply the object function by $e^{at}$.

Theorem 4. If $\mathcal{L}\{f(t)\} = \hat{f}(s)$ exists for $s > \alpha$, then $\mathcal{L}\{e^{at}f(t)\}$ exists for $s > \alpha + a$ and

$$\mathcal{L}\{e^{at}f(t)\} = \hat{f}(s - a).$$  

Proof If $\mathcal{L}\{e^{at}f(t)\} = \int_0^\infty e^{-st}e^{at}f(t) dt = \int_0^\infty e^{-(s-a)t}f(t) dt = \hat{f}(s - a)$.

This is sometimes called the first shifting theorem, since multiplication of $f(t)$ by $e^{at}$ has the effect of shifting the transform of $\hat{f}(s)$ to the right by $a$ units to get $\hat{f}(s - a)$.

Example 3. Since $\mathcal{L}\{\sin bt\} = \frac{b}{s^2 + b^2}$, we have (5) that

$$\mathcal{L}\{e^{at}\sin bt\} = \frac{b}{(s-a)^2 + b^2}.$$ 

Example 4. Since $\mathcal{L}\{t\} = \frac{1}{s^2}$, we have $\mathcal{L}\{te^{at}\} = \frac{1}{(s-a)^2}$.

We now consider what happens to the object function when we differentiate the transform function. We have

$$\frac{d}{ds}\hat{f}(s) = \frac{d}{ds} \int_0^\infty e^{-st}f(t) dt.$$ 

Assuming it is possible to differentiate under the integral sign, we obtain

$$\hat{f}'(s) = \int_0^\infty -te^{-st}f(t) dt = \int_0^\infty e^{-st}\{-tf(t)\} dt = \mathcal{L}\{-tf(t)\}.$$ 

The following theorem provides the conditions for which the above results hold.

Theorem 5. If $f(t)$ is continuous and of exponential order $\alpha$, then $\hat{f}(s)$ has derivatives of all orders and for $s > \alpha$

$$\hat{f}'(s) = \mathcal{L}\{-tf(t)\}$$
$$\hat{f}''(s) = \mathcal{L}\{t^2f(t)\}$$
$$\vdots$$
$$\hat{f}^{(n)}(s) = \mathcal{L}\{(-t)^nf(t)\}$$ 

Example 5. Since $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$, we have

$$\mathcal{L}\{te^{at}\} = -\frac{d}{ds} \frac{1}{s-a} = \frac{1}{(s-a)^2}$$
$$\mathcal{L}\{t^2e^{at}\} = \frac{d^2}{ds^2} \frac{1}{s-a} = \frac{2}{(s-a)^3}$$
$$\vdots$$
$$\mathcal{L}\{t^n e^{at}\} = (-1)^n \frac{d^n}{ds^n} \frac{1}{s-a} = \frac{n!}{(s-a)^{n+1}}$$ 

(7)
where \( n \) is a positive integer. In particular we may put \( a = 0 \) to get

\[
\mathcal{L} \{ t^n \} = \frac{n!}{(s - a)^{n+1}}
\]  

(8)

If \( F(s) \) is a given function of \( s \), we can ask if is the transform of some function \( f(t) \). It turns out that there are certain conditions that must hold for \( F(s) \) to be the transform of some function. The following theorem gives us some information.

**Theorem 6.** If \( f \) is continuous and of exponential order \( \alpha \), then \( \lim_{s \to \infty} \hat{f}(s) = 0 \). Furthermore \( |s \hat{f}(s)| \) is bounded as \( s \to \infty \).

**Proof** Since \( f \) is of exponential order \( \alpha \), we have for \( s > \alpha \)

\[
|f(t)| < M e^{\alpha t} \quad \text{and} \quad |e^{-st} f(t)| < M e^{-(s-\alpha)t}.
\]

Therefore

\[
|\hat{f}(s)| = \left| \int_0^\infty e^{-st} f(t) \, dt \right| \leq \int_0^\infty e^{-st} |f(t)| \, dt \leq M \int_0^\infty e^{-(s-\alpha)t} \, dt = \frac{M}{s-\alpha}.
\]

so that \( \hat{f}(s) \to 0 \) as \( s \to \infty \). We also have

\[
|s \hat{f}(s)| \leq \frac{sM}{s-\alpha} \leq 2M
\]

if \( s \) is big enough; thus \( |s \hat{f}(s)| \) is bounded.

From this theorem it follows that the functions \( 1, \frac{s}{s - 1} \) cannot be transforms of functions of exponential order since they do not approach 0 as \( s \) approaches \( \infty \). Also \( 1/\sqrt{s} \) cannot be the transform of a function of exponential order since \( s/\sqrt{s} = \sqrt{s} \) is unbounded as \( s \to \infty \).

**Exercises 2.5**

1. Using \( \mathcal{L} \{ \sin bt \} = \frac{b}{s^2 + b^2} \), find \( \mathcal{L} \{ \cos bt \} \).
2. Using \( \mathcal{L} \{ \cos bt \} = \frac{s}{s^2 + b^2} \), find \( \mathcal{L} \{ e^{at} \cos bt \} \).
3. Find \( \mathcal{L} \{ t \sin 3t \} \) and \( \mathcal{L} \{ t^2 \cos 3t \} \).
4. Find \( \mathcal{L} \{ te^{at} \sin bt \} \).
5. Find \( \mathcal{L}^{-1} \left\{ \frac{2}{s(s - 3)^5} \right\} \).
6. Find \( \mathcal{L}^{-1} \left\{ \frac{4}{s^2 + 5} \right\} \) and \( \mathcal{L}^{-1} \left\{ \frac{4s}{s^2 + 5} \right\} \).
7. Find \( \mathcal{L}^{-1} \left\{ \frac{2s}{s^2 + 2s + 5} \right\} \).

2.6–Partial Fractions and Use of Table

Solving differential equations using Laplace transforms involves three steps (1) taking the Laplace transform of both sides of the equation, (2) solving for the transform of the solution, and (3) finding the inverse transform of the result in (2) to get the solution. The first two steps are done without any difficulty, but the third step can be onerous and often involves a partial fraction expansion and a considerable amount of algebra.

It is worthwhile getting familiar with the table on Page 57. The left hand half of the page is convenient for finding transforms, while the right hand half of the page is more convenient for finding inverse transforms. Lines 1-14 give the transforms and inverse transforms of specific functions, Lines 15-18 deal with discontinuous functions and will be studied in Section 2.9. Lines 19-20 give the transforms of \( e^{at} f(t) \) and \( t^n f(t) \), in terms of the transform of \( f(t) \). Lines 21-24 give the transform of integrals and
this page is blank for table
derivatives, and Line 25 gives the transform of a convolution which will be discussed in Section 2.8. We start with a couple of simple examples of finding transforms.

**Example 1.** Find \( \mathcal{L}\{e^t \cos 2t\} \).

Looking at Line 10 of the table, we see that \( a = 1 \) and \( b = 2 \), thus \( \mathcal{L}\{e^t \sin 2t\} = \frac{s - 1}{(s - 1)^2 + 4} \).

**Example 2.** Find \( \mathcal{L}\{e^{(t+2)}\} \).

Line 2 contains the transform of \( e^t \), but not of \( e^{(t+2)} \). However noting that \( e^{(t+2)} = e^t e^2 \), we find, using linearity and Line 2 that

\[
\mathcal{L}\{e^{(t+2)}\} = \mathcal{L}\{e^t e^2\} = e^2 \mathcal{L}\{e^t\} = \frac{e^2}{s - 1}.
\]

We now give some simple examples illustrating the use of the table to find inverse transforms.

**Example 3.** Find \( \mathcal{L}^{-1}\left\{\frac{s + 6}{(s + 2)^3}\right\} \).

A simple trick expresses this as a sum of terms that appear in the table. \( \mathcal{L}^{-1}\left\{\frac{s + 6}{(s + 2)^3}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s + 2)^2} + \frac{4}{(s + 2)^3}\right\} = \frac{te^{-2t}}{1} + 4 \frac{e^{-2t}}{2} \), using Line 8 of the table.

**Example 4.** \( \mathcal{L}^{-1}\left\{\frac{2}{(s + 4)^2} + \frac{3}{s^4}\right\} = 2 \mathcal{L}^{-1}\left\{\frac{1}{(s + 4)^2}\right\} + 3 \mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\} = \frac{2 t^2 e^{-4t}}{2!} + \frac{3 t^3}{3!} = t^2 e^{-4t} + \frac{t^3}{2} \).

The first step uses the linearity of the inverse transform. The next step uses Line 8 of the table for the first term and Line 2 for the second.

**Example 5.** Find \( \mathcal{L}^{-1}\left\{\frac{s}{s^2 + 2s + 5}\right\} \). Note that the denominator does not have real factors. Therefore we complete the square to be able to use Lines 9 or 10 in the table. We have

\[
\mathcal{L}^{-1}\left\{\frac{s}{s^2 + 2s + 5}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{(s + 1)^2 + 4}\right\} = \mathcal{L}^{-1}\left\{\frac{s + 1 - 1}{(s + 1)^2 + 4}\right\} = \mathcal{L}^{-1}\left\{\frac{s + 1}{(s + 1)^2 + 4}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(s + 1)^2 + 4}\right\} = e^{-t} \cos 2t - e^{-t} \sin 2t
\]

It is often necessary to find the inverse transform of a rational function \( p(s)/q(s) \) where \( p, q \) are polynomials and the degree of the numerator is less than the degree of the denominator. This is done by using partial fractions. The first step is to factor the denominator into real linear or irreducible quadratic factors.

\[
p(s) q(s) = c_0 (s - a_1)^{m_1} \cdots (s - a_k)^{m_k} (s^2 + b_1 s + c_1)^{n_1} \cdots (s^2 + b_l s + c_l)^{n_l}.
\]

The partial fraction expansion of Equation (1) consists of a sum of terms obtained as follows.

1. For each factor \( (s - a)^m \) in (1), we have the terms

\[
\frac{A_1}{s - a} + \frac{A_2}{(s - a)^2} + \cdots + \frac{A_m}{(s - a)^m},
\]
2. For each factor \((s^2 + bs + c)^n\) in (1), we have the terms

\[
\frac{B_1 + C_1 s}{s^2 + bs + c} + \frac{B_2 + C_2 s}{(s^2 + bs + c)^2} + \cdots + \frac{B_n + C_n s}{(s^2 + bs + c)^n}.
\]

The coefficients \(A_i, \ B_i, \ C_i\) may be obtained by multiplying both sides by \(q(s)\) and equating coefficients of like powers of \(s\).

**Example 6.** Find the partial fraction expansion of \(\frac{s^2 + 1}{s^2(s^2 + 5s + 6)}\).

First we factor the denominator

\[
\frac{s^2 + 1}{s^2(s^2 + 5s + 6)} = \frac{s^2 + 1}{s^2(s + 3)(s + 2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s + 3} + \frac{D}{s + 2}.
\]

Clearing fractions

\[
s^2 + 1 = As(s + 3)(s + 2) + B(s + 3)(s + 2) + Cs^2(s + 2) + Ds^2(s + 3) \tag{\ast}
\]

\[
= s^3(A + C + D) + s^2(5A + B + 2C + D) + s(6A + 5B) + 6B
\]

Equating coefficients of like powers

\[
A + C + D = 0 \\
5A + B + 2C + D = 1 \\
6A + 5B = 0
\]

Solving this system yields \(A = \frac{5}{36}, \ B = \frac{1}{6}, \ C = -\frac{10}{9}, \ D = \frac{5}{4}\). Thus

\[
\frac{s^2 + 1}{s^2(s^2 + 5s + 6)} = -\frac{5}{36} + \frac{1}{6} \frac{1}{s^2} - \frac{10}{9} \frac{1}{s + 3} + \frac{5}{4} \frac{1}{s + 2}.
\]

If the inverse transform is desired, it can now be easily obtained from the table.

Perhaps a better way of obtaining the coefficients is to substitute strategically chosen values of \(s\) into (\ast). Putting \(s = 0\) yields \(B = \frac{1}{6}\), \(s = -3\) yields \(C = -\frac{10}{9}\), and \(s = -2\) produces \(D = \frac{5}{4}\). This leaves only \(A\) to determine. Substituting some small value, say \(s = 1\), we find, after a little arithmetic, \(A = -\frac{5}{36}\).

**Example 7.** Find the inverse transform of \(\frac{s + 2}{(s - 2)(s^2 + 2)^2}\).

The proper form for the partial fraction expansion is

\[
\frac{s + 2}{(s - 2)(s^2 + 2)^2} = \frac{A}{s - 2} + \frac{Bs + C}{s^2 + 2} + \frac{Cs + D}{(s^2 + 2)^2}.
\]

Crossmultiplying, we get

\[
s + 2 = A(s^2 + 2)^2 + (Bs + C)(s - 2)(s^2 + 2) + (Ds + E)(s - 2) \tag{\ast}
\]

\[
= s^4(A + B) + s^3(-2B + C) + s^2(4B + 2B - 2C + D) + s(-4B + 2C - 2D + E) + (4A - 4C - 2E).
\]
Thus \( A + B = 0, \ -2B + C = 0, \ 4B + 2B - 2C + D = 1, \ -4B + 2C - 2D + E = 0, \ 4A - 4C - 2E = 1. \) Note that we can obtain \( A \) by setting \( s = 2 \) in (*), yielding \( A = 1/9. \) The other coefficients are now easily obtained. The results are \( B = -1/9, \ C = -2/9, \ D = -2/3, \ E = -1/3, \) and the expansion is

\[
\frac{s + 2}{(s - 2)(s^2 + 2)^2} = \frac{1}{9} \frac{1}{s - 2} - \frac{1}{9} \frac{s + 2}{s^2 + 2} - \frac{1}{3} \frac{2s + 1}{(s^2 + 2)^2}
\]

For the inverse transform we use Line 3 of the table for the first term, Lines 3, 4 for the second term and Lines 13, 14 for the last term. The result is

\[
\mathcal{L}^{-1}\left\{\frac{s + 2}{(s - 2)(s^2 + 2)^2}\right\} = \frac{1}{9} e^{-2t} - \frac{1}{9} \cos \sqrt{2}t - \frac{1}{9\sqrt{2}} \sin \sqrt{2}t - \frac{1}{\sqrt{2}} t \sin \sqrt{2}t + \frac{1}{12\sqrt{2}} (\sin \sqrt{2}t - \sqrt{2} \cos \sqrt{2}t)
\]

**Example 8.** Find the inverse transform of \( \frac{5s + 3}{(s - 1)(s^2 + 2s + 5)}. \)

\[
\frac{5s + 3}{(s - 1)(s^2 + 2s + 5)} = A \frac{1}{s - 1} + \frac{Bs + c}{s^2 + 2s + 5}
\]

Clearing fractions, we write

\[
5s + 3 = A(s^2 + 2s + 5) + (Bs + c)(s - 1)
\]

\[
The result is \( A + B = 0, \ 2A - B + C = 5, \ 5A - C = 3. \) Putting \( s = 1 \) in (*) we find \( A = 1, \) we then find \( B = -1, \ C = 2. \) The partial fraction expansion is

\[
\frac{5s + 3}{(s - 1)(s^2 + 2s + 5)} = \frac{1}{s - 1} - \frac{s - 2}{s^2 + 2s + 5} = \frac{1}{s - 1} - \frac{s - 2}{(s + 1)^2 + 4} = \frac{1}{s - 1} - \frac{s + 1 - 3}{(s + 1)^2 + 4} = \frac{1}{s - 1} - \frac{s + 1}{(s + 1)^2 + 4} + \frac{3}{(s + 1)^2 + 4}
\]

The inverse transform is

\[
\mathcal{L}^{-1}\left\{\frac{5s + 3}{(s - 1)(s^2 + 2s + 5)}\right\} = e^t - e^{-t} \cos 2t + \frac{3}{2} e^{-t} \sin 2t.
\]

**Exercises 2.6**

Find transforms of

1. \( (t - 1)^2. \)

2. \( \sin(2t - 1). \)

3. \( \frac{t}{e^{2t}}. \)

Find the inverse transforms of

4. \( \frac{1}{s^2 - 3s + 2}. \)

5. \( \frac{1}{s(s + 1)^2}. \)

6. \( \frac{1}{(s + 2)(s^2 + 9)}. \)

7. \( \frac{3s}{(2s - 4)^2}. \)

8. \( \frac{s}{s^2 - 4s + 6}. \)

9. \( \frac{3s}{2s^2 - 2s + 1}. \)

10. \( \frac{1}{s(s^2 - 4s + 13)}. \)

11. \( \frac{a^2}{s(s^2 + a^2)}. \)

12. \( \frac{s}{(s + 2)^2(s^2 + 9)}. \)
13. Find the correct form (do not evaluate the coefficients) for the partial fraction expansion of
\[
\frac{3s^2 - 5s + 2}{s^2(s - 3)^3(s^2 - 2s + 3)^3}.
\]

2.7–Solution of Differential Equations

As we have seen, Laplace transforms may be used to solve initial value problems for linear differential equations with constant coefficients. The second order case is
\[
\Delta E: \quad ay'' + by' + cy = f(t)
\]
\[
IC: \quad y(0) = \alpha, \ y(0) = \beta
\]
where \( f(t) \) is of exponential order. The procedure consists of three steps

1. Take the Laplace transform of both sides of the equation.
2. Solve for \( \hat{y} \), the Laplace transform of the solution.
3. Find the inverse transform of \( \hat{y} \) to find the solution.

We illustrate the procedure with several examples.

Example 1. Solve
\[
DE: \quad \ddot{y} + 5\dot{y} + 6y = e^{7t}
\]
\[
IC: \quad y(0) = 0, \ \dot{y}(0) = 2
\]
Taking transforms, we obtain
\[
\hat{y} = \frac{2}{s^2 + 5s + 6} + \frac{1}{(s - 7)(s^2 + 5s + 6)}
\]
\[
= \frac{2}{(s + 3)(s + 2)} + \frac{1}{(s - 7)(s + 3)(s + 2)}
\]
The inverse transform of the first term appears directly in the table
\[
\mathcal{L}^{-1}\left\{\frac{2}{(s + 3)(s + 2)}\right\} = -2(e^{-3t} - 2^{-2t})
\]
We split the second term into its partial fraction expansion
\[
\frac{1}{(s - 7)(s + 3)(s + 2)} = \frac{A}{s - 7} + \frac{B}{s + 3} + \frac{C}{s + 2}
\]
Clearing fractions, we find
\[
1 = (s + 3)((s + 2)A + (s - 7)(s + 2)B + (s - 7)(s + 3)C)
\]
Setting \( s = 7 \), we obtain \( A = 1/90 \); setting \( s = -3 \), we find \( B = 1/10 \); and setting \( s = -2 \), \( C = -1/9 \). Therefore
\[
\frac{1}{(s - 7)(s + 3)(s + 2)} = \frac{1}{90} \frac{1}{s - 7} + \frac{1}{10} \frac{1}{s + 3} - \frac{1}{9} \frac{1}{s + 2}
\]
and
\[
\mathcal{L}^{-1}\left\{\frac{1}{(s - 7)(s + 3)(s + 2)}\right\} = \frac{1}{90} e^{7t} + \frac{1}{10} e^{-3t} - \frac{1}{9} e^{-2t}
\]
The solution \( y(t) \) is the sum of (i) and (ii)
\[
y(t) = \frac{1}{90} e^{7t} + \frac{1}{10} e^{-3t} - \frac{1}{9} e^{-2t}
\]
Example 2. Solve DE: $\ddot{y} + y = 3$
IC: $y(0) = 1, \dot{y}(0) = 2$

Taking transforms we have $(s^2\hat{y} - s - 2) + \hat{y} = 3/s$. Solving for $\hat{y}$ yields

$$\hat{y} = \frac{3}{s(s^2 + 1)} + \frac{s + 2}{s^2 + 1}$$

Splitting the first term into partial fractions produces

$$\frac{3}{s(s^2 + 1)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 1},$$

or

$$3 = A(s^2 + 1) + (Bs + C)s.$$

Equating coefficients of $s$ on both sides, we find $A = 3, B = -3, C = 0$. Therefore

$$\hat{y} = \frac{3}{s} - \frac{2s}{s^2 + 1} + \frac{2}{s^2 + 1}.$$

From the table we find

$$y(t) = 2 - 2\cos t + 2\sin t.$$

Example 3. Solve DE: $\ddot{y} + y = e^t \sin 2t$
IC: $y(0) = 1, \dot{y}(0) = 3$.

$$(s^2\hat{y} - s - 3) + \hat{y} = \frac{2}{(s - 1)^2 + 4}.$$  

$$\hat{y} = \frac{s + 3}{s^2 + 1} + \frac{2}{(s^2 + 1)[(s - 1)^2 + 4]}.$$  

We must find the partial fraction expansion for the second term.

$$\frac{2}{(s^2 + 1)[(s - 1)^2 + 4]} = \frac{As + B}{s^2 + 1} + \frac{Cs + D}{(s - 1)^2 + 4},$$

or

$$2 = (As + B)[(s - 1)^2 + 4] + (Cs + D)(s^2 + 1).$$

Equating coefficients of $s$ on both sides, we find $A = 1/5, B = 2/5, C = -1/5, D = 0$. Therefore

$$\hat{y} = \frac{6}{5} \frac{s}{s^2 + 1} + \frac{17}{5} \frac{1}{s^2 + 1} - \frac{1}{5} \frac{(s - 1) + 1}{(s - 1)^2 + 4}.$$  

From the table we obtain the solution

$$y(t) = \frac{6}{5} \cos t + \frac{17}{5} \sin t - \frac{1}{10} e^t \sin 2t - \frac{1}{5} e^t \cos 2t.$$

Exercises 2.7

1. $\ddot{y} + 3\dot{y} + 2y = t, \ y(0) = 1, \dot{y}(0) = 1.$
2. $\ddot{y} + 4\dot{y} + 4y = e^{-2t}, \ y(0) = \dot{y}(0) = 0.$
3. $2y'' - 2y' + y = 0, \ y(0) = 0, \ y'(0) = 1.$
4. $y'' + 5y' + 6y = e^{-2t}, \ y(0) = 1, \ y'(0) = 0.$
5. $\ddot{y} + 2\dot{y} + 2y = \sin t, \ y(0) = 0, \dot{y}(0) = 1.$
6. $\ddot{y} - y = e^{at}, \ y(0) = 1, \ y'(0) = 1, \ a \neq 1$ and $a = 1.$
7. $y''' + y' = 0, \ y(0) = 1, \ y'(0) = y''(0) = 0.$
8. $y'' - y = 0, \ y(0) = y'(0) = y''(0) = 0, \ y'''(0) = 1.$
9. $\ddot{y} - 4\dot{y} + 5y = e^{2t} \cos t, \ y(0) = \dot{y}(0) = 1.$
10. Find the general solution of $\ddot{y} + \dot{y} = t.$
11. Solve the harmonic oscillator equation \( m\ddot{x} + kx = F(t), \ x(0) = x_0, \ \dot{x}(0) = v_0 \) for the cases
   a. \( F(t) \equiv 0 \)
   b. \( F(t) \equiv 1 \)
   c. \( F(t) = F_0 \cos \omega t, \ \omega \neq \sqrt{k/m} \)
   d. \( F(t) = F_0 \sin \omega t, \ \omega = \sqrt{k/m} \).

2.8–Product of Transforms; Convolutions

When Laplace transforms are used to solve differential equations it is often necessary to find the inverse transform of the product of two transform functions \( \hat{f}(s)\hat{g}(s) \). If \( \hat{f}(s) \) and \( \hat{g}(s) \) are both rational functions, the inverse transform may be found by the method of partial fractions. However, it is useful to know whether or not the product of transform functions is itself a transform function and if so, what it the inverse transform.

Let \( f(t) \) and \( g(t) \) be continuous functions that possess the transforms \( \hat{f}(s) \) and \( \hat{g}(s) \), respectively. Let \( h(t) \) be a continuous function such that

\[
\mathcal{L}\{h(t)\} = \hat{f}(s)\hat{g}(s) = \int_0^\infty e^{-su} f(u) \, du \int_0^\infty e^{-sx} g(x) \, dx. \tag{1}
\]

if such a function exists. We assume that the right hand side can be written as an iterated integral

\[
\mathcal{L}\{h(t)\} = \int_0^\infty \left[ \int_0^\infty e^{-s(u+x)} f(u)g(x) \, du \right] dx.
\]

Making the transformation \( u + x = t, \ x = x \), we obtain

\[
\mathcal{L}\{h(t)\} = \int_0^\infty \left[ \int_x^\infty e^{-st} f(t-x)g(x) \, dt \right] dx.
\]

Assuming it is possible to change the order of integration, we find (see Figure 1)

\[
\mathcal{L}\{h(t)\} = \int_0^\infty e^{-st} \left[ \int_0^t f(t-x)g(x) \, dx \right] dt.
\]

From the uniqueness theorem we obtain

\[
h(t) = \int_0^t f(t-x)g(x) \, dx.
\]

Figure 1
The expression on the right is called the convolution of $f(t)$ and $g(t)$ and is symbolized by $f(t) * g(t)$, that is,

$$f(t) * g(t) = \int_{0}^{t} f(t - x)g(x) \, dx.$$ 

Thus we have that the multiplication of transform functions $\tilde{f}(s)\tilde{g}(s)$ corresponds to the convolution of the object functions $f(t) * g(t)$.

The above derivation was purely formal. However, in more advanced treatments, the following theorem can be proven.

**Theorem 1.** If $f$ and $g$ are continuous and of exponential order, then $f(t) * g(t) = \int_{0}^{t} f(t - x)g(x) \, dx$ exists and

$$\mathcal{L} \{f(t) * g(t)\} = \mathcal{L} \{f(t)\} \mathcal{L} \{g(t)\} = \tilde{f}(s)\tilde{g}(s).$$

or in terms of inverse transforms

$$\mathcal{L}^{-1} \{\tilde{f}(s)\tilde{g}(s)\} = f(t) * g(t) = \int_{0}^{t} f(t - x)g(x) \, dx.$$ 

**Example 1.** Verify Equation (2) for $f(t) = \sin t$, $g(t) = 1$.

According to Equation (2) we have

$$\mathcal{L} \{\sin t \ast t\} = \mathcal{L} \{\sin t\} \cdot \mathcal{L} \{t\} = \frac{1}{s^2 + 1} \cdot \frac{1}{s}.$$ 

Now $(\sin t) \ast t = \int_{0}^{t} \sin(t - x) \cdot 1 \, dx = 1 - \cos t$. Thus, we also have

$$\mathcal{L} \{\sin t \ast t\} = \mathcal{L} \{1 - \cos t\} = \frac{1}{s} - \frac{s}{s^2 + 1} = \frac{1}{s(s^2 + 1)}.$$ 

The inverse form of the convolution given in Equation (3) is very helpful in finding inverse transformations. The following two examples show that it may be simpler than partial fractions.

**Example 2.** Find $\mathcal{L}^{-1} \left\{ \frac{1}{s(s^2 + a^2)} \right\}$. We have

$$\mathcal{L}^{-1} \left\{ \frac{1}{s(s^2 + a^2)} \right\} = \mathcal{L}^{-1} \left\{ \frac{1}{s} \cdot \frac{1}{(s^2 + a^2)} \right\} = 1 \ast \left( \frac{1}{a} \sin at \right) = \int_{0}^{t} (1) \cdot \left( \frac{1}{a} \sin ax \right) \, dx = \frac{1}{a^2} (1 - \cos at).$$

**Example 3.** Find $\mathcal{L}^{-1} \left\{ \frac{1}{(s^2 + a^2)^2} \right\}$.

We have

$$\mathcal{L}^{-1} \left\{ \frac{1}{(s^2 + a^2)^2} \right\} = \mathcal{L}^{-1} \left\{ \frac{1}{(s^2 + a^2)} \cdot \frac{1}{(s^2 + a^2)} \right\} = \frac{1}{a \sin at} \ast \frac{1}{a \sin at} = \int_{0}^{t} \frac{1}{a \sin a(t - x)} \cdot \frac{1}{a \sin ax} \, dx = \frac{1}{2a^3} (\sin at - at \cos at).$$
The notation \( f \ast g \) for convolution suggests that convolution can be thought of as a new kind of multiplication of functions. In fact, convolutions has some properties similar to ordinary multiplication:

- \( f \ast g = g \ast f \), commutative law
- \( (f \ast g) \ast h = f \ast (g \ast h) \), associative law
- \( (f \ast (g + h)) = f \ast g + f \ast h \), distributive law
- \( (cf) \ast g = c(f \ast g) = c(g \ast f) \), \( c \)-constant

where \( f, g, h \) are assumed to possess transforms. These properties are easily proven using Equation 2. In particular the commutative law above states that \( f \ast g \) can be written in either of the equivalent forms

\[
f \ast g = \int_0^t f(t-x)g(x) \, dx = \int_0^t g(t-x)f(x) \, dx
\]

Convolution is really necessary when we deal with differential equations with arbitrary forcing terms.

**Example 4.**

DE: \( y'' + y = f(t) \)

IC: \( y(0) = y'(0) = 0 \).

where \( f(t) \) is assumed to have a Laplace transform. Taking transforms we have

\[(s^2 + 1)\hat{y}(s) = \hat{f}(s), \text{ or, } \hat{y}(s) = \frac{1}{s^2 + 1} \hat{f}(s).
\]

Therefore

\[y = \sin t \ast f(t) = \int_0^t \sin(t-x)f(x) \, dx,
\]

or, since convolution is commutative, we have the alternative form of the solution

\[y = \sin t \ast f(t) = \int_0^t f(t-x) \sin(x) \, dx.
\]

Finally we illustrate how to solve certain special types of integral equations where the unknown function is under the integral sign.

**Example 5.** Solve \( y(t) = t^2 + \int_0^t y(\tau) \sin(t-\tau) \, d\tau \).

Since the term with the integral is a convolution we may rewrite this equation as

\[y(t) = t^2 + y(t) \ast \sin t.
\]

Assuming \( y \) has a transform, we take the transform of both sides to get

\[\hat{y}(s) = \frac{2}{s^3} + \hat{y}(s) \frac{1}{s^2 + 1}, \text{ or, } \hat{y}(s) = \frac{2}{s^3} + \frac{2}{s^5},
\]

therefore

\[y(t) = t^2 + \frac{1}{12}t^4.
\]

To see that this is the solution, we substitute into the integral equation and find that, indeed, it does satisfy the equation.

**Exercises 2.8**

1. Using convolutions find:
   
   a. \( L^{-1}\left\{ \frac{1}{s(s-2)} \right\} \)
   b. \( L^{-1}\left\{ \frac{1}{s^2(s-2)} \right\} \)
   c. \( L^{-1}\left\{ \frac{s^2}{(s^2 + 4)^2} \right\} \)
   d. \( L^{-1}\left\{ \frac{1}{s^2(s^2 + 2)} \right\} \)
2. Solve \( DE: \ y'' - y = f(t) \)
IC: \( y(0) = y'(0) = 0 \)

3. Solve the following integral equations
   a. \( f(t) = 1 + \int_0^t f(\tau) \sin(t - \tau) \, d\tau \)
   b. \( f(t) = \sin t + \int_0^t f(\tau) \cos(t - \tau) \, d\tau \)

### 2.9–Discontinuous Forcing Functions

In the preceding sections we assumed, for simplicity, that the functions considered were continuous in \( 0 \leq t < \infty \). There is no particular difficulty in taking Laplace transforms of functions with finite or jump discontinuities, and the transform method is very efficient in solving linear differential equations with a discontinuous forcing term. Such discontinuous forcing terms appear often in practical applications. The simple matter of throwing a switch in an electrical circuit causes the voltage to jump, practically instantaneously, from one value to another. We shall generalize our treatment of transforms to piecewise continuous functions defined below.

**Definition 1.** \( f(t) \) is said to be piecewise continuous on every finite interval, if, for every \( A \), \( f(t) \) is continuous on \( 0 \leq t \leq A \) except for a finite number of points \( t_i \), at which \( f(t) \) possesses a right- and left-hand limit. The difference between the right-hand and left-hand limits, \( f(t_i^+) - f(t_i^-) \), is called the jump of \( f(t) \) at \( t = t_i \).

A graph of a typical piecewise continuous function is shown in Figure 1. A very simple and very useful discontinuous function is the **Heaviside function** or **unit step function** shown in Figure 2 and defined by

\[
H(t - t_0) = \begin{cases} 
0, & t < t_0 \\
1, & t > t_0 
\end{cases}
\]  

(1)

This function is continuous except at \( t = t_0 \) where it has a jump of 1. It is easy to compute the Laplace transform of \( H(t - t_0) \)

\[
\mathcal{L} \{ H(t - t_0) \} = \int_0^\infty H(t - t_0) e^{-st} \, dt = \int_{t_0}^\infty e^{-st} \, dt = \frac{e^{-st_0}}{s}.
\]

We can use the Heaviside function to express discontinuous functions in a convenient form. Consider the function defined by different analytic expressions in different intervals, as shown in Figure 3, i.e.,

\[
F(t) = \begin{cases} 
f(t), & t < t_0 \\
g(t), & t > t_0 
\end{cases}
\]
This can be expressed in terms of the Heaviside function as follows

\[ F(t) = f(t) + (g(t) - f(t))H(t - t_0). \]  

Equation (2) can be thought of as starting with \( f(t) \) and then, at \( t = t_0 \), switching on \( g(t) \) and switching off \( f(t) \), or 'jumping' to \( g(t) \) from \( f(t) \). This can be extended to any number of jumps. For example

\[ G(t) = \begin{cases} f(t), & t < t_0 \\ g(t), & t_0 < t < t_1 \\ h(t), & t > t_1 \end{cases} \]  

can be expressed as

\[ G(t) = f(t) + (g(t) - f(t))H(t - t_0) + (h(t) - g(t))H(t - t_1). \]  

**Example 1.** Write the function in Figure 4 in terms of Heaviside functions.

First we write the function in the usual way

\[ f(t) = \begin{cases} t, & 0 \leq t \leq 1 \\ -t + 2, & 1 \leq t \leq 2 \\ 0, & t \geq 2 \end{cases} \]

Now, using the method illustrated in Equation (4), we have

\[ f(t) = t + (-t + 2 - t)H(t - 1) + (0 - (-t + 2))H(t - 2) = t + (2 - 2t)H(t - 1) + (t - 2)H(t - 2). \]
To find the Laplace transform of a function such as those defined in equations (2) or (4), we need to find the Laplace transform of a function of the form \( f(t)H(t - t_0) \). The answer is given in the following theorem.

**Theorem 1.** If \( \mathcal{L} \{ f(t) \} \) exists then \( \mathcal{L} \{ f(t)H(t - t_0) \} \) exists and is given by

\[
\mathcal{L} \{ f(t)H(t - t_0) \} = e^{-t_0s} \mathcal{L} \{ f(t) \}. \tag{5}
\]

**Proof** \( \mathcal{L} \{ f(t)H(t - t_0) \} = \int_0^\infty e^{-st} f(t)H(t - t_0) \, dt = \int_{t_0}^\infty e^{-st} f(t) \, dt \). Letting \( t = t_0 + \tau \), we find

\[
\int_0^\infty e^{-s(\tau + t_0)} f(\tau + t_0) \, d\tau = e^{-st_0} \int_0^\infty e^{-st} f(\tau + t_0) \, d\tau = e^{-st_0} \mathcal{L} \{ f(t + t_0) \}.
\]

**Example 2.** Find the Laplace transform of

\[
f(t) = \begin{cases} t, & 0 < t < 2 \\ 1, & t > 2 \end{cases}
\]

In terms of the Heaviside function, we have

\[ f(t) = t + (1-t)H(t-2) \]

and according to Equation (5)

\[
\mathcal{L} \{ f(t) \} = \frac{1}{s} + e^{-2s} \mathcal{L} \{ 1 - (t+2) \} = \frac{1}{s} + e^{-2s} \mathcal{L} \{ -1 - t \} = \frac{1}{s} - e^{-2s} \left( \frac{1}{s} + \frac{1}{s^2} \right).
\]

In taking Laplace transforms, functions need to be defined for \( 0 \leq t < \infty \). However, for the purposes of this next theorem, we assume that \( f(t) \) is defined for all \( t \), but \( f(t) = 0 \) for \( t < 0 \). Consider the function given by

\[
f_s(t) = f(t - t_0)H(t - t_0) = \begin{cases} f(t - t_0), & t > t_0 \\ 0, & t < t_0 \end{cases}
\]

The function \( f_s(t) \) is the function \( f(t) \) shifted to the right by an amount \( t_0 \) as shown in Figure 5.

The following theorem tells us how to find the transform of the shifted function \( f_s(t) \).

**Theorem 2.** If \( \hat{f}(s) = \mathcal{L} \{ f(t) \} \) exists, then \( \mathcal{L} \{ f_s(t) \} \) exists and is given by

\[
\mathcal{L} \{ f_s(t) \} = \mathcal{L} \{ f(t - t_0)H(t - t_0) \} = e^{-st_0} \hat{f}(s). \tag{6}
\]

**Proof**

\[
\mathcal{L} \{ f(t - t_0)H(t - t_0) \} = \int_0^\infty e^{-st} f(t - t_0)H(t - t_0) \, dt
\]

\[
= \int_{t_0}^\infty e^{-st} f(t - t_0) \, dt.
\]
Make the substitution \( t - t_0 = \tau \) to find

\[
\mathcal{L} \{ f(t - t_0)H(t - t_0) \} = \int_0^\infty e^{-s(\tau + t_0)} f(\tau) \, d\tau \\
= e^{-st_0} \int_0^\infty e^{-s\tau} f(\tau) \, d\tau \\
= e^{-st_0} \hat{f}(s).
\]

The above theorem is be particularly useful in finding inverse transforms of functions involving \( e^{-as} \).

**Example 3.** Find the inverse transform of \( \frac{e^{-2s}}{s^2} \). Let \( \hat{f}(s) = 1/s^4 \), then from Line 2 of the table, \( f(t) = \frac{t^3}{6} \). Thus, according to Equation (6), we have

\[
\mathcal{L}^{-1} \left\{ \frac{e^{-2s}}{s^2} \right\} = \frac{(t-2)^3}{6} H(t-2).
\]

We now consider a differential equation with a discontinuous forcing term

**DE:** \( ay'' + by' + cy = f(t) \)

**IC:** \( y(0) = \alpha, \ y'(0) = \beta \)

For simplicity assume that \( f(t) \) is continuous except for a jump discontinuity at \( t = t_0 \). First of all we must consider what we mean by a solution, since it is clear from the DE that \( y''(t) \) will not exist at \( t_0 \). However, a unique solution exists in the interval \( 0 \leq t \leq t_0 \); therefore the left hand limits \( y(t_0^-) \), and \( y'(t_0^-) \) exist. Using these as new initial values at \( t = t_0 \), a unique solution can be found for \( t \geq t_0 \). We defined the function obtained by this piecing-together process to be the solution. Note that \( y \) and \( y' \) are continuous, even at \( t_0 \), whereas \( y'' \) has a jump discontinuity at \( t_0 \).

**Example 4.**

**DE:** \( y'' + 5y' + 6y = f(t) \)

**IC:** \( y(0) = y'(0) = 0 \)

where \( f(t) = 1, \ t < 2, \ f(t) = -1, \ t > 2 \). Solving \( y'' + 5y' + 6y = 1 \), \( y(0) = y'(0) = 0 \) for \( 0 \leq t \leq 2 \) we find

\[
y = \frac{1}{6} - \frac{1}{2} e^{-2t} + \frac{1}{3} e^{-3t}, \ 0 \leq t \leq 2.
\]
We now solve \( y'' + 5y' + 6y = -1, \ y(2) = \frac{1}{6} - \frac{1}{2}e^{-4} + \frac{1}{3}e^{-6}, \ y'(2) = e^{-4} - e^{-6} \) for \( t \geq 2 \). The general solution of the DE is \( y = c_1e^{-2t} + c_2e^{-3t} - \frac{1}{6} \). Satisfying the initial conditions at \( t = 2 \) we find

\[
y(t) = \frac{1}{2}e^{-2t}(e^t - 1) + \frac{1}{3}e^{-2t}(1 - e^t), \quad t \geq 2
\]

The function defined by (i) and (ii) is the solution. Note that \( y \) and \( y' \) are continuous at \( t = 2 \), whereas \( y''(2^+) - y''(2^-) = -2 \).

The Laplace transform is an efficient tool for handling differential equations with a discontinuous forcing term \( f(t) \), provided that \( f(t) \) possesses a transform. It is not difficult to show that if \( f(t) \) is piecewise continuous and of exponential order, then it has a Laplace transform. Also the formulas for the transform of derivatives still hold if \( y, y', \ldots y^{(n-1)} \) are continuous and of exponential order but \( y^{(n)} \) is piecewise continuous. Using these facts let us redo Example 4 using transforms.

**Example 5.** We write the right hand side in terms of Heaviside functions: \( f(t) = 1 + (-1 - 1)H(t - 2) = 1 - 2H(t - 2) \). Our problem can now be written as

\[
y'' + 5y' + 6y = 1 - 2H(t - 2), \ y(0) = y'(0) = 0.
\]

Taking transforms, we obtain

\[
(s^2 + 5s + 6)\hat{y} = \frac{1}{s} - \frac{2e^{-2s}}{s}
\]

\[
\hat{y} = \frac{1}{s(s^2 + 5s + 6)} - \frac{2e^{-2s}}{s(s^2 + 5s + 6)}.
\]

We find

\[
\mathcal{L}^{-1}\left(\frac{1}{s(s^2 + 5s + 6)}\right) = \mathcal{L}^{-1}\left(\frac{1}{s(s + 3)(s + 2)}\right) = \frac{1}{6} - \frac{1}{2}e^{-2t} + \frac{1}{3}e^{-3t} \equiv g(t).
\]

Therefore, using Equation (6)

\[
\mathcal{L}^{-1}\left(\frac{e^{-2s}}{s(s^2 + 5s + 6)}\right) = g(t - 2)H(t - 2),
\]

and the solution is

\[
y = \frac{1}{6} - \frac{1}{2}e^{-3t} + \frac{1}{3}e^{-3t} + \left(\frac{1}{6} - \frac{1}{2}e^{-2(t-2)} + \frac{1}{3}e^{-3(t-2)}\right)H(t - 2),
\]

which agrees with the solution obtained in Example 4.

**Exercises 2.9**

For problems 1-4, write in terms of Heaviside functions, sketch, and find the Laplace transform

1. \( f(t) = \begin{cases} t, & 0 \leq t < 5 \\ 1, & t > 5 \end{cases} \)

2. \( f(t) = \begin{cases} 0, & 0 \leq t < \pi \\ \cos t, & t > \pi \end{cases} \)

3. \( f(t) = \begin{cases} 0, & 0 \leq t < 1 \\ 2, & 1 < t < 2 \\ 0, & t > 2 \end{cases} \)

4. \( f(t) = \begin{cases} -1, & 0 \leq t < 3 \\ 1, & 3 < t < 5 \\ t - 2, & t > 5 \end{cases} \)
5. Write in the form (3) and sketch.
   a. \( f(t) = t + (1 - t)H(t - 2) + t^2H(t - 3) \)
   b. \( f(t) = H(t) + H(t - 1) - H(t - 2) \).

6. Find \( \mathcal{L}\{t^2H(t - 1)\} \).

7. Find \( \mathcal{L}\{e^tH(t - 2)\} \).

8. Find \( \mathcal{L}\{\sin tH(t - 2)\} \).

9. Find \( \mathcal{L}^{-1}\left\{\frac{1 - e^{-3t}}{s^2}\right\} \).

10. Find \( \mathcal{L}^{-1}\left\{\frac{3e^{-\pi t}}{s^2 + 4}\right\} \).

11. Find \( \mathcal{L}^{-1}\left\{\frac{e^{-as}}{s^2 - 1}\right\} \).

12. Find the transforms of the functions in the following diagrams

13. Solve \( y'' - 3y' + 2y = f(t), \ y(0) = y'(0) = 0 \) where
    \[
    f(t) = \begin{cases} 
    0, & 0 \leq t < 1 \\
    2, & 1 < t < 2 \\
    0, & t \geq 2 
    \end{cases}
    \]

14. Solve \( y'' + y = f(t), \ y(0) = y'(0) = 0 \) where
    \[
    f(t) = \begin{cases} 
    1, & 0 \leq t < 1 \\
    0, & t > 1 
    \end{cases}
    \]

2.10–The Weighting Function and Impulse Functions

Consider a physical device or system that is governed by an \( n \)th order linear differential equation with constant coefficients

\[
\text{DE: } a_0y^{(n)}(t) + a_1y^{(n-1)}(t) + \cdots + a_ny(t) = f(t), \ t \geq 0
\]

\[
\text{IC: } y(0) = b_0, \ y'(0) = b_1, \ldots, y^{(n-1)}(0) = b_{n-1}.
\]

where the \( a_i \) and \( b_i \) are given constants, \( a_0 \neq 0 \), and \( f(t) \) is a given continuous function. This differential equation could be the model of an electric circuit, a mechanical mass–spring system, or an economic system. The function \( f(t) \) can be thought of as the input or excitation to the system and \( y(t) \) as the output or response of the system. We indicate this by the simple block diagram shown in Figure 1. In a mechanical system the input could be an external force, and the output a displacement; in an electrical system the input could be a voltage, and the output the current in some branch of the circuit.

![Figure 1](image-url)
The output of the system for a given initial conditions can conveniently be obtained by Laplace transforms. For simplicity we consider here the second order case

\[
\text{DE: } ay'' + by' + cy = f(t) \\
\text{IC: } y(0) = \alpha, \quad y'(0) = \beta
\]

where the system parameters, \( a \neq 0, \ b, \ c, \) are constants, and the input \( f(t) \) is a continuous function of exponential order. Taking transforms of the DE, we find

\[
a \{ s^2 \hat{y}(s) - sa - \beta \} + b \{ s\hat{y}(s) - \alpha \} + c\hat{y}(s) = \hat{f}(s),
\]

or,

\[
(as^2 + bs + c)\hat{y}(s) = \hat{f}(s) + (as + b)\alpha + a\beta. \tag{3}
\]

Letting

\[
p(s) = as^2 + bs + c, \quad \text{and} \quad q(s) = (as + b)\alpha + a\beta
\]

we can rewrite (3) as

\[
p(s)\hat{y}(s) = \hat{f}(s) + q(s). \tag{5}
\]

We note that \( p(s) \) is the characteristic polynomial of the differential equation and \( q(s) \) is a polynomial of degree 1 which depends on \( \alpha \) and \( \beta \), that is, on the initial conditions. If the system is in the zero-initial state \( (\alpha = \beta = 0) \), then \( q(s) \equiv 0 \).

From (3) we find the transform \( \hat{y}(s) \) is

\[
\hat{y}(s) = \frac{\hat{f}(s)}{p(s)} + \frac{q(s)}{p(s)} \tag{6}
\]

It is customary to define the transfer function \( Y(s) \) of the system by

\[
Y(s) = \frac{1}{p(s)} = \frac{1}{as^2 + bs + c}.
\]

Thus (6) becomes

\[
\hat{y}(s) = Y(s)\hat{f}(s) + Y(s)q(s).
\]

If we define

\[
y_0(t) = \mathcal{L}^{-1} \left\{ Y(s)\hat{f}(s) \right\}, \tag{7}
\]

\[
y_1(t) = \mathcal{L}^{-1} \left\{ Y(s)q(s) \right\}, \tag{8}
\]

then the output \( y(t) \) is given by

\[
y(t) = y_0(t) + y_1(t)
\]

It is easy to see that \( y_0(t) \) is a particular solution of (2) and \( y_1(t) \) is a solution of the corresponding homogeneous DE. The function \( y_0(t) \) is the output due to the input \( f(t) \) when the initial state is the zero state we call \( y_0(t) \) the zero-state response or the response due to the input. The function \( y_1(t) \) is the response of the system if the input \( f(t) \equiv 0 \) and is called the zero-input response or the response due to the initial state.

**The zero-state response and the weighting function**

The zero-state response is given by

\[
y_0(t) = \mathcal{L}^{-1} \left\{ Y(s)\hat{f}(s) \right\}.
\]
We know that \( Y(s) \) is a rational function. If \( \hat{f}(s) \) is also a rational function, we could find the zero state response by partial fractions. Even if \( \hat{f}(s) \) is not a rational function, we may proceed using convolutions. For this purpose we define the weighting function by

\[
    w(t) = \mathcal{L}^{-1}\{Y(s)\}.
\]

Since \( Y(s) \) is a proper rational function the weighting function can be found by partial fractions. The zero-state response can now be given by any of the forms

\[
    y_0(t) = w(t) * f(t) \tag{9}
\]

\[
    y_0(t) = \int_{0}^{t} w(\tau)f(t - \tau)d\tau \tag{10}
\]

\[
    y_0(t) = \int_{0}^{t} w(t - \tau)f(\tau)d\tau \tag{11}
\]

From (10) we see that the zero-state response at time \( t \) is the weighted integral of the input the input \( \tau \) units in the past, namely, \( f(t - \tau) \), is weighted by \( w(\tau) \). Therefore knowledge of the weighting function completely determines the zero-state response for a given input.

![Graphs of \( w(\tau) \) for different memory](image)

**Figure 2**

A graph of \( w(\tau) \) gives useful information about the response of the system. For instance, if \( w(\tau) \) is as in Figure 2(a), the weighting function is almost zero for \( \tau \geq 2 \); therefore the values of the input more than two time units in the past do not appreciably affect the output. We say the system *remembers* the input for about two time units or that the system has a *memory* of about two time units. The weighting function of Figure 2(b) has a memory of about eight units; the function in Figure 2(c) has an infinite memory. In general, we do not expect reasonable practical systems to have weighting functions like that in Figure 2(c); rather, we expect that inputs far in the past do not appreciable affect the output.

**Example 1.** Consider the system governed by the differential equation

\[
    y''(t) + 5y'(t) + 6y(t) = f(t),
\]

with zero initial conditions. The characteristic polynomial is

\[
    p(s) = s^2 + 5s + 6 = (s + 3)(s + 2).
\]
The transfer function is
\[ Y(s) = \frac{1}{p(s)} = \frac{1}{(s + 3)(s + 2)} = \frac{1}{s + 2} - \frac{1}{s + 3}. \]

The weighting function is
\[ w(t) = \mathcal{L}^{-1}\{Y(s)\} = e^{-2t} - e^{-3t}. \]

Therefore the zero-state response is
\[ y_0(t) = \int_0^t (e^{-2\tau} - e^{-3\tau}) f(t - \tau) d\tau. \]

The zero state response for a given input can be obtained from the above; for instance, if \( f(t) = H(t) \), the unit step function, then
\[ y_0(t) = \int_0^t (e^{-2\tau} - e^{-3\tau}) d\tau = \frac{e^{-3t}}{3} - \frac{e^{-2t}}{2} + \frac{1}{6}. \]

The graph of \( w(\tau) \) is close to that shown in Figure 2(a). Thus the system has a memory of about 2 units, and inputs far in the past have little effect on the output.

The zero-input response

The zero-input response is given by Equations (8) and (4)
\[
\begin{align*}
y_1(t) &= \mathcal{L}^{-1}\{Y(s)q(s)\} \\
 &= \mathcal{L}^{-1}\{Y(s)((as + b)\alpha + a\beta)\} \\
 &= \mathcal{L}^{-1}\{Y(s)(b\alpha + a\beta)\} + \mathcal{L}^{-1}\{a\alpha sY(s)\} \\
\end{align*}
\]

(12)

We know that \( \mathcal{L}^{-1}\{Y(s)\} = w(t) \), therefore, for the first term in (12) we have
\[ \mathcal{L}^{-1}\{Y(s)(b\alpha + a\beta)\} = (b\alpha + a\beta) w(t). \]

Recall from Section 2.5, Theorem 1, that \( \mathcal{L}\{w'(t)\} = s\mathcal{L}\{w(t)\} - w(0) = sY(s) - w(0). \) In Problem 5 below we show that \( w(0) = 0 \), thus \( \mathcal{L}^{-1}\{sY(s)\} = w(t). \) The zero-input response is now easily obtained
\[ y_1(t) = (b\alpha + a\beta) w(t) + a\alpha w'(t). \] (13)

Note that the zero-input response is given entirely in terms of the weighting function \( w(t) \) and its derivative. If we pick the particular initial conditions, \( y(0) = \alpha = 0 \), and \( y'(0) = \beta = 1/a \), we find from Equation (13) that \( y_1(t) = w(t) \). Thus the weighting function can be characterized as the zero-input response due to the initial conditions
\[ w(0) = 0, \text{ and } w'(0) = 1/a. \]

Example 2. Consider the system of Example 1, where the system parameters are \( a = 1, b = 5, c = 6 \). The weighting function is \( w(t) = e^{-2t} - e^{-3t} \). We find that \( w(0) = 0 \) and \( w'(0) = 1 \). Since \( a = 1 \), this is in accordance with (13). If the initial conditions are
\[ y(0) = 4, \text{ and } y'(0) = -2. \]
Putting $\alpha = 4$ and $\beta = -2$ into Equation (13) we obtain the zero-input response

$$y_1(t) = 18w(t) + 4w'(t) = 10e^{-2t} + 6e^{-3t}.$$

The weighting function and the unit impulse response

If we try to find an input $f(t)$ such that the zero-state response in the weighting function, we are led to the equation

$$w(t) = w(t) * f(t),$$

or, taking transforms

$$Y(s) = Y(s) \hat{f}(s).$$

Therefore $\hat{f}(s) \equiv 1$; this is impossible (see Page 55). However, let us consider an input $f(t)$ whose Laplace transform is 'close to' 1. Such an input is the function $\delta_\epsilon(t)$ defined by

$$\delta_\epsilon(t) = \begin{cases} 
\frac{1}{\epsilon}, & 0 \leq t < \epsilon \\
0, & t > \epsilon 
\end{cases}$$

and is shown in Figure 3.

If $\epsilon$ is small $\delta_\epsilon(t)$ represents a function that is very large for a small time interval but the area under $\delta_\epsilon(t) = 1$, i.e. $\int_0^\infty \delta_\epsilon(t) \, dt = 1$. The limit of $\delta_\epsilon(t)$ as $\epsilon \to 0$ is often called a unit impulse function or delta function, and denoted by $\delta(t)$. However $\delta(t)$ is not a function in the usual sense since, according to the definition $\delta(0)$ would have to be infinite. For the moment, let us work with $\delta_\epsilon(t)$. It is easy to calculate the Laplace transform of $\delta_\epsilon(t)$

$$\mathcal{L}\{\delta_\epsilon(t)\} = \int_0^\infty e^{-st} \delta_\epsilon(t) \, dt = \frac{1}{\epsilon} \int_0^\epsilon e^{-st} \, dt = \frac{1 - e^{-s\epsilon}}{s\epsilon}. $$

Furthermore, we find that as $\epsilon \to 0$ we have

$$\lim_{\epsilon \to 0} (\mathcal{L}\{\delta_\epsilon(t)\}) = \lim_{\epsilon \to 0} \frac{1 - e^{-s\epsilon}}{s\epsilon} = 1,$$
Section 2.10–The Weighting Function and Impulse Functions

where, for example, the limit can be calculated by l’Hospital’s Rule. Therefore for small \( \epsilon \), \( \mathcal{L}\{\delta_{\epsilon}(t)\} \) is ‘close to’ 1. Now let \( w_{\epsilon}(t) \) be the zero-state response to the input \( \delta_{\epsilon}(t) \). We have

\[
w_{\epsilon}(t) = w(t) * \delta_{\epsilon}(t) = \int_{0}^{t} w(t - \tau) \delta_{\epsilon}(\tau) d\tau = \frac{1}{\epsilon} \int_{0}^{t} w(t - \tau) d\tau
\]

It is easy to show that

\[
\lim_{\epsilon \to 0} w_{\epsilon}(t) = w(t).
\]

Therefore the weighting function is the limit of the zero-state response to the input \( \delta_{\epsilon}(t) \) as \( \epsilon \to 0 \). Briefly, but less precisely, we say that \textit{the weighting function is the zero-state response to a unit impulse}.

Think for a moment of a mass-spring system which is sitting at rest. If we give the mass a sudden blow with a hammer, we will impart a force to it which looks roughly like that in Figure 3, that is, the force will be large for a short period of time and zero thereafter. If the area under the force-time curve is equal to \( A \), then the force is approximately \( A \delta(t) \). The displacement of the system for \( t \geq 0 \) will be approximately \( Aw(t) \). This is an experimental way of finding the weighting function for a mechanical system. Once the weighting function is found, the response to any input is determined, as we have seen above.

\textbf{Unit impulse functions or delta functions}

To treat impulse functions in a careful manner requires more advanced mathematical concepts that we suppose here. However, it is rather easy to use Laplace transforms to solve systems such as (2) where the forcing function is an impulse function. We think of the unit impulse, \( \delta(t) \), as a \textit{generalized function}. We define operation on \( \delta(t) \) by means of appropriate limit operations on \( \delta_{\epsilon}(t) \). The following are easy to establish.

\[
\mathcal{L}\{\delta(t)\} = \lim_{\epsilon \to 0} \mathcal{L}\{\delta_{\epsilon}(t)\} = 1.
\]

\[
\mathcal{L}\{\delta(t - t_0)\} = \lim_{\epsilon \to 0} \mathcal{L}\{\delta_{\epsilon}(t - t_0)\} = e^{-st_0}.
\]

\[
\int_{-\infty}^{\infty} \delta(t - t_0) f(t) dt = \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \delta_{\epsilon}(t - t_0) f(t) dt = f(t_0).
\]

The last property is known as \textit{sifting property} of \( \delta(t) \).

\textbf{Example 3.} Solve \( y'' + 4y = \delta(t) \), \( y(0) = y'(0) = 0 \).

Taking the transform of both sides we find

\[
s^2 \tilde{y} + 4 \tilde{y} = 1, \quad \tilde{y} = \frac{1}{s^2 + 1}.
\]

\[
y(t) = \mathcal{L}^{-1} \left\{ \frac{1}{s^2 + 1} \right\} = \sin t.
\]

Note that \( y(t) \) is just the weighting function.

\textbf{Example 4.} Solve \( y'' + 4y = \delta(t - 2) \), \( y(0) = 0 \), \( y'(0) = 1 \).

Taking transforms we find

\[
s^2 \tilde{y} - 1 + \tilde{y} = e^{-2s}, \quad \tilde{y} = \frac{1}{s^2 + 1} + \frac{e^{-2s}}{s^2 + 1}.
\]
Therefore

\[ y(t) = \sin t + H(t - 2) \sin(t - 2). \]

**Exercises 2.10**

In problems 1–3 find the weighting function and use it to find the zero-state response and the zero-input response.

1. \( y'' + y = f(t), \ y(0) = y'(0) = 1 \).
2. \( y'' - y = f(t), \ y(0) = 0, \ y'(0) = 1 \).
3. \( y'' + y' - 2y = f(t), \ y(0) = 1, \ y'(0) = 0 \).

4. Show that the weighting function \( w(t) \) has the property that \( w(0) = 0 \).

**Hint:** Write \( Y(s) = \frac{1}{as^2 + bs + c} = \frac{1}{s} \frac{s}{as^2 + bs + c} \) and note that

\[ \mathcal{L}^{-1} \left\{ \frac{s}{as^2 + bs + c} \right\} = h(t) \text{ is some function that can be obtained by partial fractions.} \]

Solve the following using properties of impulse functions.

5. \( y'' - 4y = \delta(t - 1) + \delta(t - 2), \ y(0) = y'(0) = 0 \).
6. \( y'' + 4y' + 4y = \delta(t - 2), \ y(0) = 0, \ y'(0) = 1 \).
7. \( y' - y = \delta(t - 2), \ y(0) = 5 \).
8. \( y'' - y = \delta(t) + H(t - 2), \ y(0) = y'(0) = 0 \).
9. \( y^{(iv)} - y = \delta(t - 2), y(0) = 1, \ y'(0) = y''(0) = y'''(0) = 0 \).
CHAPTER III

MATRICES AND SYSTEMS OF EQUATIONS

3.0–Introduction

The first objective of this chapter is to develop a systematic method for solving systems of linear algebraic equations.

Consider the system of two linear equations in the three unknowns \( x, y, \) and \( z \)

\[
\begin{align*}
x - 2y + 3z &= -5 \\
2x - 4y + 7z &= -12
\end{align*}
\]

(1)

By a solution of the system (1) we mean an ordered triple of numbers, written as the array \([x, y, z]\), which satisfy both equations in (1). Thus \([1, 0, -2]\) is a solution while \([-5, 0, 0]\) is not; the latter triple satisfies only the first equation. We consider the triple of numbers as a single entity, called a vector which we denote by a single letter \( w \).

\[
w = [x, y, z].
\]

(2)

The numbers \( x, y, z \) are called the first, second and third components of \( w \), respectively. When the components are written in a horizontal row as in (2), the vector is called a row vector. We could equally as well have written the components in a vertical column

\[
v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}
\]

(3)

which we call a column vector. We shall adopt the usual convention in matrix theory and write solution vectors as column vectors as in (3). If we have a system of equations with \( n \) unknowns the solution vector has \( n \) components. We shall study the algebra of \( n \)-vectors in the next section.

Let us start by considering three examples of systems of two equations in two unknowns. Although the examples are extremely simple they illustrate the various situations that can arise.

Example 1. Consider the system

\[
\begin{align*}
x + y &= 5 \\
2x - y &= 4
\end{align*}
\]

(4)

We shall use the method of elimination to find all possible solution vectors \( v = \begin{bmatrix} x \\ y \end{bmatrix} \). Suppose the components, \( x \) and \( y \), satisfy the system (4). By multiplying the first equation by \(-2\) and adding to the second equation we get a new second equation, \(-3y = -6\), where \( x \) has been eliminated. Thus any solution of the system (4) is also a solution of

\[
\begin{align*}
x + y &= 5 \\
-3y &= -6
\end{align*}
\]

(5)

Conversely, if \( x \) and \( y \) satisfy (5), we may multiply the first equation by \( 2 \) and add it to the second equation to retrieve the original second equation, \(2x - y = 4\). Therefore the systems (4) and (5) have the same solution vectors. However the system (5) is easy to solve. The second equation yields \( y = 2 \) which may be back-substituted into the first equation to get \( x = 3 \). The system (5) and thus the system (4) has a unique solution vector, namely

\[
v = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.
\]
Geometrically, each equation in the system (4) represents a straight line as shown in Figure 1(a). Since the lines are not parallel, they intersect in a unique point, the point $(3, 2)$.

**Example 2.**

\[
\begin{align*}
  x + 2y &= 2 \\
 2x + 4y &= 4
\end{align*}
\]

Multiplying the first equation by $-2$ and adding to the second equation produces

\[
\begin{align*}
  x + 2y &= 2 \\
 0 \cdot x + 0 \cdot y &= 0
\end{align*}
\]

The second equation in (7) puts no restriction on $x$ and $y$, thus it is only necessary to satisfy the first equation to produce a solution. This fact is already obvious in system (6) since the second equation is simply twice the first. We may assign any value to $y$, say $y = t$, and then $x = 2 - 2t$. Thus, there are infinitely many solutions of (6), given by

\[
v = \begin{bmatrix} 2 - 2t \\ t \end{bmatrix},
\]

where $t$ is arbitrary. Geometrically, both equations in (6) represent the same straight line as shown in Figure 1(b). Any point on this straight line is a solution. The vectors given in (8), in component form are $x = 2 - 2t$ and $y = t$; these are just the parametric equations of the line $x + 2y = 2$.

**Example 3.**

\[
\begin{align*}
  x + y &= 5 \\
 2x + 4y &= 5
\end{align*}
\]

Clearly it is impossible for two numbers $x$ and $y$ to add up to 4 and 5 at the same time. The system (9) has no solutions; we call such a system *inconsistent*. If we try to eliminate $x$ from the second equation we obtain

\[
\begin{align*}
  x + y &= 5 \\
 0 \cdot x + 0 \cdot y &= 2
\end{align*}
\]

The inconsistency clearly shows up in the second equation in (10). Geometrically, the two equations in (9) represent two parallel lines as seen in Figure 1(c). These lines never intersect, and the equations have no solution.

We see that a system of two equations in two unknowns may possess no solutions, exactly one solution or infinitely many solutions. We shall see later that the same three situations may occur for $n$ equations in $n$ unknowns.
A general system of \( m \) equations in \( n \) unknowns may be written
\[
\begin{align*}
    a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
    a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
    \vdots & \quad \vdots & \quad \vdots \\
    a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m.
\end{align*}
\] (11)

The coefficients \( a_{ij} \) and the right hand sides \( b_i \) are assumed to be given real or complex numbers, and the \( x_i \) are the unknowns. In this double subscript notation for the coefficients, \( a_{ij} \) stands for the coefficient, in the \( i^{th} \) equation, of the \( j^{th} \) unknown, \( x_j \). Using summation notation the system (11) may be written
\[
\sum_{j=1}^{n} a_{ij}x_j = b_i, \quad i = 1, 2, \ldots, m.
\] (12)

By a solution of the system we mean the \( n \)-vector
\[
\mathbf{v} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}
\]
whose components \( x_i \) satisfy all of the equations of the system (12).

If the system (12) has at least one solution vector it is said to be consistent; if no solution vector exists it is said to be inconsistent. By systematically exploiting the method of elimination we shall shortly develop a procedure for determining whether or not a system is consistent and, if consistent, to find all solution vectors.

If in the system (12) all of the right hand sides \( b_i = 0 \), the system is called homogeneous, otherwise it is called \( x \). A homogeneous system
\[
\sum_{j=1}^{n} a_{ij}x_j = 0, \quad i = 1, 2, \ldots, m.
\]
always has a solution, namely, \( x_1 = 0, x_2 = 0, \ldots, x_n = 0 \); this is called the trivial solution. Thus homogeneous equations are always consistent. The fundamental questions concerning homogeneous equations are whether or not nontrivial solutions exists, and how to find these solutions.

**Exercises 3.0**

For the systems in problems 1 and 2, find all solutions and write the solutions in vector form.

1. a. \( 2x - 3y = 2 \) \quad b. \( 2x + 2y = 5 \) \quad c. \( x + y = 0 \) \quad d. \( 2x - 2y = 4 \)
   \[ \begin{align*}
   x - 2y &= 7 \\
   4x + 4y &= 7 \\
   x - y &= 0
   \end{align*} \]
2. a. \( x - 2y + 3z = -5 \) \quad b. \( x - 2y + 3z = 0 \)
   \[ \begin{align*}
   2x - 4y + 7z &= -1 \\
   2x - 4y + 7z &= 0
   \end{align*} \]
3. What are the restrictions on the values of \( b_1 \) and \( b_2 \), if any, for the following systems to be consistent:
   a. \( 2x + 3y = b_1 \) \quad b. \( 2x - 3y = b_1 \)
   \[ \begin{align*}
   x - 3y &= b_2 \\
   -2x + 3y &= b_2
   \end{align*} \]
4. Consider one equation in one unknown, \( ax = b \). Discuss the existence and uniqueness of solutions. Consider three cases (i) \( a \neq 0 \), (ii) \( a = 0, b = 0 \) and (iii) \( a = 0, b \neq 0 \).
3.1–The Algebra of n–vectors

The solution of a system of equations with \( n \) unknowns is an ordered \( n \)-tuple of real or complex numbers. The set of all ordered \( n \)-tuples of complex numbers we denote by \( \mathbb{C}^n \). If \( \mathbf{x} \in \mathbb{C}^n \) we call \( \mathbf{x} \) a \( n \)-vector or simply a vector and write

\[
\mathbf{x} = \begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{bmatrix}
\]

where \( x_i \) is called the \( i^{th} \) component of \( \mathbf{x} \). The algebra of \( n \)-vectors is a generalization of the algebra of vectors with two or three components with which the reader is undoubtedly familiar.

**Definition 1.** Let \( \mathbf{x} \) and \( \mathbf{y} \) be vectors in \( \mathbb{C}^n \) and let \( \alpha \) be a scalar (a complex number), we define multiplication by a scalar, \( \alpha \mathbf{x} \), and the sum of two vectors, \( \mathbf{x} + \mathbf{y} \), by

\[
\alpha \mathbf{x} = \alpha \begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{bmatrix} = \begin{bmatrix}
\alpha x_1 \\
\alpha x_2 \\
\vdots \\
\alpha x_n
\end{bmatrix}, \quad \mathbf{x} + \mathbf{y} = \begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{bmatrix} + \begin{bmatrix}
y_1 \\
y_2 \\
\vdots \\
y_n
\end{bmatrix} = \begin{bmatrix}
x_1 + y_1 \\
x_2 + y_2 \\
\vdots \\
x_n + y_n
\end{bmatrix}
\]

The difference of two vectors is defined by \( \mathbf{x} - \mathbf{y} = \mathbf{x} + (-1)\mathbf{y} \).

**Example 1.**

\[
\begin{bmatrix}
1 \\
-2 \\
3
\end{bmatrix} + \begin{bmatrix}
-1 \\
0 \\
-2
\end{bmatrix} = \begin{bmatrix}
-1 \\
-2 \\
1
\end{bmatrix}, \quad \begin{bmatrix}
1 + i \\
-2 + 5i
\end{bmatrix} = \begin{bmatrix}
1 \\
0
\end{bmatrix} + i \begin{bmatrix}
-1 \\
0
\end{bmatrix}
\]

The zero vector is the vector with all of its components equal to zero; we denote it by \( \mathbf{0} \). Note that \( 0\mathbf{x} = \mathbf{0} \) for all \( \mathbf{x} \in \mathbb{C}^n \). It is also worthwhile noting that if \( \mathbf{x}, \mathbf{y} \in \mathbb{C}^n \) and \( \mathbf{x} = \mathbf{y} \) then \( x_i = y_i, i = 1, 2, \ldots, n \). Thus one vector equation is equivalent to \( n \) scalar equations.

The properties of addition and multiplication by a scalar are given by

**Theorem 1.** Let \( \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{C}^n \) and \( \alpha, \beta \) be scalars, then the following hold

1. \( \mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{y}) + \mathbf{z} \) (associative law of multiplication)
2. \( \mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x} \) (commutative law of addition)
3. \( \mathbf{x} + \mathbf{0} = \mathbf{x} \)
4. \( \mathbf{x} - \mathbf{x} = \mathbf{0} \)
5. \( (\alpha \beta)\mathbf{x} = \alpha(\beta \mathbf{x}) \)
6. \( \alpha(\mathbf{x} + \mathbf{y}) = \alpha \mathbf{x} + \alpha \mathbf{y} \)
7. \( (\alpha + \beta)\mathbf{x} = \alpha \mathbf{x} + \beta \mathbf{x} \)

The proof of this theorem follows immediately from Definition 1 and the properties of complex numbers.

Sometimes we may wish to restrict ourselves the components of vectors and the scalar multipliers to be real numbers. In this case we denote the set of \( n \)-tuples of real numbers by \( \mathbb{R}^n \). Definition 1 and Theorem 1 still apply in this case.

**Example 2.** Find all real solutions of the single linear equation in two unknowns

\[
x + 2y = 3
\]

It is clear that we may take \( y \) to be an arbitrary real number, say \( y = t \), and then \( x = 3 - 2t \), so that the general solution is given in vector form by

\[
\mathbf{v} = \begin{bmatrix}
x \\
y
\end{bmatrix} = \begin{bmatrix}
3 - 2t \\
t
\end{bmatrix} = \begin{bmatrix}
3 \\
0
\end{bmatrix} + t \begin{bmatrix}
-2 \\
1
\end{bmatrix}
\]
where \( t \) is arbitrary. The solution is the sum of a fixed vector, \( \begin{bmatrix} 3 \\ 0 \end{bmatrix} \) and a multiple of \( \begin{bmatrix} -2 \\ 1 \end{bmatrix} \). Geometrically the tips of the solution vectors lie on the line \( x + 2y = 3 \) as shown in Figure 1.

Note that \( \begin{bmatrix} 3 \\ 0 \end{bmatrix} \) is a particular solution of (i), and \( t \begin{bmatrix} -2 \\ 1 \end{bmatrix} \) is the general solution of the associated homogeneous equation \( x + 2y = 0 \). This is an instance of a general theorem we will consider later.

Sums of multiples of vectors will occur often in our work. It is convenient to give such expressions a name.

**Definition 2.** Let \( \{ \mathbf{v}^1, \mathbf{v}^2, \ldots, \mathbf{v}^k \} \) be a set of vectors in \( \mathbb{C}^n \) (or \( \mathbb{R}^n \)), and let \( \alpha_1, \alpha_2, \ldots, \alpha_k \) be scalars. The expression

\[
\alpha_1 \mathbf{v}^1 + \alpha_2 \mathbf{v}^2 + \cdots + \alpha_k \mathbf{v}^k
\]

is called a **linear combination** of \( \mathbf{v}^1, \mathbf{v}^2, \ldots, \mathbf{v}^k \) with **weights** \( \alpha_1, \alpha_2, \ldots, \alpha_k \).

**Example 3.** Let \( x = \begin{bmatrix} 2 \\ -5 \end{bmatrix}, \ u = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \ v = \begin{bmatrix} 1 \\ 6 \end{bmatrix} \). Express \( x \) as a linear combination of \( u \) and \( v \).

We must find weights \( \alpha, \beta \) so that \( x = \alpha u + \beta v \), or

\[
\begin{bmatrix} 2 \\ -5 \end{bmatrix} = \alpha \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 6 \end{bmatrix} = \begin{bmatrix} 3\alpha + \beta \\ \alpha + 6\beta \end{bmatrix}.
\]

Therefore

\[
\begin{align*}
3\alpha + \beta &= 2 \\
\alpha + 6\beta &= -5
\end{align*}
\]

These equations can be solved to yield the weights \( \alpha = 1, \beta = -1 \). Thus \( x = u - v \), as can be readily verified.

**Definition 3.** The standard unit vectors in \( \mathbb{C}^n \) (or \( \mathbb{R}^n \)) are defined by

\[
\mathbf{e}^1 = \begin{bmatrix} 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{e}^2 = \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}, \ldots, \quad \mathbf{e}^n = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}, \quad (3)
\]
The vector $e^i$ has its $i$th component equal to 1 and all other components equal to 0. If $x$ is any vector in $\mathbb{C}^n$ (or $\mathbb{R}^n$), we have

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + x_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}. $$

or

$$x = x_1 e^1 + x_2 e^2 + \cdots + x_n e^n. $$

Thus every vector $x$ can be expressed as a linear combination of the unit vectors $e^i$ weighted by the components of $x$.

Let us now analyze the solutions of a single linear equation in $n$ unknowns

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b. \quad (4)$$

We consider three cases.

Case i. Not all $a_i = 0$. Suppose for simplicity that $a_1 \neq 0$, then Equation (4) can be written

$$x_1 = \frac{b}{a_1} - \left( \frac{a_2}{a_1} x_2 + \cdots + \frac{a_n}{a_1} x_n \right). \quad (5)$$

It is clear that $x_2, x_3, \ldots, x_n$ can be given arbitrary values, and then Equation (5) determines $x_1$. The variables $x_2, x_3, \ldots, x_n$ whose values are arbitrary are called free variables, while the variable $x_1$ is called a basic variable. In vector form the general solution is

$$x = \begin{bmatrix} \frac{b}{a_1} - \frac{a_2}{a_1} x_2 - \cdots - \frac{a_n}{a_1} x_n \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \frac{b}{a_1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -\frac{a_2}{a_1} \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + x_n \begin{bmatrix} -\frac{a_n}{a_1} \\ 0 \\ \vdots \\ 1 \end{bmatrix},$$

where $x_2, x_3, \ldots, x_n$ are arbitrary.

Case ii. All $a_i = 0$, but $b \neq 0$. In this case Equation (4) becomes

$$0 x_1 + 0 x_2 + \cdots + 0 x_n = b \neq 0.$$

This equation can never be satisfied for any choice of the $x_i$. The equation has no solution or is inconsistent.

Case iii. All $a_i = 0$, and $b = 0$. In this case we have

$$0 x_1 + 0 x_2 + \cdots + 0 x_n = 0.$$

Each $x_i$ can be arbitrarily assigned, thus, every variable is a free variable. Every vector $x$ is a solution. In vector form we can write the general solution as

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 e^1 + \cdots + x_n e^n,$$
where the \( e_i \) are the standard unit vectors given in (3).

Finally we mention that we could have written vectors as rows rather than as columns. In fact, when we discuss matrices, we will often treat the rows of a matrix as a vector.

**Exercises 3.1**

1. Given \( x = \begin{bmatrix} 1 - 2i \\ -i \\ 3 \\ 4 + 5i \end{bmatrix} \), write \( x \) in the form \( u + iv \) where \( u \) and \( v \) are real.

2. In each of the following cases express \( x \) as a linear combination of \( u \) and \( v \), if possible.
   
   a. \( x = \begin{bmatrix} 1 \\ 3 \\ \end{bmatrix} \), \( u = \begin{bmatrix} 1 \\ -1 \\ \end{bmatrix} \), \( v = \begin{bmatrix} 1 \end{bmatrix} \)
   
   b. \( x = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \), \( u = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \), \( v = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \)
   
   c. \( x = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \), \( u = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \), \( v = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \)
   
   d. \( x = \begin{bmatrix} 1 \\ -2 \\ \end{bmatrix} \), \( u = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} \), \( v = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \)

3. Find the general solution in vector form
   
   a. \( x_1 + 3x_2 - x_3 = 7 \)
   
   b. \( x_1 + 2x_2 + 0x_3 = 2 \)
   
   c. \( 0x_1 + 0x_2 = 0 \)

4. Given the row vectors
   
   \[ r = [1, \ -1, \ 0, \ 2] \], \( s = [1, \ -1, \ 0, \ 0] \), \( t = [0, \ 0, \ 0, \ 1] \),

   express \( r \) as a linear combination of \( s \) and \( t \), if possible.

### 3.2–Matrix Notation for Linear Systems

A single linear equation in one unknown, \( x \), is written as

\[
ax = b. \tag{1}
\]

We shall develop a notation so that a general linear system can be written in a compact form similar to (1). For this purpose we need the concept of a matrix.

**Definition 1.** A matrix is a rectangular array of numbers. If a matrix has \( m \) rows and \( n \) columns, it is said to have order (or size) \( m \times n \) (read ‘\( m \) by \( n \)’).

**Example 1.**

\[
A = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 6.4 & -4 \end{bmatrix}, \quad c = \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix}, \quad r = [2, \ 0, \ 8],
\]

\( A \) is a \( 2 \times 3 \) matrix, \( c \) is a \( 3 \times 1 \) column matrix, and \( r \) is a \( 1 \times 3 \) row matrix.

The number located in the \( i \)th row and \( j \)th column of a matrix is called the \( (i, j) \)th element of the matrix. For example the \( (2, 3) \)th element in the matrix \( A \) above is \( -4 \). A matrix consisting of a single row is called a row matrix and is identified with the row vector having the same elements, similarly, a matrix with a single column is called a column matrix and is identified with the column vector having the same elements. A \( 1 \times 1 \) matrix whose single element is \( a \) is identified with the scalar \( a \).
A matrix is usually denoted by a single letter such as \( A \), or \( A_{m \times n} \), if the order is to be indicated. To describe the elements of a general \( m \times n \) matrix we need to use a double subscript notation.

\[
A_{m \times n} = \begin{bmatrix}
    a_{11} & a_{12} & \cdots & a_{1n} \\
    a_{21} & a_{22} & \cdots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix}.
\]

(2)

Note that \( a_{ij} \) is the element in the \( i^{th} \) row and \( j^{th} \) column. We shall also write in the abbreviated form

\[
A = [a_{ij}].
\]

Before considering the general case let us consider a single linear equation in \( n \) unknowns

\[
a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b.
\]

(4)

We define the matrices

\[
A = [a_1, a_2, \ldots, a_n], \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.
\]

(5)

Our object is to define the product \( Ax \) so that we may write Equation (4) in the compact form \( Ax = b \); this leads us to the following definition

**Definition 2.**  (Row-column product rule). If \( A \) is a \( 1 \times n \) row matrix and \( x \) is a \( n \times 1 \) column matrix, the product \( Ax \) is the \( 1 \times 1 \) matrix given by

\[
Ax = [a_1, a_2, \ldots, a_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = [a_1 x_1 + a_2 x_2 + \cdots + a_n x_n].
\]

(6)

With this definition Equation (4) can be written simply as

\[
Ax = b.
\]

(7)

where we have identified the scalar \( b \) with the \( 1 \times 1 \) matrix \([b]\).

Now let us consider \( m \) equations in \( n \) unknowns.

\[
a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n = b_1 \\
a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n = b_2 \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n = b_m
\]

(8)

Define the following matrices

\[
A = \begin{bmatrix}
    a_{11} & a_{12} & \cdots & a_{1n} \\
    a_{21} & a_{22} & \cdots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.
\]
The matrix $A$ is called the **coefficient matrix**. Again our object is to define the matrix product $Ax$ so that (8) can be written in the compact form $Ax = b$. Notice that the left hand side of the $i^{th}$ equation in (8) is the matrix product of the $i^{th}$ row of $A$ and the column matrix $x$. Let us write $r_i(A)$ for the $i^{th}$ row of $A$. The matrix $A$ can be considered as an array of $n$-dimensional row vectors.

$$ A = \begin{bmatrix} r_1(A) \\ r_2(A) \\ \vdots \\ r_m(A) \end{bmatrix}, \quad r_1(A) = [a_{11}, a_{12}, \ldots, a_{1n}] \\ r_2(A) = [a_{21}, a_{22}, \ldots, a_{2n}] \\ \vdots \\ r_m(A) = [a_{m1}, a_{m2}, \ldots, a_{mn}] $$

**Definition 3.** The product of an $m \times n$ matrix $A$ by the $n \times 1$ column matrix $x$ is the $m \times 1$ column matrix $Ax$ defined by

$$ Ax = \begin{bmatrix} r_1(A) \\ r_2(A) \\ \vdots \\ r_m(A) \end{bmatrix} x = \begin{bmatrix} r_1(A)x \\ r_2(A)x \\ \vdots \\ r_m(A)x \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix} $$

With this definition the linear system (8) can be written as

$$ Ax = b. $$

**Example 2.**

$$ \begin{bmatrix} 2 & -2 & 3 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 2(-1) + (-2)3 + 3 \cdot 2 \\ 1(-1) + 0 \cdot 3 + 2 \cdot 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix} $$

**Example 3.** Consider the linear equations

$$ x_1 - 2x_2 = 1 $$
$$ x_1 + 7x_2 + x_3 = 0. $$

a. Write in matrix form.
b. Using matrix multiplication determine if $x_1 = 1, \ x_2 = 1, \ x_3 = -1$ is a solution.
c. Determine if $x_1 = 1, \ x_2 = 0, \ x_3 = -1$ is a solution.

**Solution**

a. $$ \begin{bmatrix} 1 & -2 & 0 \\ 1 & 7 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} $$

b. $$ \begin{bmatrix} 1 & -2 & 0 \\ 1 & 7 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 0 \end{bmatrix}, $$

Therefore $x_1 = 1, \ x_2 = 1, \ x_3 = -1$ is not a solution.

c. $$ \begin{bmatrix} 1 & -2 & 0 \\ 1 & 7 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, $$

Therefore $x_1 = 1, \ x_2 = 0, \ x_3 = -1$ is a solution.
Matrix notation and matrix multiplication are often useful in organizing data and performing certain types of calculations. The following is a simple example.

Example 4. A toy manufacturer makes two types of toys, a toy truck and a toy plane. The toy truck requires 3 units of steel, 5 units of plastics and 6 units of labor. The toy plane requires 2 units of steel, 6 units of plastic and 7 units of labor. Suppose the price of steel is $2 per unit, plastic is $1 per unit and labor is $3 per unit. We put the resources needed for each toy in a matrix \( R \) and the prices in a column matrix \( p \).

\[
R = \begin{bmatrix}
3 & 5 & 6 \\
2 & 6 & 7
\end{bmatrix}
\text{truck} \quad \text{plane}
\]

\[
p = \begin{bmatrix}
2 \\
1 \\
3
\end{bmatrix}
\text{unit price of steel} \quad \text{unit price of plastic} \quad \text{unit price of labor}
\]

The components of the product \( Rp \) give the cost of making the truck and plane.

\[
Rp = \begin{bmatrix}
3 & 5 & 6 \\
2 & 6 & 7
\end{bmatrix}
\begin{bmatrix}
2 \\
1 \\
3
\end{bmatrix}
= \begin{bmatrix}
29 \\
31
\end{bmatrix}
\]

We see that the toy truck costs 29 dollars to produce, and the toy plane 31 dollars.

It is important to realize that one cannot multiply any matrix \( A \) by any column vector \( x \); the number of columns in the left hand factor \( A \) must equal the number of rows in the right hand factor \( x \): \( A_{m \times n} x_{n \times 1} = b_{m \times 1} \). Later we will define multiplication of a matrix \( A \) by a matrix \( B \), where \( B \) has more than one column. However, in order for \( AB \) to be defined, the number of columns in \( A \) must equal the number of rows in \( B \).

Finally we define two special matrices. First, the zero matrix of order \( m \times n \) is a matrix all of whose entries are zero. We denote a zero matrix by \( O_{m \times n} \) or simply by \( O \), if the order is understood. It is easy to show that \( O_{m \times n} x_{n \times 1} = 0_{m \times 1} \), for all \( x \).

Second, the identity matrix of order \( n \times n \) is the matrix

\[
\begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{bmatrix}
\]

The identity matrix has all of its diagonal elements equal to 1 and its off diagonal elements equal to 0. If the order is understood, we simply use \( I \) to denote the identity matrix. The following property is easy to prove

\[
Ix = x, \quad \text{for all } x,
\]

where of course \( I \) and \( x \) must have compatible orders in order that the multiplication is defined.

Exercises 3.2

1. Compute \[
\begin{bmatrix}
1 & -3 & 0 \\
1 & -1 & 1
\end{bmatrix}
\begin{bmatrix}
2 \\
1 \\
4
\end{bmatrix}.
\]
2. Consider the system of equations:
\[\begin{align*}
2x_1 + x_2 &= 2 \\
2x_1 - x_2 &= 1 \\
2x_1 + x_2 &= 3
\end{align*}\]

a. Write the equations in matrix form.

b. Determine, using matrix multiplication if \( x = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \) is a solution.

c. Determine, using matrix multiplication if \( x = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \) is a solution.

3. If \( Ax = \begin{bmatrix} 0 \\ 3 \end{bmatrix} \) and \( x \) is a 4-dimensional vector, what is the order of \( A \)?

4. In Example 4, change the unit prices of steel, plastic and labor to 2, 2, 1, respectively. What is the cost of making a toy truck? A toy plane?

5. Prove equation (9).

6. Prove or give a counterexample: if \( A \) and \( B \) are \( 2 \times 2 \), and \( A \begin{bmatrix} 2 \\ 5 \end{bmatrix} = B \begin{bmatrix} 2 \\ 5 \end{bmatrix} \), then \( A = B \).

### 3.3–Properties of Solutions of Linear Systems

Consider a system of \( m \) linear equations in \( n \) unknowns:

\[
\begin{align*}
    a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
    a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
    \vdots & \quad \vdots \quad \vdots \\
    a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{align*}
\]

We may write system (1) in the matrix form

\[
A \mathbf{x} = \mathbf{b},
\]

where

\[
A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\
            & a_{22} & \cdots & a_{2n} \\
            &       & \vdots & \cdots \\
            & a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}
\]

The \( m \times n \) matrix \( A \) is called the coefficient matrix, the \( n \times 1 \) column matrix or vector \( \mathbf{x} \) is called the unknown vector or solution vector and the \( m \times 1 \) vector \( \mathbf{b} \) is called the right hand side vector. If there exists a vector \( \mathbf{x} \) such that \( A \mathbf{x} = \mathbf{b} \) for given \( A \) and \( \mathbf{b} \), the system is called consistent, otherwise the system is called inconsistent. If \( \mathbf{b} = \mathbf{0} \) the system \( A \mathbf{x} = \mathbf{0} \) is called homogeneous, otherwise \( A \mathbf{x} = \mathbf{b} \) with \( \mathbf{b} \neq \mathbf{0} \) is called nonhomogeneous.

Before considering properties of solutions, we need the following simple, but important, property of the matrix product \( A \mathbf{x} \): If \( A \) is an \( m \times n \) matrix, \( \mathbf{x} \) and \( \mathbf{y} \) are \( n \times 1 \) column vectors and \( \alpha \) and \( \beta \) are scalars then

\[
A(\alpha \mathbf{x} + \beta \mathbf{y}) = \alpha A \mathbf{x} + \beta A \mathbf{y}.
\]

This is called the linearity property. Equation (4) may be proved without difficulty by writing out both sides.

First we consider properties of solutions of the homogeneous system \( A \mathbf{x} = \mathbf{0} \).
**Property 1.** The homogeneous equation always has the trivial solution, \( x = 0 \) (i.e. \( x_1 = x_2 = \ldots = x_n = 0 \)).

This is obvious since \( A0 = 0 \), therefore homogeneous systems are always consistent. The fundamental question for a homogeneous system \( Ax = 0 \) is “When does a nontrivial solution exist?” We shall answer this question shortly. However, if a nontrivial solution exists, the following property shows us how to find infinitely many such solutions.

**Property 2.** If \( u \) and \( v \) are solutions of \( Ax = 0 \) then \( \alpha u + \beta v \) are also solutions for arbitrary scalars \( \alpha \) and \( \beta \).

**Proof** \( A(\alpha u + \beta v) = \alpha Au + \beta Av = 0 + 0 = 0 \).

In other words, arbitrary linear combinations of solutions of homogeneous equations are also solutions.

**Example 1.** Consider \( Ax = 0 \) where \( A = \begin{bmatrix} 1 & 1 & -1 & 2 \\ 1 & 1 & -1 & 3 \end{bmatrix} \). Show \( u = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \) and \( v = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \) are solutions. Also verify that \( \alpha u + \beta v \) are solutions.

Since \( Au = \begin{bmatrix} 1 & 1 & -1 & 2 \\ 1 & 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \) and \( Av = \begin{bmatrix} 1 & 1 & -1 & 2 \\ 1 & 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \), we see that \( u \) and \( v \) are solutions. Also \( \alpha u + \beta v = \begin{bmatrix} \beta \\ \alpha + \beta \\ 0 \end{bmatrix} \) and \( A(\alpha u + \beta v) = \begin{bmatrix} 1 & 1 & -1 & 2 \\ 1 & 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} \beta \\ \alpha + \beta \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \).

One should always bear in mind that a nonhomogeneous system need not be consistent. A simple example of an inconsistent system is:

\[ \begin{align*}
  x_1 - x_2 + x_3 &= 1 \\
  x_1 - x_2 + x_3 &= 2
\end{align*} \]

We now consider the properties of solutions for a consistent nonhomogeneous system \( Ax = b \).

**Property 3.** If \( u \) and \( v \) are solutions of the nonhomogeneous system \( Ax = b \), then \( u - v \) is a solution of the corresponding homogeneous system \( Ax = 0 \).

**Proof** \( A(u - v) = Au - Av = b - b = 0 \).

**Example 2.** Consider \( Ax = b \) where \( A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \) and \( b = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \).

\( u = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \) is a solution since \( Au = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \) and \( v = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \) is also a solution since \( Av = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \). However \( u - v = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \) is a solution of \( Ax = 0 \) since \( A(u - v) = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \).

**Property 4.** If \( x^p \) is a particular solution of \( Ax = b \) and \( x^h \) is a solution of \( Ax = 0 \), then \( x^p + x^h \) is also a solution of \( Ax = b \). Furthermore if \( x^h \) is the general solution of \( Ax = 0 \) then \( x^p + x^h \) is the general solution of \( Ax = b \).

**Proof** Since \( Ax^p = b \) and \( Ax^h = 0 \), it follows that \( A(x^p + x^h) = b + 0 = b \), thus \( x^p + x^h \) is a solution. Now to complete the second part of the proof we must show that if \( x \) is any solution of
Ax = b, it can be written in the form x = \(x^p + x^h\). We know that \(x^p\) is a solution of \(Ax = b\), and by Property 4., we have that \(x - x^p\) is a solution of \(Ax = 0\). Therefore we must have \(x - x^p = x^h\), since \(x^h\) is the general solution of \(Ax = 0\) and thus \(x = x^p + x^h\) as desired.

**Example 3.** Consider the single linear equation \(x_1 + x_2 - 3x_3 = 2\). It is clear that \(x_2\) and \(x_3\) may be taken as arbitrary or free variables and then \(x_1\), called a basic variable, is determined in terms of \(x_2\) and \(x_3\) by \(x_1 = 2 - x_2 + 3x_3\). The general solution in vector form is therefore

\[
x = \begin{bmatrix} 2 - x_2 + 3x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}.
\]

Letting \(x^p = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}\) and \(x^h = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}\), we see that \(x^p\) is a (particular) solution of the equation (obtained by setting \(x_2 = x_3 = 0\)), \(x^h\) is the general solution of the corresponding homogeneous equation \(x_1 + x_2 - 3x_3 = 0\) and \(x = x^p + x^h\) is the general solution of the nonhomogeneous equation.

**Property 5.** If \(x^1\) is a solution of \(Ax = b^1\) and \(x^2\) is a solution of \(Ax = b^2\) then \(\alpha_1 x^1 + \alpha_2 x^2\) is a solution of \(Ax = \alpha_1 b^1 + \alpha_2 b^2\).

**Proof** \(A(\alpha_1 x^1 + \alpha_2 x^2) = \alpha_1 Ax^1 + \alpha_2 Ax^2 = \alpha_1 b^1 + \alpha_2 b^2\).

**Example 4.** Let \(x^1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\), \(x^2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\) and \(A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & -1 \end{bmatrix}\).

1. Verify that \(Ax^1 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}\) and \(Ax^2 = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}\).

2. Find a solution of \(Ax = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\).

**Solution**

1. \(Ax^1 = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}\), \(Ax^2 = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}\).

2. Note that \(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} - \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}\), thus \(x = 2x^1 - x^2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}\) must be a solution of \(Ax = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\), as can be readily verified.

**Exercises 3.3**

1. Let \(x^1 = \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}\) and \(x^2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}\) be solutions of \(Ax = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\). Without attempting to find the matrix \(A\), answer the following

   a. Find infinitely many solutions of \(Ax = 0\). 
   b. Find one solution of \(Ax = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}\). 
   c. Find infinitely many solutions of the equation in b.
2. Let \( \mathbf{x}^1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \) be a solution of \( A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) and \( \mathbf{x}^2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \) a solution of \( A\mathbf{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \).

a. Find a solution of \( A\mathbf{x} = \begin{bmatrix} -3 \\ 6 \end{bmatrix} \).

b. Find a solution of \( A\mathbf{x} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \) for arbitrary values of \( b_1 \) and \( b_2 \).

3. If \( \mathbf{x}^1 \) and \( \mathbf{x}^2 \) are solutions of \( A\mathbf{x} = \mathbf{b} \), what equation does \( \mathbf{x}^1 + \mathbf{x}^2 \) satisfy?

4. Explain why a system of equations \( A\mathbf{x} = \mathbf{b} \) cannot have exactly two distinct solutions.

3.4–Elementary Operations, Equivalent Systems

Two systems of \( m \) equations in \( n \) unknowns, \( A\mathbf{x} = \mathbf{b} \) and \( \hat{A}\mathbf{x} = \hat{\mathbf{b}} \) are called equivalent if they have the same set of solutions, or equivalently, if every solution of \( A\mathbf{x} = \mathbf{b} \) is a solution of \( \hat{A}\mathbf{x} = \hat{\mathbf{b}} \) and vice-versa. There are three elementary operations on a system of equations which produce an equivalent system. They are type I: interchange of two equations, type II: multiply an equation by a nonzero constant, and type III: add a multiple of one equation to another. These are used in the method of Gaussian elimination to produce an equivalent system which is easy to solve.

Instead of performing elementary operations on the equations of a system \( A\mathbf{x} = \mathbf{b} \), we may perform elementary row operations on the augmented matrix \( [A \mid \mathbf{b}] \). These elementary row operations are:

Type I. Interchange two rows \( (r_i \leftrightarrow r_j) \).

Type II. Multiply a row by a non-zero constant \( (c\mathbf{r}_i, c \neq 0) \).

Type III. Add a multiple of one row to another \( (c\mathbf{r}_i + r_j) \).

**Definition 1.** If a matrix \( C \) is obtained from a matrix \( B \) by a sequence of elementary row operations, then \( C \) is said to be row equivalent to \( B \).

The following theorem shows the connection between row equivalence and solving systems of equations.

**Theorem 1.** If the augmented matrix \( [\hat{A} \mid \hat{\mathbf{b}}] \) is row equivalent to the augmented matrix \( [A \mid \mathbf{b}] \), the corresponding systems \( \hat{A}\mathbf{x} = \hat{\mathbf{b}} \) and \( A\mathbf{x} = \mathbf{b} \) are equivalent (i.e. have the same solutions).

**Proof** It is obvious that elementary row operations of types I and II do not change the solutions of the corresponding systems. If the row operation of type III is applied to the matrix \( [A \mid \mathbf{b}] \), to produce \( [\hat{A} \mid \hat{\mathbf{b}}] \), only the \( j^{th} \) rows differ, say the new \( j^{th} \) row is \( r'_j = c\mathbf{r}_i + r_j \). By performing the operation \( -c\mathbf{r}_i + r'_j = r_j \), the original \( j^{th} \) row is obtained. This implies that any solution of \( A\mathbf{x} = \mathbf{b} \) is a solution of \( \hat{A}\mathbf{x} = \hat{\mathbf{b}} \) and conversely.

The following example illustrates how we may use elementary operations on a system to obtain another system that is easy to solve. Consider the system

\[
\begin{align*}
x_1 + x_2 + x_3 + x_4 &= 4 \\
x_1 + x_2 - x_3 - 3x_4 &= -2 \\
3x_1 + 3x_2 + x_3 - x_4 &= 6
\end{align*}
\]

The corresponding augmented matrix is

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 4 \\
1 & 1 & -1 & -3 & -2 \\
3 & 3 & 1 & -1 & 6
\end{bmatrix}.
\]
We use the circled element as a pivot and zero out the elements below it by doing the operations $-r_1 + r_2$ and $-3r_1 + r_3$ to obtain:

$$\begin{bmatrix}
1 & 1 & 1 & 1 & 4 \\
0 & 0 & 2 & -4 & -6 \\
0 & 0 & -2 & -4 & -6
\end{bmatrix}. \quad (1)$$

Next use the circled $-2$ as a pivot and perform $-r_2 + r_3$:

$$\begin{bmatrix}
1 & 1 & 1 & 1 & 4 \\
0 & 0 & 2 & -4 & -6 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}. \quad (2)$$

As we shall see in the next section this matrix is in a row echelon form. If we write down the equations corresponding to this matrix, it will be clear how to obtain the solutions. The equations are

$$\begin{align*}
x_1 + x_2 + x_3 + x_4 &= 4 \\
-2x_3 - 4x_4 &= -6,
\end{align*}$$

We see that we may take $x_4$ and $x_2$ as arbitrary, or free variables, and then we may obtain $x_3$ and $x_1$, called basic variables, in terms of the free variables. The process of solving the second equation for $x_3$, and substituting it into the first equation, and solving for $x_1$ is called back substitution. Rather than doing this we shall perform some further row operations on the matrix in (3). First we make the second pivot equal to one by performing $-\frac{1}{2}r_2$

$$\begin{bmatrix}
1 & 1 & 1 & 1 & 4 \\
0 & 0 & 1 & 2 & 3 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}. \quad (3)$$

Finally we perform $-r_2 + r_1$ to get

$$\begin{bmatrix}
1 & 1 & 0 & -1 & 1 \\
0 & 0 & 1 & 2 & 3 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}. \quad (4)$$

This matrix is said to be in reduced row echelon form. If we write the equations corresponding to this augmented matrix we get

$$\begin{align*}
x_1 + x_2 - x_4 &= 1 \\
x_3 + 2x_4 &= 3
\end{align*}$$

It is now clear that the basic variables $x_1$, $x_3$, corresponding to the pivot columns, may be solved in terms of the free variables, $x_2$ and $x_4$, corresponding to the columns without pivots (and to the left of the vertical line). We have the so called terminal equations

$$\begin{align*}
x_1 &= 1 - x_2 + x_4 \\
x_3 &= 3 - 2x_4
\end{align*}$$

In vector form the solutions are

$$x = \begin{bmatrix} 1 - x_2 + x_4 \\ x_2 \\ 3 - 2x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}.$$
It is a good idea to get into the habit of checking solutions

\[
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & -1 & -3 \\
3 & 3 & 1 & -1
\end{bmatrix}
\begin{bmatrix}
1 \\
0 \\
3 \\
0
\end{bmatrix}
+ x_2
\begin{bmatrix}
-1 \\
1 \\
0 \\
0
\end{bmatrix}
+ x_4
\begin{bmatrix}
1 \\
0 \\
-2 \\
1
\end{bmatrix}
= 
\begin{bmatrix}
4 \\
-2 \\
-6 \\
6
\end{bmatrix}
\]

Exercises 3.4

1. If the matrix \( B \) is obtained from \( A \) by performing an elementary operation of type I, what operation must be performed on \( B \) to get back \( A \)? How about an operation of type II?

2. a. Show \( A \) is row equivalent to itself.
   
   b. Show that if \( A \) is row equivalent to \( B \) then \( B \) is row equivalent to \( A \).
   
   c. Show that if \( A \) is row equivalent to \( B \) and \( B \) is row equivalent to \( C \) then \( A \) is row equivalent to \( C \).

3.5–Row Echelon Form (REF) and Reduced Row Echelon Form (RREF)

**Definition 1.** A matrix is said to be in **row echelon form** (REF) if the following two conditions hold:

1. The first nonzero entries (called pivots) in each row move to the right as you move down the rows.
2. Any zero rows are at the bottom.

**Example 1.**

\[
A = \begin{bmatrix}
0 & 2 & 2 & 4 & -2 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & -2 & 6
\end{bmatrix}, \quad B = \begin{bmatrix}
0 & 2 & 2 & 4 & -2 \\
0 & 0 & 0 & -2 & 6 \\
0 & 0 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

The matrix \( A \) is not in row echelon form; it violates both conditions. The matrix \( B \) is in REF.

**Definition 2.** 2. A matrix is said to be in **reduced row echelon form** (RREF) if it is in row echelon form and in addition:

3. All the pivots = 1.
4. The elements above (and therefore below) each pivot = 0.

**Example 2.**

\[
C = \begin{bmatrix}
0 & 1 & -3 & 2 & 1 \\
0 & 0 & 0 & 1 & -3 \\
0 & 0 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}, \quad D = \begin{bmatrix}
0 & 1 & -3 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

The matrix \( C \) is not in RREF, although it is in REF, while the matrix \( D \) is in RREF.

**Theorem 1.** Every matrix can be reduced to REF or RREF by a sequence of elementary row operations.

**Proof** The algorithm for the reduction is:

**Part I. Reduction to REF (The Forward Pass):**

Step 1. (a) Find the leftmost nonzero column and pick a nonzero entry. If there is none you are through.
Section 3.5–Row Echelon Form (REF) and Reduced Row Echelon Form (RREF)  

(b) Bring this nonzero entry or pivot to the first row by interchanging rows.
(c) Zero out all entries below this pivot by adding a multiple of this row to all rows below it.

Step 2. Ignore the first row of the matrix obtained in Step 1. and repeat Step 1 on the remaining matrix.

Step 3. Ignore the first two rows and repeat Step 1. Continue in this manner until the remaining matrix consists entirely of zero rows or you run out of rows.

Clearly this produces a matrix in REF. To get the matrix into RREF proceed to the next part of the algorithm:

Part II. Reduction to RREF (The Backward Pass): Starting with the pivot in the last nonzero row and working upwards do

Step 4. Zero out all elements above the pivot.

Step 5. Make the pivot equal to 1 by dividing each element in the row by the pivot.

This produces a matrix in RREF.

One example of the reduction was given in the last section. Here is another example.

\[
A = \begin{bmatrix}
0 & 2 & 3 & 1 \\
3 & 1 & -1 & 1 \\
2 & -2 & -1 & -1
\end{bmatrix}
\]

There are many different ways to use elementary operations to get to a REF. For this simple integer matrix, using hand calculations, we have chosen a way that avoids fractions until the last few steps.

\[
\begin{align*}
& r_1 \leftrightarrow r_2 \\
& -r_3 + r_1, \\
& -2r_1 + r_3
\end{align*}
\]

\[
\begin{bmatrix}
3 & 1 & -1 & 1 \\
0 & 2 & 3 & 1 \\
2 & -2 & -1 & -1
\end{bmatrix}
\]

This is now in a REF. Now we proceed to the RREF:

\[
\begin{align*}
& \frac{1}{11} r_3 \\
& -3r_3 + r_2, \\
& \frac{1}{2} r_2
\end{align*}
\]

\[
\begin{bmatrix}
1 & 3 & 0 & 2 \\
0 & 2 & 3 & 1 \\
0 & 0 & 1 & -1/11
\end{bmatrix}
\]

One last step and we have the RREF.

\[
-3r_2 + r_1
\]

\[
\begin{bmatrix}
1 & 0 & 0 & 1/11 \\
0 & 1 & 0 & 7/11 \\
0 & 0 & 1 & -1/11
\end{bmatrix}
\]

Although there are many different row echelon forms for a given matrix, the RREF is unique. We present the following theorem without proof.

**Theorem 2.** If \( A \) is a matrix, the RREF of \( A \) is unique, that is, it is independent of the sequence of elementary row operations used to obtain it. In particular the number of nonzero rows in the RREF is also unique.

It follows that the positions of the pivots, and the number of nonzero rows in any REF are unique.
Definition 3. If $A$ is a matrix, the number of nonzero rows in any REF for $A$ is called the rank of $A$, denoted by rank $(A)$. The columns of a REF that contain the pivots are called basic columns of $A$.

The following properties of the rank of a matrix follow easily from the definitions and Theorem 1.

Corollary. If $A$ is an $m \times n$ matrix, then
1. rank $(A) = $ the number of nonzero rows in any REF for $A$.
2. rank $(A) = $ the number of basic columns in any REF for $A$.
3. rank $(A) = $ the number of pivots in any REF for $A$.
4. rank $(A) \leq m$ (the number of rows in $A$).
5. rank $(A) \leq n$ (the number of columns in $A$).
6. If $A$ is row equivalent to $B$, then rank $(A) = $ rank $(B)$.

Example 3. The 0 matrix has rank 0. There are no basic columns.

Example 4. The $n \times n$ identity matrix $I$ has rank $n$. Every column is basic.

Example 5. Consider $A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 7 & 6 \end{bmatrix}$.

To find the rank we reduce it to REF:

$$-2r_1 + r_2 \begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & -2 \end{bmatrix}$$

Since there are two nonzero rows in a REF $(A)$, we have rank $(A) = 2$. The first two columns are of $A$ are basic.

Although the rank of a matrix is defined in terms of its RREF, the rank tells us something about the matrix itself. As we shall see later, the rank of a matrix is the number of ‘independent’ rows in the matrix, which is also equal to the number of ‘independent columns’, in fact the basic columns of the original matrix form an ‘independent’ set.

Exercises 3.5

1. What is the rank of a $m \times n$ matrix all of whose rows are identical and nonzero?
2. What is the rank of a $m \times n$ matrix all of whose columns are identical and nonzero?
3. Describe the RREF for the matrices in the preceding two exercises.
4. By inspection, find the rank of the following:
   a. $\begin{bmatrix} 2 & 3 & 4 & 5 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 4 & -8 \end{bmatrix}$
   b. $\begin{bmatrix} 2 & 3 & 4 & 5 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 4 & -9 \end{bmatrix}$
5. What is the RREF of an $n \times n$ matrix of rank $n$?

3.6–Solution of Systems of Equations

It is now easy to describe how to solve a system of equations. Given a system $Ax = b$, form the augmented matrix $[A | b]$ and find a REF of $[A | b]$. From a REF $(A)$ one can see whether or not the equations are consistent.

Theorem 1. The system $Ax = b$ is consistent if and only if any one of the following equivalent conditions hold:
Section 3.6–Solution of Systems of Equations

(a) The matrix REF (\([A | b]\)) contains no ‘bad rows’ of the form \([0, 0, \cdots, 0 | c]\) with \(c \neq 0\).

(b) The last column in REF (\([A | b]\)) is not a basic column.

(c) \(\text{rank}(A) = \text{rank}([A | b]).\)

Proof (a) A bad row \([0, 0, \cdots, 0 | c]\) with \(c \neq 0\) corresponds to an ‘equation’

\[
0 \cdot x_1 + 0 \cdot x_2 + \cdots + 0 \cdot x_n = c, \ c \neq 0.
\]

Obviously no choice of the variables will satisfy this equation, thus the system is inconsistent. (b) and (c) are simply restatements of (a).

If the equations are consistent and \(r = \text{rank}(A)\), then the variables corresponding to the \(r\) basic columns are basic variables, the remaining \(n - r\) variables are free variables or arbitrary variables. We may solve for the basic variables in terms of the free variables. This can be done from any REF by back substitution (this is called solution by Gaussian elimination) or from the RREF using the ‘terminal equations’ (this is called Gauss-Jordan reduction). The general solution can then be written down in vector form. Here are several examples.

**Example 1.** Consider the system

\[
\begin{align*}
x_1 + x_2 &= 1 \\
x_1 + 3x_2 &= 3 \\
x_1 + 4x_2 &= 0.
\end{align*}
\]

We proceed to reduce the augmented matrix to REF

\[
\begin{bmatrix}
1 & 1 & 1 \\
1 & 3 & 3 \\
1 & 4 & 0
\end{bmatrix},
- r_1 + r_2
\begin{bmatrix}
1 & 1 & 1 \\
0 & 2 & 2 \\
0 & 3 & -1
\end{bmatrix},
- \frac{3}{2} r_1 + r_3
\begin{bmatrix}
1 & 1 & 1 \\
0 & 2 & 2 \\
0 & 0 & -4
\end{bmatrix}.
\]

The last row of the last augmented matrix is a ‘bad row’ thus the system is inconsistent.

**Example 2.** Consider the system

\[
\begin{align*}
x_1 + x_2 + x_3 + x_4 &= 4 \\
x_1 + x_2 - x_3 - 2x_4 &= -1.
\end{align*}
\]

We proceed to reduce the augmented matrix to REF

\[
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & -1 & -2 \\
1 & 1 & -1 & -2
\end{bmatrix},
- r_1 + r_2
\begin{bmatrix}
1 & 1 & 1 & 1 \\
0 & 0 & -2 & -3 \\
0 & 0 & -2 & -3
\end{bmatrix},
- \frac{3}{2} r_1 + r_3
\begin{bmatrix}
1 & 1 & 1 & 1 \\
0 & 0 & -2 & -3 \\
0 & 0 & -2 & -3
\end{bmatrix}.
\]

The last matrix is in REF, it is clear here that the basic variables are \(x_1\) and \(x_3\), the other variables are free. Instead of solving for \(x_3\) and back substituting into the first equation, we shall reduce the matrix to RREF.

\[
- \frac{1}{2} r_2
\begin{bmatrix}
1 & 1 & 1 & 1 \\
0 & 0 & 1 & 3/2 \\
0 & 0 & 1 & 3/2
\end{bmatrix},
- r_2 + r_1
\begin{bmatrix}
1 & 1 & 0 & -1/2 \\
0 & 0 & 1 & 3/2 \\
0 & 0 & 1 & 3/2
\end{bmatrix}.
\]

The terminal equations are

\[
\begin{align*}
x_1 &= \frac{3}{2} - x_2 + x_4/2 \\
x_3 &= \frac{5}{2} - 3x_4/2.
\end{align*}
\]

and the general solution in vector form is

\[
x = \begin{bmatrix}
\frac{3}{2} - x_2 + x_4/2 \\
x_2 \\
\frac{5}{2} - 3x_4/2 \\
x_4
\end{bmatrix} = \begin{bmatrix}
\frac{3}{2} \\
0 \\
\frac{5}{2} \\
0
\end{bmatrix} + x_2 \begin{bmatrix}
-1 \\
1 \\
0 \end{bmatrix} + x_4 \begin{bmatrix}
1/2 \\
0 \\
-3/2 \\
1
\end{bmatrix}.
\]
Letting

\[
\mathbf{x}^p = \begin{bmatrix}
\frac{3}{2} \\
0 \\
\frac{5}{2} \\
0
\end{bmatrix}, \quad \mathbf{v}^1 = \begin{bmatrix}
-1 \\
1 \\
0 \\
0
\end{bmatrix}, \quad \mathbf{v}^2 = \begin{bmatrix}
\frac{1}{2} \\
0 \\
-\frac{3}{2} \\
1
\end{bmatrix},
\]

we note that \( \mathbf{x}^p \) is a particular solution of the nonhomogeneous system, \( \mathbf{v}^1 \) and \( \mathbf{v}^2 \) are solutions of the corresponding homogeneous system, \( \mathbf{x}^h = x_2 \mathbf{v}^1 + x_4 \mathbf{v}^2 \) is the general solution of the homogeneous system and \( \mathbf{x}^p + \mathbf{x}^h \) is the general solution of the nonhomogeneous system.

\textit{Example 3.} Consider the system

\[
\begin{align*}
x_1 + 2x_2 + x_3 &= 3 \\
3x_1 - x_2 - 3x_3 &= -1 \\
2x_1 + 3x_2 + x_3 &= 4
\end{align*}
\]

The augmented matrix and its RREF are (leaving out the details)

\[
[A \mid \mathbf{b}] = \begin{bmatrix}
1 & 2 & 1 & 3 \\
3 & -1 & -3 & -1 \\
2 & 3 & 1 & 4
\end{bmatrix}, \quad \text{RREF} ([A \mid \mathbf{b}]) = \begin{bmatrix}
1 & 0 & 0 & 3 \\
0 & 1 & 0 & -2 \\
0 & 0 & 1 & 4
\end{bmatrix}.
\]

The terminal equations are simply \( x_1 = 3, x_2 = -2 \) and \( x_3 = 4 \), thus the unique solution vector is

\[
\mathbf{x} = \begin{bmatrix}
3 \\
-2 \\
4
\end{bmatrix}.
\]

If we are dealing with a homogeneous system \( A \mathbf{x} = \mathbf{0} \), it is not necessary to write the the augmented matrix \([A \mid \mathbf{0}]\) since the last column will always remain a zero column when reducing to REF; one simply reduces \( A \) to REF.

\textit{Example 4.}

\[
\begin{align*}
x_1 + x_2 &= 0 \\
x_1 - x_2 &= 0 \\
2x_1 + 2x_2 &= 0
\end{align*}
\]

We have (again leaving out the details)

\[
A = \begin{bmatrix}
1 & 1 \\
1 & -1 \\
2 & 2
\end{bmatrix}, \quad \text{RREF} (A) = \begin{bmatrix}
1 & 0 \\
0 & 1 \\
0 & 0
\end{bmatrix}.
\]

The RREF corresponds to the equations \( x_1 = 0, x_2 = 0 \) and \( 0 = 0 \), thus the only solution is the trivial solution \( \mathbf{x} = \mathbf{0} \).

\textit{Example 5.}

\[
\begin{align*}
x_1 + x_2 + x_3 &= 0 \\
x_1 - x_2 + x_3 &= 0 \\
2x_1 + 2x_3 &= 0
\end{align*}
\]

We have

\[
A = \begin{bmatrix}
1 & 1 & 1 \\
1 & -1 & 1 \\
2 & 0 & 2
\end{bmatrix}, \quad \text{RREF} (A) = \begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

The terminal equations are

\[
\begin{align*}
x_1 &= -x_3 \\
x_2 &= 0,
\end{align*}
\]
and the general solution is
\[ x = \begin{bmatrix} -x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}. \]

We now collect some facts about solutions in the following theorem.

**Theorem 2.** If the \( m \times n \) system \( Ax = b \) is consistent and \( r = \text{rank } (A) = \text{rank } ([A \mid b]) \) then
a. if \( r = n \) there is a unique solution.
b. if \( r < n \) there are infinitely many solutions with \( n - r \) free variables. The general solution has the form
\[ x = x^p + x_{t_1}v^1 + x_{t_2}v^2 + \cdots + x_{t_{n-r}}v^{n-r} \]
where \( x_{t_i} \) is the \( i \)th free variable, \( x^p \) is a particular solution of \( Ax = b \) and the \( v^i \) are solutions of the homogeneous equation \( Ax = 0 \) and the general solution of \( Ax = 0 \) is
\[ x^h = x_{t_1}v^1 + x_{t_2}v^2 + \cdots + x_{t_{n-r}}v^{n-r} \]

Although the above theorem holds for any system, it is worthwhile to state the conclusions of the theorem for homogeneous systems.

**Theorem 3.** Consider the \( m \times n \) homogeneous system \( Ax = 0 \) where \( r = \text{rank } (A) \), then
(a) if \( r = n \) the system has only the trivial solution \( x = 0 \).
(b) if \( r < n \) there are infinitely many solutions given in terms of the \( n - r \) free variables. The general solution can be written
\[ x^h = x_{t_1}v^1 + x_{t_2}v^2 + \cdots + x_{t_{n-r}}v^{n-r} \]
where the \( x_{t_i} \) are free variables and the \( v^i \) are solutions of \( Ax = 0 \).

Thus we see that a non trivial solution of \( Ax = 0 \) exists if and only if the rank of \( A \) is less than the number of unknowns. There is one very simple but important case where this occurs given in the following theorem.

**Theorem 4.** The \( m \times n \) system \( Ax = 0 \) with \( m < n \) (more unknowns than equations) always has a nontrivial solution.

*Proof* rank \((A) \leq m \) but \( m < n \), therefore rank \((A) < n \).

**Exercises 3.6**

In problems 1–5 determine whether or not the systems are consistent. If consistent find the general solution in vector form and check.

1. \( 3x + 4y = 4 \)
2. \( x - y - z + w = 0 \)
3. \( 3x_1 - 6x_2 + 7x_3 = 0 \)
4. \( 9x + 12y = 6 \)
   \( x + 2y - z - w = 1 \)
   \( x - 2y + z/2 + w/2 = 0 \)
   \( 2x_1 - x_2 + x_3 = 1 \)
   \( 7x_1 + x_2 - 6x_3 = 2 \)
   \( 2x_1 + 2x_2 - 4x_3 = 1 \)

5. \( x_1 + x_2 + 2x_3 + 2x_5 = 1 \)
6. \( x_2 + x_3 + x_5 = -2 \)
7. \( 2x_2 + 3x_3 + x_4 + 4x_5 = 3 \)
8. \( x_1 + x_2 + 2x_3 + 2x_5 = 1 \)

In problems 6–8 do only as much work as necessary to determine whether or not the equations are consistent.

6. \[
\begin{bmatrix}
2 & -4 & 2 \\
-5 & 10 & 1
\end{bmatrix}
\]
7. \[
\begin{bmatrix}
1 & -5 & 9 & 1 \\
2 & 6 & 7 & 6
\end{bmatrix}
\]
8. \[
\begin{bmatrix}
1 & 0 & -1 & 1 \\
0 & 1 & -1 & 2 \\
0 & 1 & 1 & 0
\end{bmatrix}
\]

9. Solve $Ax = 0$ for the following matrices $A$, write the solution in vector form and check.
   a. $\begin{bmatrix} 0 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix}$  
   b. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$  
   c. $\begin{bmatrix} 1 & 3 & 0 & 5 & 6 \\ -2 & -6 & 2 & -14 & -8 \\ 1 & 3 & 2 & 3 & 10 \end{bmatrix}$

10. For each of the following matrices $A$ do only as much work as necessary to determine whether or not $Ax = 0$ has a nontrivial solution.
   a. $\begin{bmatrix} 1 & 2 & \sqrt{2} \\ 2 & 1 & 4 \end{bmatrix}$  
   b. $\begin{bmatrix} -2 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$  
   c. $\begin{bmatrix} 1 & -2 \\ 2 & 3 \\ 5 & 6 \\ 9 & 27 \end{bmatrix}$

11. For each coefficient matrix below find the values of the parameter $a$, if any, for which the corresponding homogeneous systems have a nontrivial solution.
   a. $\begin{bmatrix} 1 & 2 \\ -2 & a \end{bmatrix}$  
   b. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 4 & 5 & a \end{bmatrix}$

12. Can a system of 3 equations in 5 unknowns have a unique solution? Explain.

3.7–More on Consistency of Systems

For a given matrix $A$ it is possible for the system $Ax = b$ to have a solution for some right hand side vectors $b$ and not have a solution for other $b$. We wish to characterize those vectors $b$ for which $Ax = b$ is consistent.

Example 1. Find the conditions on $b_1$, $b_2$, $b_3$, if any, such that the following system is consistent.

\[
\begin{align*}
x_1 + 2x_2 + 3x_3 &= b_1 \\
2x_1 + 3x_2 + 4x_3 &= b_2 \\
3x_1 + 4x_2 + 5x_3 &= b_3 
\end{align*}
\]

The augmented matrix and a REF are

\[
[A | b] = \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 2 & 3 & 4 & b_2 \\ 3 & 4 & 5 & b_3 \end{bmatrix}, \quad \text{REF}([A | b]) = \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 0 & 1 & 2 & 2b_1 - b_2 \\ 0 & 0 & 0 & b_3 - 2b_2 + b_1 \end{bmatrix}.
\]

We see that the system has a solution if and only if $b_3 - 2b_2 + b_1 = 0$.

The following theorem shows one situation when we can guarantee that $Ax = b$ is consistent for every $b$.

Theorem 1. If $A$ is a $m \times n$ matrix then the system $Ax = b$ is consistent for every $b$ if and only if $\text{rank}(A) = m$.

Proof If $\text{rank}(A) = m$ then every row of any $\text{REF} (A)$ contains a pivot, and thus there cannot be any ‘bad’ rows in $\text{REF} ([A | b])$ and the system must be consistent. Conversely if the system is consistent for every $b$, there can not be a zero row in a $\text{REF} (A)$ and thus $\text{rank}(A) = m$.

The existence of solutions in the important special case when the number of equations is the same as the number of unknowns deserves separate mention.

Theorem 2. If $A$ is an $n \times n$ matrix, the system $Ax = b$ has a unique solution for every $b$ if and only if any one of the following hold

1. $\text{rank}(A) = n$.
2. $\text{RREF}(A) = I$ or $A$ is row equivalent to $I$. 

3. $Ax = 0$ has only the trivial solution.
4. Every column of $A$ is a basic column.

Proof From Theorem 1 we know that $Ax = b$ has a solution for every $b$ if and only if $\text{rank } (A) = n = \text{the number of rows}$. From Theorem 2 of the last section we have that the solution is unique if and only if $\text{rank } (A) = n = \text{the number of columns}$. This proves statement 1. However we already know that statement 1 is equivalent to statements 2, 3 and 4.

The following theorem for the corresponding homogeneous system follow immediately from Theorem 2.

**Theorem 3.** If $A$ is an $n \times n$ matrix, the system $Ax = 0$ has a nontrivial solution if and only if any one of the following hold
1. $\text{rank } (A) < n$.
2. $\text{RREF } (A) \neq I$ or $A$ is not row equivalent to $I$.
3. $Ax = b$ does not have a solution for every $b$, and if a solution exists for a particular $b$, it is not unique.
4. At least one column of $A$ is not a basic column.

There is another way to look at the matrix product $Ax$ which will need to get our last characterization of consistency. If $A$ is an $m \times n$ matrix and $x$ is an $n \times 1$ vector we have

$$Ax = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}$$

We may factor out the $x_i$’s to obtain

$$Ax = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} x_1 + \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} x_2 + \cdots + \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} x_n. \tag{1}$$

Thus $Ax$ is a linear combination of the columns of $A$ weighted by the components of $x$. If we use the notation $c_j(A)$ for the $j^{th}$ column of $A$ we have

$$Ax = c_1(A)x_1 + c_2(A)x_2 + \cdots + c_n(A)x_n. \tag{2}$$

From Equation 2 we can deduce the following simple criterion for existence of a solution of $Ax = b$

**Theorem 4.** If $A$ is an $m \times n$ matrix then $Ax = b$ has a solution if and only if $b$ is a linear combination of the columns of $A$.

Proof From Equation 2, $Ax = b$ can be written as

$$c_1(A)x_1 + c_2(A)x_2 + \cdots + c_n(A)x_n = b. \tag{3}$$

Clearly a solution exists if and only if $b$ is a linear combination of the columns of $A$.

**Example 2.** Suppose $A$ is a $3 \times 4$ matrix and $b = -3c_1(A) + 5c_3(A)$. What is a solution to $Ax = b$?

From Equation 3, we see that $Ax = b$ can be written

$$c_1(A)x_1 + c_2(A)x_2 + c_3(A)x_3 + c_4(A)x_4 = b.$$
Clearly this equation is satisfied by \( x_1 = -3, x_3 = 5 \) and the other \( x_i \)'s = 0, so a solution vector is

\[
\mathbf{x} = \begin{bmatrix} -3 \\ 0 \\ 5 \\ 0 \end{bmatrix}.
\]

For the homogeneous system \( A\mathbf{x} = 0 \), Equation 3 still holds with \( \mathbf{b} = 0 \)

\[
c_1(A)x_1 + c_2(A)x_2 + \cdots + c_n(A)x_n = 0
\]

(4)

The following result follows immediately

**Theorem 5.** If \( A \) is an \( m \times n \) matrix., \( A\mathbf{x} = 0 \) has a nontrivial solution if and only if at least one column of \( A \) is a linear combination of the other columns.

**Example 3.** Suppose \( A \) is a \( 3 \times 4 \) matrix and \( c_1(A) = 5c_3(A) - 3c_2(A) \). What is a nontrivial solution to \( A\mathbf{x} = 0 \)?

From Equation 3, we see that \( A\mathbf{x} = 0 \) can be written

\[
c_1(A)x_1 + c_2(A)x_2 + c_3(A)x_3 + c_4(A)x_4 = 0.
\]

Rewriting the given linear relation among the columns we have

\[
c_1(A) + 3c_2(A) - 5c_4(A) = 0.
\]

Comparing the two we find a nontrivial solution is \( \mathbf{x} = \begin{bmatrix} 1 \\ 3 \\ 0 \\ -5 \end{bmatrix} \).

In order to improve Theorem 5, we must take a closer look at the information about the columns of a matrix \( B \) that can be obtained from the columns of \( \text{RREF} (B) \). An example will make it clear. Let

\[
B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 3 \\ 3 & 3 & 1 & -1 \\ 4 & -2 & 6 \end{bmatrix}.
\]

Then \( \hat{B} = \text{RREF} (B) \) is

\[
\hat{B} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}.
\]

The homogeneous system \( B\mathbf{x} = 0 \) can be written

\[
c_1(B)x_1 + c_2(B)x_2 + \cdots + c_n(B)x_n = 0
\]

(5)

and the system \( \hat{B}\mathbf{x} = 0 \) can be written

\[
\hat{c}_1(B)x_1 + \hat{c}_2(B)x_2 + \cdots + \hat{c}_n(B)x_n = 0
\]

(6)

Since the two systems have the same solutions, Equations 5 and 6 show that if any column of \( B \) is a linear combination of the other columns, the corresponding column of \( \hat{B} \) is the same linear combination of the columns of \( \hat{B} \) and conversely. Looking at \( \hat{B} \), it is easy to see that

\[
\hat{c}_2(B) = \hat{c}_1(B),
\]
\[
\hat{c}_4(B) = -\hat{c}_1(B) + 2\hat{c}_3(B),
\]

and \( \hat{c}_5(B) = \hat{c}_1(B) + 3\hat{c}_3(B) \).
Thus we must have, as can easily be verified

\[ c_2(B) = c_1(B), \]
\[ c_4(B) = -c_1(B) + 2c_3(B), \]
\[ \text{and } c_5(B) = c_1(B) + 3c_3(B). \]

In other words, every non–basic column of \( \hat{B} \) is a linear combination of the basic columns of \( \hat{B} \) which lie to the left of it and therefore every non–basic column of \( B \) is a linear combination of the basic columns of \( B \) which lie to the left of it. This is true generally as is indicated in the following theorem.

**Theorem 6.** In any matrix, the non–basic columns are linear combination of the basic columns that lie to the left of it.

The improved version of Theorem 3 may now be stated.

**Theorem 7.** If \( A \) is an \( m \times n \) matrix then \( Ax = b \) has a solution if and only if \( b \) is a linear combination of the basic columns of \( A \).

**Proof** We know that \( Ax = b \) is consistent if and only if \( b \) is not a basic column. Therefore the result follows from Theorem 4.

**Example 4.** Consider the system \( Ax = b \) where

\[
\begin{bmatrix}
1 & 2 & 3 \\
2 & 3 & 4 \\
3 & 4 & 5
\end{bmatrix}
\]

Determine those vectors \( b \) for which the system is consistent.

**Solution** By computing any REF \( (A) \) we find that the first two columns are basic columns. Therefore a solution exists if and only if \( b \) is a linear combination of the first two columns of \( A \)

\[ b = \alpha \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \beta \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} \alpha + 2\beta \\ 2\alpha + 3\beta \\ 3\alpha + 4\beta \end{bmatrix}. \tag{7} \]

We solved this same example by a different method in Example 1 where we found that the components of \( b \) must satisfy \( b_3 - 2b_2 + b_1 = 0 \). We find that the components of \( b \) given in Equation (7) do indeed satisfy this relation.

**Exercises 3.7**

1. If \( A \) is an \( m \times n \) matrix and all the columns of \( A \) are identical and nonzero, describe those columns \( b \) for which \( Ax = b \) is consistent.

2. If \( A \) is an \( m \times n \) matrix and all the rows of \( A \) are identical and nonzero, describe those columns \( b \) for which \( Ax = b \) is consistent.

3. Find the conditions on \( b_1 \), \( b_2 \), \( b_3 \) if any, so that the following systems have solutions. Find the solutions and check.

\[ \begin{align*}
x_1 + x_2 + x_3 &= b_1 \\
x_1 - x_2 + x_3 &= b_2 \\
x_1 + x_2 - x_3 &= b_3
\end{align*} \]

a. \[ x_1 + x_2 - x_3 = b_1 \]

b. \[ 2x_1 - x_2 + x_3 = b_2 \]

\[ 4x_1 + x_2 - x_3 = b_3 \]
4. Suppose $A$ is $m \times n$ and $\text{rank}(A) = r$. What conditions must exist among the three numbers $r$, $m$ and $n$ in each of the following cases.

   a. $Ax = 0$ has only the trivial solution.

   b. $Ax = b$ has a solution for some $b$ but does not have a solution for some other $b$, but when a solution exists it is unique.

   c. $Ax = b$ has a solution for every $b$, but the solution is not unique.

5. If $A$ is an $3 \times 4$ matrix and all the columns of $A$ are identical and nonzero, what is a nontrivial solution of $Ax = 0$?

6. If $A$ is $3 \times 5$ and $c_1(A) = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ and rank $(A) = 1$, describe those vectors $b$ for which $Ax = b$ has a solution.

7. Suppose $A$ and $\text{RREF}(A)$ are

   \[ A = \begin{bmatrix} 2 & -4 & -8 & 6 & 3 \\ 0 & 1 & 3 & 2 & 3 \\ 3 & -2 & 0 & 0 & 8 \end{bmatrix}, \quad \text{RREF}(A) = \begin{bmatrix} 1 & 0 & 2 & 0 & 4 \\ 0 & 1 & 3 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1/2 \end{bmatrix}, \]

   a. Express each nonbasic column of $A$ as a linear combination of basic columns.

   b. For what vectors $b$ does $Ax = b$ have a solution?

3.8–Matrix Algebra

Suppose $A$ is an $m \times n$ matrix. For each column vector $x \in \mathbb{C}^n$ (or $\mathbb{R}^n$), the product $Ax$ is a vector in $\mathbb{C}^m$ (or $\mathbb{R}^m$). Thus we may think of multiplication by $A$ (or $A$ itself) as a transformation (or function) which transforms vectors in $\mathbb{C}^n$ into vectors in $\mathbb{C}^m$.

Since $A0 = 0$ we know that the zero vector in $\mathbb{C}^n$ is transformed into the zero vector in $\mathbb{C}^m$. We also have the “linearity” property

\[ A(\alpha x + \beta y) = \alpha Ax + \beta Ay \] (1)

for all $x$, $y \in \mathbb{C}^n$ and all scalars $\alpha$, $\beta$. Thus linear combinations of $x$ and $y$ are transformed into the same linear combinations of $Ax$ and $Ay$. This is illustrated in Figure 1.

![Figure 1](image-url)
Thinking of $A$ as a transformation provides us with a natural way to define various operations on matrices. First we need the following fact.

**Theorem 1.** If $A$ and $B$ are $m \times n$ matrices, then $A = B$ if and only if $Ax = Bx$, for all $x$.

**Proof** Clearly if $A = B$, then $Ax = Bx$. Conversely if $Ax = Bx$ then

$$Ax = c_1(A)x_1 + \ldots + c_n(A)x_n = Bx = c_1(B)x_1 + \ldots + c_n(B)x_n$$

where $c_j(A)$ and $c_j(B)$ are the $j^{th}$ columns of $A$ and $B$ respectively. Setting $x_1 = 1$ and $x_2 = x_3 = \ldots = x_n = 0$ we see that $c_1(A) = c_1(B)$. In a similar way we can show that $c_j(A) = c_j(B)$ for $j = 2, \ldots, n$. Thus $A = B$.

**Definition 1.** The sum of two $m \times n$ matrices $A$, $B$ is the $m \times n$ matrix $A + B$ defined by

$$(A + B)x = Ax + Bx \quad \text{for all } n \times 1 \text{ vectors } x.$$  \hfill (2)

**Theorem 2.** If $A = [a_{ij}]$ and $B = [b_{ij}]$ are $m \times n$ matrices then

$$A + B = [a_{ij} + b_{ij}]$$  \hfill (3)

Theorem 2 states that to add two matrices one simply adds corresponding elements.

**Proof**

$$(A + B)x = Ax + Bx$$

$$= \begin{bmatrix} a_{11}x_1 + \ldots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + \ldots + a_{mn}x_n \end{bmatrix} + \begin{bmatrix} b_{11}x_1 + \ldots + b_{1n}x_n \\ \vdots \\ b_{m1}x_1 + \ldots + b_{mn}x_n \end{bmatrix}$$

$$= \begin{bmatrix} (a_{11} + b_{11})x_1 + \ldots + (a_{1n} + b_{1n})x_n \\ \vdots \\ (a_{m1} + b_{m1})x_1 + \ldots + (a_{mn} + b_{mn})x_n \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} + b_{11} & \ldots & a_{1n} + b_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} + b_{m1} & \ldots & a_{mn} + b_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$  

The result now follows from Theorem 1.

Notice that only matrices of the same size can be added.

**Definition 2.** If $A$ is an $m \times n$ matrix and $\alpha$ is a scalar then $\alpha A$ is the $m \times n$ matrix defined by

$$(\alpha A)x = \alpha (Ax) \text{ for all } x.$$  \hfill (4)

**Theorem 3.** If $A = [a_{ij}]$ then

$$\alpha A = \alpha [a_{ij}] = [\alpha a_{ij}].$$  \hfill (5)

This states that to multiply a matrix by a scalar, each element is multiplied by the scalar. The proof of Theorem 3 is left to the reader.
Chapter III–MATRICES AND SYSTEMS OF EQUATIONS

Example 1.

\[
\begin{bmatrix}
1 & -3 & 4 \\
0 & 5 & 6
\end{bmatrix} + 2 \begin{bmatrix}
0 & 1 & 4 \\
5 & -6 & 7
\end{bmatrix} = \begin{bmatrix}
1 & -3 & 4 \\
0 & 5 & 6
\end{bmatrix} + \begin{bmatrix}
0 & 2 & 8 \\
10 & -12 & 14
\end{bmatrix} = \begin{bmatrix}
1 & -1 & 12 \\
10 & -7 & 20
\end{bmatrix}
\]

The properties in the following theorem follow easily from Theorems 2 and 3.

Theorem 4. If \( A, B, C, O \) are \( m \times n \) matrices and \( \alpha, \beta \) are scalars then

1. \( A + B = B + A \) (commutative law of addition)
2. \( (A + B) + C = A + (B + C) \) (associative law of addition)
3. \( A + O = A \)
4. \( (\alpha + \beta)A = \alpha A + \beta A \)
5. \( \alpha O = O \)
6. \( \alpha A = O \), if \( \alpha = 0 \).

Suppose \( x \in \mathbb{C}^p \). If \( B \) is an \( n \times p \) matrix then \( Bx \in \mathbb{C}^n \). If \( A \) is an \( m \times n \) matrix we may form \( A(Bx) \in \mathbb{C}^n \). This leads us to the following definition.

Definition 3. If \( A \) is an \( m \times n \) and \( B \) is an \( n \times p \) matrix, \( AB \) is defined to be the \( m \times p \) matrix such that

\[
(AB)x = A(Bx) \quad \text{for all } x \in \mathbb{C}^p.
\] (6)

Note that the product \( AB \) is only defined if the number of columns in the left hand factor is the same as the number of rows in the right hand factor. The diagram in Figure 2 should be kept in mind.

![Diagram](image)

Figure 2

Theorem 5. If \( A \) is an \( m \times n \) and \( B \) is an \( n \times p \) matrix then the product \( AB \) is an \( m \times p \) matrix and

a. The \( (i, j)^{th} \) element of \( AB = r_i(A)c_j(B) \), where

\[
r_i(A)c_j(B) = [a_{i1} \ldots a_{in}] \begin{bmatrix} b_{1j} \\ \vdots \\ b_{nj} \end{bmatrix} = \sum_{k=1}^{n} a_{ik}b_{kj}, \quad \begin{cases} i = 1, \ldots, m \\ j = 1, \ldots, p \end{cases}
\]

b. \( c_j(AB) = Ac_j(B) \), \( j = 1, \ldots, p \), or, \( AB = [Ac_1(B), \ldots, Ac_p(B)] \).

c. \( r_i(AB) = r_i(A)B \), \( i = 1, \ldots, m \), or \( AB = \begin{bmatrix} r_1(A)B \\ \vdots \\ r_m(A)B \end{bmatrix} \).
Before proving this theorem let us look at a numerical example

**Example 2.** Let

\[
A = \begin{bmatrix} 1 & -1 & 2 & 0 \\ -2 & 0 & 1 & 3 \\ 1 & 5 & 2 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & -1 \\ 2 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.
\]

Using (a) in Theorem 5 we have

\[
C = AB = \begin{bmatrix} 1 & -1 & 2 & 0 \\ -2 & 0 & 1 & 3 \\ 1 & 5 & 2 & -2 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 2 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ -5 & 5 \\ 15 & 7 \end{bmatrix}.
\]

where for instance \(c_{32} = r_3(A)c_2(B) = [1 \ 5 \ 2 \ -2] \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix} = 7.\)

Using (b) of Theorem 5, let us compute the 2\(^{nd}\) column of \(AB\)

\[
c_2(AB) = Ac_2(B) = \begin{bmatrix} 1 & -1 & 2 & 0 \\ -2 & 0 & 1 & 3 \\ 1 & 5 & 2 & -2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \\ 7 \end{bmatrix}.
\]

Using (c) of Theorem 5, let us find the 3\(^{rd}\) row of \(AB\).

\[
r_3(AB) = r_3(A)B = [1 \ 5 \ 2 \ -2] \begin{bmatrix} 3 & -1 \\ 2 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = [15 \ 7].
\]

**Proof of Theorem 5.** We first prove b.

\[
(AB)x = A(Bx) = A(c_1(B)x_1 + \ldots + c_p(B)x_n)
\]

\[
= A(c_1(B)x_1) + \ldots + A(c_p(B)x_n)
\]

\[
= (Ac_1(B))x_1 + \ldots + (Ac_p(B))x_n
\]

\[
= [Ac_1(B), Ac_2(B), \ldots, Ac_p(B)]x
\]

thus we have

\[
AB = [Ac_1(B), \ldots, Ac_p(B)].
\]

To prove a., we look at the \(j\)\(^{th}\) column of \(AB\), \(c_j(AB) = Ac_j(B)\). We want the \(i\)\(^{th}\) element in this \(j\)\(^{th}\) column

\[
c_j(AB) = \begin{bmatrix} r_1(A) \\ \vdots \\ r_m(A) \end{bmatrix}, \quad c_j(B) = \begin{bmatrix} r_1(A)c_j(B) \\ \vdots \\ r_m(A)c_j(B) \end{bmatrix}.
\]

Thus the \((i,j)\)\(^{th}\) element of \(AB = r_i(A)c_j(B)\).

To prove c., we can write the \(i\)\(^{th}\) row of \(AB\)

\[
r_i(AB) = [r_i(A)c_1(B), r_i(A)c_2(B), \ldots, r_i(A)c_p(B)]
\]

\[
= r_i(A)[c_1(B), c_2(B), \ldots, c_p(B)] = r_i(A)B.
\]
Theorem 6. If \( A, B, C \) are of the proper orders so that the indicated products are defined then

1. \( A(BC) = (AB)C \) (associative law of multiplication)
2. \( A(B + C) = AB + AC \) distributive law
3. \( (A + B)C = AC + BC \) distributive law
4. \( A(\alpha B) = \alpha(AB) = (\alpha A)B \) for any scalar \( \alpha \)
5. \( AO = O \)

We shall prove item 1 and then leave the other proofs to the reader.

Proof of item 1. Let \( D = A(BC) \) and \( E = (AB)C \). We shall show that \( Dx = Ex \), for all \( x \). We have, using Definition 3 repeatedly,

\[
Dx = (A(BC))x = A((BC)x) = A(B(Cx)) = (AB)(Cx) = ((AB)C)x = Ex.
\]

There are several important properties that hold for ordinary algebra which do not hold for matrix algebra:

1. \( AB \) is not necessarily the same as \( BA \). Note that if \( A \) is \( m \times n \) and \( B \) is \( n \times p \) then \( AB \) is defined but \( BA \) is not defined unless \( m = p \). Even if \( AB \) and \( BA \) are both defined, they need not be equal. For example

\[
A = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}
\]

\[
AB = \begin{bmatrix} 2 & -2 \\ 7 & -1 \end{bmatrix}, \quad BA = \begin{bmatrix} 1 & -3 \\ 4 & 0 \end{bmatrix}
\]

2. If \( AB = 0 \) it does not necessarily follow that either \( A = 0 \) or \( B = 0 \). For example

\[
\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
\]

3. If \( AB = AC \) it does not necessarily follow that \( B = C \). Note that \( AB = AC \) is equivalent to \( AB - AC = 0 \) or to \( A(B - C) = 0 \). From fact 2 we cannot conclude that \( B = C \) even if \( A \neq 0 \).

4. If \( BA = CA \) it does not necessarily follow that \( B = C \).

Theorem 7. If \( A \) is an \( m \times n \) matrix then

\[
AI_n = A \quad (7)
\]

\[
I_m A = A. \quad (8)
\]

The proof is left to the reader.

Exercises 3.8

1. In a., b. find, if possible (i) \( 5A - 6B \), (ii) \( AB \) and (iii) \( BA \).
   a. \( A = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 0 \\ 2 & 1 \end{bmatrix} \)
   b. \( A = \begin{bmatrix} 2 & -1 \\ -1 & 0 \\ 3 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 1 & 5 & 2 \\ 0 & 4 & -1 & 3 \end{bmatrix} \)

2. Compute, if possible
   a. \( [1, -2] \begin{bmatrix} 2 \\ 3 \end{bmatrix} = [-1, 5] \), b. \( [2, 3] \begin{bmatrix} 1 & -2 \\ -1 & 5 \end{bmatrix} \), c. \( \begin{bmatrix} 1 & 3 & 2 \\ -2 & 5 & 7 \end{bmatrix} \begin{bmatrix} 0 & 0 \end{bmatrix} \), d. \( [x_1, x_2] \begin{bmatrix} 3 & 5 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \)
3. If \( A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \), find \( A^2 \equiv AA \) and \( A^3 \equiv A^2A \).

4. Prove or give a counterexample
   
   a. If the 1st and 3rd rows of \( A \) are the same then the 1st and 3rd rows of \( AB \) are the same.
   
   b. If the 1st and 3rd columns of \( A \) are the same then the 1st and 3rd columns of \( AB \) are the same.
   
   c. If the 1st and 3rd columns of \( B \) are the same then the 1st and 3rd columns of \( AB \) are the same.
   
   d. If the 2nd column of \( B = 0 \) then the 2nd column of \( AB = 0 \).
   
   e. If the first column of \( A = 0 \) then the first column of \( AB = 0 \).

5. Suppose \( A \) is an \( n \times n \) matrix and \( u, \ v \) are \( n \times 1 \) column vectors such that

\[
Au = 3u - 2v, \quad Av = 2u - 3v
\]

Let the \( n \times 2 \) matrix \( T \) be defined by \( T = [u, \ v] \). Find a matrix \( B \) such that \( AT = TB \).

6. Write out \((A + B)^2 = (A + B)(A + B)\).

7. If \( A, \ S, \ T \) are \( n \times n \) matrices and \( TS = I \) simplify \((SAT)^3\).

8. If \( A_{6 \times 4} = [a_{ij}], \ B_{4 \times 6} = [b_{ij}], \ C = AB \) and \( D = BA \),
   
   a. write the expressions for \( c_{23} \) and \( c_{66} \), if possible,
   
   b. write the expressions for \( d_{13} \) and \( d_{56} \), if possible.

3.9–Transposes, Symmetric Matrices, Powers of Matrices

**Definition 1.** If \( A \) is an \( m \times n \) matrix, the transpose of \( A \), denoted by \( A^T \) is the \( n \times m \) matrix formed by interchanging the rows and columns of \( A \).

**Example 1.**

   
   (a) If \( x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \) then \( x^T = [1 \ 2 \ 3] \)
   
   (b) If \( A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 5 \end{bmatrix}, \quad A^T = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 3 & 5 \end{bmatrix} \)
   
   (c) If \( x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \) in \( \mathbb{R}^3 \) then the length of \( x \) denoted by \( |x| \) is defined by

\[
|x| = \sqrt{x_1^2 + x_2^2 + x_3^2},
\]

Note that \( |x|^2 = x^T x \).

   
   (d) The equation of a central conic has the form

\[
ax^2 + bxy + cy^2 = 1.
\]

This can be written

\[
[x, y] \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 1.
\]
In matrix notation the equation becomes
\[ z^T A z = 1 \]
where \( z = \begin{bmatrix} x \\ y \end{bmatrix}, \ A = \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix} \). Note that \( A = A^T \).

**Theorem 1.** If \( A \) is an \( m \times n \) matrix, then

1. \( [A^T]_{ij} = [A]_{ji} \), \( i = 1, \ldots, n \), \( j = 1, \ldots, m \)
2. \( r_i(A^T) = c_i(A), \ i = 1, \ldots, n \)
3. \( c_j(A^T) = r_j(A), \ j = 1, \ldots, m \)
4. \( (A^T)^T = A \)
5. If \( B \) is an \( m \times n \) matrix \( (A + B)^T = A^T + B^T \).

**Proof** Items (1), (2), (3) are simply restatements of the definition; the proofs of the other two items are left up to the reader.

We now consider how to take the transpose of a product of matrices.

**Theorem 2.** If \( A \) is an \( m \times n \), \( B \) is \( n \times p \) and \( x \) is \( n \times 1 \), then

1. \( (Ax)^T = x^T A^T \)
2. \( (AB)^T = B^T A^T \).

**Proof** Item (1) can be proved by writing out both sides. For the proof of (2) we have

\[ (AB)^T = (A[c_1(B), \ldots, c_p(B)]^T \]
\[ = [Ac_1(B), \ldots, Ac_p(B)]^T \]
\[ = \begin{bmatrix} (Ac_1(B))^T \\ \vdots \\ (Ac_p(B))^T \end{bmatrix} \begin{bmatrix} c_1(B)^T A^T \\ \vdots \\ c_p(B)^T A^T \end{bmatrix} = \begin{bmatrix} r_1(B^T)A^T \\ \vdots \\ r_m(B^T)A^T \end{bmatrix} = B^T A^T. \]

**Definition 2.** A matrix (necessarily square) is called symmetric if

\[ A = A^T \]  \hspace{1cm} (1)

and skew-symmetric if

\[ A = -A^T. \]  \hspace{1cm} (2)

**Example 2.**

\[ A = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 3 & 4 \\ 3 & 4 & -6 \end{bmatrix} = A^T \text{ is symmetric} \]

\[ B = \begin{bmatrix} 0 & 1 & 3 \\ -1 & 0 & -2 \\ -3 & 2 & 0 \end{bmatrix} = -B^T \text{ is skew-symmetric}. \]

**Example 3.** Show that \( AA^T \) and \( A^T A \) are symmetric. Note that if \( A \) is \( m \times n \) then \( AA^T \) is \( m \times m \) and \( A^T A \) is \( n \times n \). To show that \( B = AA^T \) is symmetric we must show \( B = B^T \). We have

\[ B^T = (AA^T)^T = (A^T)^T A^T = AA^T = B. \]

The symmetry of \( A^T A \) is shown in the same way.
Powers of a Square Matrix

Square matrices of the same order can always be multiplied. In particular, we can multiply an \( n \times n \) matrix \( A \) by itself

\[
A^2 = A \cdot A
\]

If we write \( A^m \) for the product of a matrix by itself \( m \)-times (\( m \) a positive integer), the following law holds,

\[
A^k A^m = A^{k+m}, \quad k, m \text{ positive integers}
\]

A diagonal matrix is a matrix where all the elements are zero except along the main diagonal

\[
D = \begin{bmatrix}
\lambda_1 & 0 & \cdots & 0 \\
0 & \lambda_2 & \cdots & 0 \\
& \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda_n
\end{bmatrix}
\]

Powers of a diagonal matrix are easy to find.

\[
D^k = \begin{bmatrix}
\lambda_1^k & 0 & \cdots & 0 \\
0 & \lambda_2^k & \cdots & 0 \\
& \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda_n^k
\end{bmatrix}
\]

A diagonal matrix with all its diagonal values equal is called a scalar matrix. If \( \Lambda \) is a scalar matrix, then

\[
\Lambda = \begin{bmatrix}
\lambda & 0 & \cdots & 0 \\
0 & \lambda & \cdots & 0 \\
& \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda
\end{bmatrix} = \lambda \begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
& \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{bmatrix} = \lambda I
\]

where \( I \) is the identity matrix. Note that \( A \Lambda = \lambda AI = \lambda A \) and \( \Lambda A = \lambda A \) so that \( \Lambda A = A \Lambda \) for every matrix \( A \).

Example 4. Consider the set of three simultaneous linear difference equations of the first order

\[
\begin{align*}
x_{n+1} &= a_{11}x_n + a_{12}y_n + a_{13}z_n \\
y_{n+1} &= a_{21}x_n + a_{22}y_n + a_{23}z_n \\
z_{n+1} &= a_{31}x_n + a_{32}y_n + a_{33}z_n
\end{align*}
\]

where \( n = 0, 1, 2, \ldots \). Let

\[
w^n = \begin{bmatrix}
x_n \\ y_n \\ z_n
\end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{bmatrix}
\]

Equation (i) can be written in matrix form

\[
w^{n+1} = Aw^n, \quad n = 0, 1, 2, \ldots
\]

If \( w^0 \) is given, then

\[
\begin{align*}
w^1 &= Aw^0 \\
w^2 &= Aw^1 = A^2w^0 \\
& \quad \vdots \\
w^n &= A^nw^0
\end{align*}
\]
So that \( w^n = A^n w^0 \) is the solution of \((ii)\). Thus it would be useful to have systematic methods of finding the \( n^{th} \) power of a matrix, \( A^n \). We will develop such methods later.

**Example 5.** Consider a second order difference equation

\[
x_{n+2} + ax_{n+1} + bx_n = 0, \quad n = 0, 1, 2, \ldots
\]  

We can reduce such an equation to a system of two first order equations as follows.

Let \( x_{n+1} = y_n \), then

\[
y_{n+1} = x_{n+2} = -ax_{n+1} - bx_n = -ay_n - bx_n
\]

This can be written as a system of first order equations

\[
\begin{align*}
x_{n+1} &= y_n \\
y_{n+1} &= -bx_n - ay_n
\end{align*}
\]  

\((ii)\)

To write \((ii)\) in matrix form let

\[
z^n = \begin{bmatrix} x_n \\ y_n \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix}
\]

Now \((ii)\) is equivalent to

\[
z^{n+1} = Az^n
\]

Similarly, a difference equation of the \( k^{th} \) order may be reduced to a system of \( k \) first order equations, of the form

\[
z^{n+1} = Az^n, \quad n = 0, 1, 2, \ldots
\]

where \( A \) is a \( k \times k \) matrix, and the solution is

\[
z^n = A^n z^0.
\]

**Exercises 3.9**

1. Verify the ‘reverse order rule’ \((AB)^T = B^T A^T\) if

\[
A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}.
\]

2. If \( P, A \) are \( n \times n \) matrices and \( A \) is symmetric, prove that \( P^T A P \) is symmetric.

3. If \( A, B \) are \( n \times n \) symmetric matrices, under what condition on \( A \) and \( B \) is the product \( AB \) a symmetric matrix. Give an example to show the product of symmetric matrices is not always symmetric.

4. If \( A \) is square, show that \((A + A^T)/2\) is symmetric \((A - A^T)/2\) is skew-symmetric.

5. If \( A \) is skew symmetric, show that \( a_{ii} = 0 \).

6. Write the third order difference equation below as a system of first order difference equations in matrix form

\[
x_{n+3} + 2x_{n+2} + 3x_n = 0, \quad n = 0, 1, 2, \ldots
\]

Hint: Let \( x_{n+1} = y_n, \ y_{n+1} = z_n, \) write an equation for \( z_{n+1} \), then find the matrix \( A \) such that \( w^{n+1} = Aw^n \) where \( w^n = \begin{bmatrix} x_n \\ y_n \\ z_n \end{bmatrix} \).
7. If \( A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \), find a formula for \( A^n \) for arbitrary \( n \).

8. A square matrix \( P \) is called orthogonal if \( PP^T = I \). If \( P, Q \) are orthogonal matrices, show that the product \( PQ \) is also an orthogonal matrix.

9. If \( I, A \) are \( n \times n \) matrices, simplify

\[
(I - A) (I + A + A^2 + \ldots + A^n).
\]

10. Prove or give a counterexample (all matrices are \( n \times n \))

   a. \( (A + B)(A - B) = A^2 - B^2 \)
   b. \( (A + B)^2 = A^2 + 2AB + B^2 \)
   c. \( (I - A)(I + A + A^2) = (I + A + A^2)(I - A) \).

11. a. Show that \( A^2 = I \) if \( A = \begin{bmatrix} -1 & -2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix} \).

   b. Compute \( A^{20} \) and \( A^{30} \).

12. a. Show that \( U^2 = U \) if

\[
U = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}
\]

   b. Compute \( U^n \).

### 3.10–The Inverse of a Square Matrix

Let us look again at the simple scalar equation

\[
ax = b
\] (1)

If \( a \neq 0 \) we may multiply both sides by \( a^{-1} \) and use the fact that \( a^{-1}a = 1 \) to get the tentative solution \( x = a^{-1}b \). To check that this is indeed the solution we substitute \( x = a^{-1}b \) into (1)

\[
ax = a(a^{-1}b) = (aa^{-1})b = b,
\]

since \( aa^{-1} = 1 \). Thus the key point in solving (1) is the existence of a number \( a^{-1} \) such that

\[
a^{-1}a = aa^{-1} = 1.\] (2)

We now extend this idea to square matrices.

**Definition 1.** Let \( A \) be a \( n \times n \) matrix. If there exists an \( n \times n \) matrix \( X \) such that

\[
AX =XA = I
\] (3)

then \( X \) is called the *inverse* of \( A \) and we write \( X = A^{-1} \). If \( A^{-1} \) exists, \( A \) is called *nonsingular* or *invertible*. If \( A^{-1} \) does not exist \( A \) is called *singular*.

**Example 1.** Find \( A^{-1} \), if it exists, for

\[
A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}.
\]
If we let $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the condition $AX = I$ is
\[
\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a + 2c & b + 2d \\ 2a + 3c & 2b + 3d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\]
This yields the two systems
\[
\begin{align*}
\begin{cases}
a + 2c = 1 \\
2a + 3c = 0,
\end{cases}
\quad \text{and} \quad
\begin{cases}
b + 2d = 0 \\
2b + 3d = 1.
\end{cases}
\end{align*}
\]
The unique solutions are $a = -3$, $c = 2$, $b = 2$, $d = -1$. Thus the only candidate for $A^{-1}$ is
\[
X = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}.
\]
We know that this satisfies $AX = I$; we need only check that $XA = I$
\[
\begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\]
Therefore we have a unique inverse
\[
A^{-1} = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}.
\]

**Example 2.** Find $A^{-1}$, if it exists, for
\[
A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}.
\]
Let $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The condition $AB = I$ is
\[
\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a + 2c & b + 2d \\ 2a + 4c & 2b + 4d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\]
This yields
\[
\begin{align*}
\begin{cases}
a + 2c = 1 \\
2a + 4c = 0,
\end{cases}
\quad \text{and} \quad
\begin{cases}
b + 2d = 0 \\
2b + 4d = 1.
\end{cases}
\end{align*}
\]
Both of these systems are inconsistent (one would be enough). Thus $A^{-1}$ does not exist.

These examples show that a square matrix may or may not have an inverse. However a square matrix cannot have more than one inverse.

**Theorem 1.** If $A^{-1}$ exists, it must be unique.

**Proof.** Let $B, C$ be two inverses of $A$. We know that $AB = BA = CA = AC = I$. It follows that
\[
B = IB = (CA)B = C(AB) = CI = C.
\]
Let us consider the system of $n$ equations in $n$ unknowns
\[
Ax = b
\]
where $A$ is an $n \times m$ matrix and $b$ is an $n \times 1$ matrix. If $A^{-1}$ exists we may multiply both sides of (4) on the left to get

\[
A^{-1}Ax = A^{-1}b
\]

\[
Ix = A^{-1}b
\]

\[
x = A^{-1}b
\]

where we have used the fact that $A^{-1}A = I$. This shows that if (4) has a solution, the solutions must be given by $x = A^{-1}b$. To see that this is indeed the solution, we substitute (5) into (4)

\[
Ax = A(A^{-1}b) = (AA^{-1})b = Ib = b,
\]

where we have used the fact that $AA^{-1} = I$. Thus we have proved

**Theorem 2.** If $A^{-1}$ exists then $Ax = b$ has a unique solution for every $b$ and the solution is $x = A^{-1}b$.

**Example 3.** Suppose $A^{-1}$ is known and is given by

\[
A^{-1} = \begin{bmatrix} 1 & -1 \\ 2 & 5 \end{bmatrix}.
\]

(a) Solve $A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ for $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

(b) Solve $[y_1, y_2]A = [-1, 4]$ for $y = [y_1, y_2]$.

**Solution**

(a) $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A^{-1} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 17 \end{bmatrix}$.

(b) $y = [y_1, y_2] = [-1, 4]A^{-1} = [-1, 4] \begin{bmatrix} 1 & -1 \\ 2 & 5 \end{bmatrix} = [7, 21]$.

A matrix may not have an inverse. The following theorem, which will be proved later, tells us when an inverse exists.

**Theorem 3.** If $A$ is an $n \times n$ matrix, $A^{-1}$ exists if and only if rank $A = n$.

Our immediate aim is to develop a method for finding inverses. Given an $n \times n$ matrix $A$, the definition requires us to find a matrix $X$ satisfying the two conditions $AX = I$ and $XA = I$. Actually, as indicated in the following theorem, only one of these conditions is needed.

**Theorem 4.** If $A$ is an $n \times n$ then $A^{-1}$ exists if and only if there is a matrix $X$ satisfying $AX = I$. Furthermore, if $AX = I$ then $X = A^{-1}$.

We will prove this theorem later. For now we concentrate on a method for finding inverses. According to Theorem 4, we need only find a matrix $X$ satisfying

\[
AX = I.
\]

Let us introduce the notations

\[
X = [x^1, x^2, \ldots, x^n], \quad I = [e^1, e^2, \ldots, e^n]
\]

where $x^i$ is $i^{th}$ column of $X$ and $e^i$ is the $i^{th}$ column of $I$. Equation (6) is now

\[
A[x^1, x^2, \ldots, x^n] = [e^1, e^2, \ldots, e^n].
\]
Using properties of matrix multiplication we get

\[ [Ax^1, Ax^2, \ldots, Ax^n] = [e^1, e^2, \ldots, e^n]. \]  

(8)

Setting corresponding columns equal we arrive at

\[ Ax^1 = e^1, \ Ax^2 = e^2, \ldots, \ Ax^n = e^n. \]  

(9)

Thus \( AX = I \) holds if and only if equations (9) hold for the columns of \( X \). Since each of the equations in (9) have the same coefficient matrix \( A \), we may solve all of the \( n \) systems in (9) at once by using the augmented matrix

\[ [A \mid e^1, e^2, \ldots, e^n] = [A \mid I]. \]  

(10)

We know that \( A^{-1} \) exists if and only if rank \( A = n \), or equivalently, RREF \( (A) = I \). Thus if \( A^{-1} \) exists the reduced row–echelon form of (10) will be

\[ [I \mid x^1, x^2, \ldots, x^n] = [I, A^{-1}]. \]  

(11)

If \( A^{-1} \) does not exist then rank \( (A) < n \) so that at some stage in the reduction of \([A \mid I] \) to row echelon form, a zero row will appear in the ‘A’ part of the augmented matrix.

**Example 4.** Find \( A^{-1} \) and \( B^{-1} \) if possible where

\[
A = \begin{bmatrix} 1 & 4 & 3 \\ 2 & 5 & 4 \\ 1 & -3 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}
\]

\[
[A \mid I] = \begin{bmatrix} 1 & 4 & 3 & 1 & 0 & 0 \\ 2 & 5 & 4 & 0 & 1 & 0 \\ 1 & -3 & -2 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 & 3 & 1 & 0 & 0 \\ 0 & -3 & -2 & -2 & 1 & 0 \\ 0 & -7 & -5 & -1 & 0 & 1 \end{bmatrix}
\]

\[
\rightarrow \begin{bmatrix} 1 & 0 & 1/3 & -5/13 & 1 & 0 \\ 0 & 1 & 2/3 & 2/3 & -1/3 & 0 \\ 0 & 0 & -1/3 & 11/3 & -7/3 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 2 & -1 & 1 \\ 0 & 1 & 0 & 8 & -5 & 2 \\ 0 & 0 & 1 & -11 & 7 & -3 \end{bmatrix}.
\]

Thus \( A^{-1} = \begin{bmatrix} 2 & -1 & 1 \\ 8 & -5 & 2 \\ -11 & 7 & -3 \end{bmatrix} \). The reader may check that \( AA^{-1} = A^{-1}A = I \)

\[ [B \mid I] = \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 3 & 4 & 0 & 1 & 0 \\ 3 & 4 & 5 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -2 & -2 & 1 & 0 \\ 0 & -2 & -4 & -3 & 0 & 1 \end{bmatrix}
\]

\[ \rightarrow \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -2 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \end{bmatrix}. \]

Because of the zero row on the left hand side of the augmented matrix, \( B^{-1} \) does not exist.

**Proof of Theorem 3.** Suppose \( A^{-1} \) exists, then from \( Ax = 0 \) we find \( A^{-1}Ax = 0 \) or \( x = 0 \) which implies that rank \( A = n \). Conversely if rank \( A = n \), then \( Ax = b \) has a unique solution for every \( b \). In particular, let \( x^i \) be the solution of \( Ax = e^i \), for \( i = 1, \ldots, n \). Let \( X = [x^1, \ldots, x^n] \) then it follows that \( AX = I \). To show that \( AX = I \), let \( G = AXA - I \) then \( AG = AXA - A = 0 \). If \( g^i = i^{th} \) column of \( G \) it follows that \( Ag^i = 0 \). However, since \( A \) has rank \( n \), we conclude that \( g^i = 0 \). Thus \( G = 0 \) and \( AX = I \).

**Proof of Theorem 4.** If \( A^{-1} \) exists then rank \( A = n \). Then as in the proof above we can construct a matrix \( X \) such that \( AX = I \). Conversely, if there exists an \( X \) such that \( AX = I \), then \( Ax^i = e^i \)
where \( x^i = i^{th} \) column of \( X \) and \( e^i = i^{th} \) column of \( I \). If \( b \) is any column vector we have \( b = \sum_{i=1}^{n} b_i e^i \).

If \( x = \sum_{i=1}^{n} b_i x^i \), we see that

\[
Ax = A \sum_{i=1}^{n} b_i x^i = \sum_{i=1}^{n} b_i Ax^i = \sum_{i=1}^{n} b_i e^i = b.
\]

This shows that \( Ax = b \) has a solution for every \( b \) which implies that rank \( A = n \). According to Theorem 3 this means that \( A^{-1} \) exists.

It is worthwhile to stop for a moment and summarize the various equivalent conditions for \( n \) equations in \( n \) unknowns to have a unique solution.

**Theorem 5.** Let \( A \) be an \( n \times n \) matrix, then the following statements are all equivalent

(a) \( A^{-1} \) exists
(b) rank \( A = n \)
(c) \( \text{RREF}(A) = I \)
(d) \( Ax = 0 \) implies \( x = 0 \)
(e) \( Ax = b \) has a unique solution for every \( b \).

We now consider several properties of inverses.

**Theorem 6.** If \( A, B \) are nonsingular matrices then

\[
(A^{-1})^{-1} = A \quad (12)
\]

\[
(AB)^{-1} = B^{-1}A^{-1} \quad (13)
\]

\[
(A^T)^{-1} = (A^{-1})^T \quad (14)
\]

**Proof.** Property (12) follows from the definition of inverse. To prove (13), let \( X = B^{-1}A^{-1} \), then

\[
(AB)X = (AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AA^{-1} = I.
\]

According to Theorem (4), \( X = B^{-1}A^{-1} \) must be the inverse of \( AB \). For (14), let \( X = (A^{-1})^T \), then

\[
A^TX = A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I.
\]

**Example 5.** Simplify

\[
(A^T(B^{-1}A^{-1})^T)^{-1} = (A^T(A^{-1})^T(B^{-1})^T)^{-1} = (A^T)^{-1}(B^{-1})^{-1} = ((B^{-1})^T)^{-1} = B^T.
\]

**Exercises 3.10**

For problems 1–6 find the inverses, if they exist, and check.

1. \[
\begin{bmatrix}
1 & 2 \\
3 & 5
\end{bmatrix}
\]
2. \[
\begin{bmatrix}
2 & -3 \\
-4 & 6
\end{bmatrix}
\]
3. \[
\begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9
\end{bmatrix}
\]
4. \[
\begin{bmatrix}
1 & 1 & 1 \\
1 & 2 & 2 \\
1 & 2 & 3
\end{bmatrix}
\]
5. \[
\begin{bmatrix}
1 & 2 & 3 & 4 \\
2 & 3 & 4 & 5 \\
3 & 4 & 5 & 6 \\
6 & 7 & 8 & 9
\end{bmatrix}
\]
6. \[
\begin{bmatrix}
1 & -2 & 0 & 3 \\
-1 & 3 & 1 & 2 \\
2 & -4 & -1 & -1 \\
3 & -3 & 0 & 4
\end{bmatrix}
\]
7. Suppose $A^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 2 \\ 1 & 3 & 5 \end{bmatrix}$.

a. Solve $Ax = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ for $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

b. Solve $y^TA = [1, -1, 2]$ for $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

8. Simplify $(B^{-1}(B^T A)^T)^{-1} B^{-1} A^T$.

9. Simplify $(T^{-1}AT)^3$.

10. Simplify $A(A^{-1} + B^{-1})B$.


12. If $A$, $B$, $C$, $X$ are $n \times n$ matrices and $A^{-1}$, $B^{-1}$, $C^{-1}$ exist, solve the following equation for $X$ and check

$$B^{-1}XCA = AB.$$ 

13. Find the matrix $X$ such that $X = AX + B$ where

$$A = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 0 & 3 \end{bmatrix}.$$ 

14. Determine when $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is nonsingular.

15. If $A = \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix}$ find the values of $\lambda$ for which $A - \lambda I$ is singular.

16. If $A$, $B$ are $n \times n$ matrices and $A^{-1}$ exists, prove

a. if $AB = O$ then $B = O$ 

b. if $CA = O$ then $C = O$ 

c. if $AB = AC$ then $B = C$.

3.11–Linear Independence and Linear Dependence

**Definition 1.** The set of vectors $\{v^1, \ldots, v^k\}$, where each $v^i$ is an $n$-dimensional vector, is called linearly dependent (LD) if there exist scalars $\alpha_1, \ldots, \alpha_k$, not all zero, such that

$$\alpha_1 v^1 + \alpha_2 v^2 + \ldots + \alpha_k v^k = 0$$

(1)

For a set to be LD at least one of the $\alpha_1$ in Equation (1) must be non zero. If, say, $\alpha_1 \neq 0$, then we may solve for $v^1$ as a linear combination of the others

$$v^1 = -\frac{\alpha_2}{\alpha_1} v^2 \ldots - \frac{\alpha_k}{\alpha_1} v^k$$

In general we can state a set of vectors is LD if and only if at least one of the vectors is linear combinations of the others.

**Definition 2.** A set of vectors $\{v^1, \ldots, v^k\}$ is called linearly independent (LI) if it is not linearly dependent. That is, the set is LI if

$$\alpha_1 v^1 + \alpha_2 v^2 + \ldots + \alpha_k v^k = 0 \text{ implies } \alpha_1 = \alpha_2 = \ldots = \alpha_k = 0$$

(2)
From the definition it follows that a set is LI if and only if none of the vectors is a linear combination of the others.

**Example 1.** Any set containing the zero vector is LD.

Let the set be \( \{0, \mathbf{v}^2, \ldots, \mathbf{v}^k\} \) then we have the obvious equality

\[
1 \cdot 0 + 0 \cdot \mathbf{v}^2 + \ldots + 0 \cdot \mathbf{v}^k = 0
\]

From definition (1), the set is LD.

We emphasize that linear independence or dependence is a property of a set of vectors, not individual vectors.

The simplest case is if the set contains only one vector, say the set \( \{\mathbf{v}\} \). If \( \mathbf{v} = 0 \) then, since \( 1 \cdot 0 = 0 \), the set \( \{0\} \) is LD. If \( \mathbf{v} \neq 0 \), one can show that \( \alpha \mathbf{v} = 0 \) implies \( \alpha = 0 \) so that the set \( \{\mathbf{v}\} \) is LI.

A set of two vectors, \( \{\mathbf{v}^1, \mathbf{v}^2\} \) is LD if and only if one of the vectors is a multiple of the other. For vectors with real components this can be determined by inspection.

**Example 2.** The set \( \{\mathbf{v}^1, \mathbf{v}^2\} \) where

\[
\mathbf{v}^1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \quad \mathbf{v}^2 = \begin{bmatrix} -2 \\ 4 \\ -6 \end{bmatrix}
\]

is LD since \( \mathbf{v}^2 = -2 \mathbf{v}^1 \).

**Example 3.** The set \( \{\mathbf{v}^1, \mathbf{v}^2\} \) where

\[
\mathbf{v}^1 = \begin{bmatrix} 1 \\ 2 \\ -3 \\ 4 \end{bmatrix}, \quad \mathbf{v}^2 = \begin{bmatrix} 2 \\ 4 \\ -6 \\ 7 \end{bmatrix}
\]

is LI since neither vector is a multiple of the other.

**Example 4.** The set \( \{\mathbf{v}^1, \mathbf{v}^2\} \), where

\[
\mathbf{v}^1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}^2 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}
\]

is LD since it contains the zero vector. Note that \( \mathbf{v}^1 \) is a multiple of \( \mathbf{v}^2 \), namely \( \mathbf{v}^1 = 0 \cdot \mathbf{v}^2 \), but \( \mathbf{v}^2 \) is not a multiple of \( \mathbf{v}^1 \).

If a set contains three or more vectors, one must usually resort to the definition to determine whether the set is LI or LD.

**Example 5.** Is the set \( \{\mathbf{v}^1, \mathbf{v}^2, \mathbf{v}^3\} \) LI or LD where

\[
\mathbf{v}^1 = \begin{bmatrix} -1 \\ -2 \\ 0 \\ -3 \end{bmatrix}, \quad \mathbf{v}^2 = \begin{bmatrix} 3 \\ 2 \\ 4 \\ 1 \end{bmatrix}, \quad \mathbf{v}^3 = \begin{bmatrix} 7 \\ 6 \\ 8 \\ 5 \end{bmatrix}
\]

Consider

\[
\alpha_1 \mathbf{v}^1 + \alpha_2 \mathbf{v}^2 + \alpha_3 \mathbf{v}^3 = \mathbf{0}.
\]

\((i)\)
If this vector equation has a nontrivial solution (the unknowns are $\alpha_1$, $\alpha_2$, $\alpha_3$) the set is LD; if it has only the trivial solutions the set is LI. Equation (i) can be written

$$[v^1, v^2, v^3] \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ or}$$

$$\begin{bmatrix} -1 & 3 & 7 \\ -2 & 2 & 6 \\ 0 & 4 & 8 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$  \hspace{1cm} (ii)

All we need to know is whether or not a nontrivial solution exists. This is determined by the rank of coefficient matrix in (ii). If the rank = number of columns = 3, then only the trivial solution exists and the set of vectors is LI. If the rank is less than the number of columns, a non trivial solution exists and the set of vectors is LD. We start to reduce the coefficient matrix in (ii) to echelon form

$$\begin{bmatrix} -1 & 3 & 7 \\ -2 & 2 & 6 \\ 0 & 4 & 8 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 7 \\ -2 & 2 & 6 \\ 0 & 4 & 8 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 7 \\ 0 & -4 & -8 \\ 0 & 4 & 8 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 7 \\ 0 & -4 & -8 \\ 0 & 0 & 0 \end{bmatrix}.$$  \hspace{1cm} (ii)

At this point it is clear that the rank = 2 < 3 so the set of vectors is LD.

By the same reasoning as in the above example one can prove the following

**Theorem 1.** Let $\{v^1, \ldots, v^k\}$ be a set of $n$ dimensional column vectors and let $A$ be the $n \times k$ matrix whose columns are the $v^i$:

$$A = [v^1, \ldots, v^k]$$

If rank $A < k$ the set is LD. If rank $A = k$ the set is LI.

**Example 6.** Determine whether $\{v^1, v^2, v^3\}$ is LD or LI where

$$v^1 = [1, 0, 0]^T, \quad v^2 = [2, -2, 0]^T, \quad v^3 = [3, 4, 3]^T$$

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & 4 \\ 0 & 0 & 3 \end{bmatrix}$. It is clear that rank $A = 3$ so the set is LI.

**Example 7.** Assume the set $\{u, v\}$ is LI where $u, v$ are $n$-dimensional vectors. Determine whether the set $\{u + v, u - v\}$ is LI or LD.

Since we are not given specific vectors, we cannot use Theorem 1; we must go back to the definition. Consider the vector equation

$$\alpha(u + v) + \beta(u - v) = 0.$$  \hspace{1cm} (i)

If we can show $\alpha, \beta$ must be zero, the set $\{u + v, u - v\}$ is LI, otherwise it is LD. Now (i) can be written

$$(\alpha + \beta)u + (\alpha - \beta)v = 0.$$  \hspace{1cm} (ii)

Since we know that $\{u, v\}$ is LI each coefficient must be zero. Thus $\alpha + \beta = 0$ and $\alpha - \beta = 0$. These equations are easily solved yielding $\alpha = \beta = 0$ so the set is LI.
It is instructive to consider the geometric meaning of independence for vectors in the plane, \( \mathbb{R}^2 \). A set of two vectors \( \{ \mathbf{u}, \mathbf{v} \} \) in \( \mathbb{R}^2 \) is LD if and only if one vector is a multiple of the other. This means the vectors must be collinear, as shown in Figure 1. The set \( \{ \mathbf{u}, \mathbf{v} \} \) is LI if the vectors are not collinear. However, if we consider a set of three vectors \( \{ \mathbf{u}, \mathbf{v}, \mathbf{w} \} \), the set must be LD. This is shown above where \( \mathbf{w} \) can be written as a linear combination of \( \mathbf{u}, \mathbf{v} \) using the parallelogram law.

Let us prove that any set of vectors in \( \mathbb{R}^2 \) containing three or more vectors must be LD.

Let the vectors be \( \{ \mathbf{v}^1, \ldots, \mathbf{v}^m \} \) where \( m > 2 \) and the \( \mathbf{v}^i \) are \( 2 \times 1 \) vectors. Consider the \( 2 \times m \) matrix

\[
A = [\mathbf{v}^1, \ldots, \mathbf{v}^m]
\]

clearly rank \( A \leq 2 < m \) = number of columns. Thus by Theorem 1, the set is LD.

Similarly one can show that a set of three-dimensional vectors in \( \mathbb{R}^3 \), \( \{ \mathbf{v}^1, \mathbf{v}^2, \mathbf{v}^3 \} \) is LI if and only if the vectors are not coplanar. Further any set of vectors in \( \mathbb{R}^3 \) containing four or more vectors must be LD.

Let us consider vectors in \( \mathbb{C}^n \). It is easy to construct sets containing \( n \) vectors that are LI. One example is the set \( \{ \mathbf{e}^1, \ldots, \mathbf{e}^n \} \) where \( \mathbf{e}^i = i^{th} \) column of the identity matrix of order \( n \). Since \( I = [\mathbf{e}^1, \ldots, \mathbf{e}^n] \) has rank \( n \), the set is LI. Another example is the set \( \{ \mathbf{v}^1, \ldots, \mathbf{v}^n \} \) where

\[
\mathbf{v}^i = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{v}^2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \quad \mathbf{v}^n = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}
\]

It is clear that the matrix \( A = [\mathbf{v}^1, \ldots, \mathbf{v}^n] \) has rank \( n \) so the set is LI. In fact the columns of any \( n \times n \) matrix of rank \( n \) form a LI set. For sets containing more than \( n \) vectors we have

**Theorem 2.** Any set of vectors in \( \mathbb{C}^n \) containing more than \( n \) vectors must be LD.

**Proof** Let the set be \( \{ \mathbf{v}^1, \ldots, \mathbf{v}^m \} \) where \( m \times n \) and each \( \mathbf{v}^i \) is \( n \times 1 \) column matrix. Consider the matrix

\[
A_{n \times m} = [\mathbf{v}^1, \ldots, \mathbf{v}^m]
\]

clearly rank \( A \leq n < m = \) the number of columns, thus by Theorem 1, the set is LD.

**Definition 3.** Any set of \( n \) vectors in \( \mathbb{C}^n \) which is LI is called a basis for \( \mathbb{C}^n \).

There are of course infinitely many different bases. The choice of the basis depends on the problem at hand. A basis is like a reference frame that we can use to describe all vectors.
Theorem 3. If \( \{v^1, \ldots, v^n\} \) is a basis for \( \mathbb{C}^n \) and \( x \) is any vector in \( \mathbb{C}^n \) then there exists unique scalars \( \alpha_i \) such that

\[
x = \alpha_1 v^1 + \alpha_2 v^2 + \ldots + \alpha_n v^n.
\]

The scalar \( \alpha_i \) is called the \( i^{th} \)-coordinate of \( x \) relative to the given basis.

Proof Assume \( v^i \) are column vectors then (3) is equivalent to

\[
[v^1, \ldots, v^n] \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = x.
\]

This is a system of \( n \) equations for the \( n \) unknowns \( \alpha_1, \ldots, \alpha_n \) (\( x \) is given). Since the set \( \{v^1, \ldots, v^n\} \) is LI the \( n \times n \) coefficient matrix \( A = [v^1, \ldots, v^n] \) has rank \( n \). Therefore, there exists a unique solution for the \( \alpha_i \) for every \( x \).

Example 8. Given

\[
x = \begin{bmatrix} 22 \\ -14 \\ 0 \end{bmatrix}, \quad v^1 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \quad v^2 = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}, \quad v^3 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix},
\]

show that \( \{v^1, v^2, v^3\} \) is a basis for \( \mathbb{R}^3 \) and find the coordinates of \( x \) relative to this basis.

\[
\alpha_1 v^1 + \alpha_2 v^2 + \alpha_3 v^3 = x
\]

or

\[
\begin{bmatrix} 2 & 3 & -1 \\ 3 & -2 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 22 \\ -14 \\ 0 \end{bmatrix}
\]

We now reduce the augmented matrix to echelon form.

\[
\begin{align*}
\begin{bmatrix} 2 & 3 & -1 & 22 \\ 3 & -2 & 2 & -14 \\ 1 & 1 & 1 & 0 \end{bmatrix} & \rightarrow \begin{bmatrix} 1 & 1 & 1 & 0 \\ 3 & -2 & 2 & -14 \\ 1 & 1 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & -5 & -1 & -14 \\ 0 & 1 & -3 & 22 \end{bmatrix} \\
& \rightarrow \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & -3 & 22 \\ 0 & -5 & -1 & -14 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 4 & -22 \\ 0 & 1 & -3 & 22 \\ 0 & 0 & -16 & 96 \end{bmatrix} \\
& \rightarrow \begin{bmatrix} 1 & 0 & 4 & -22 \\ 0 & 1 & -3 & 22 \\ 0 & 0 & 1 & -6 \end{bmatrix}
\end{align*}
\]

Since the rank of coefficient matrix is 3 the set \( \{v^1, v^2, v^3\} \) is LI and forms a basis. The coordinates of \( x \) relative to this basis are \( \alpha_1 = 2, \alpha_2 = 4, \alpha_3 = -6 \). With these values, it can be checked that (*) holds.

Rank and Linear Independence

Recall that the rank of a matrix \( A \) is defined to be the number of nonzero rows in the row echelon form of \( A \). However, the rank also tells us something about the rows of \( A \) and the columns of \( A \) as indicated in the following theorem.

Theorem 4. Suppose \( A \) is an \( m \times n \) matrix of rank \( r \) then

(1) the maximal number of rows in any LI set of rows is \( r \)
(2) the maximal number of columns in any LI set of columns is \( r \).
We shall not prove this theorem but will illustrate it with an example.

**Example 9.** Consider the matrix \( A \) given by

\[
A = \begin{bmatrix}
1 & -1 & -2 & 3 & 4 & 3 \\
1 & -1 & 1 & -1 & 1 & -2 \\
2 & -2 & -1 & 2 & 5 & 1 \\
1 & -1 & 0 & 0 & 0 & 0 \\
4 & -4 & -1 & 2 & 5 & 1
\end{bmatrix}.
\]

By finding the row echelon form of \( A \), one finds that \( r = \text{rank}(A) = 3 \). According to the theorem, three of the rows are independent. One can show that \( \{r_1(A), r_2(A), r_4(A)\} \) is an LI set of rows. The other two rows can be expressed as linear combinations of these three rows.

\[
r_3(A) = r_1(A) + r_2(A) \\
r_5(A) = r_1(A) + r_2(A) + 2r_4(A)
\]

As far as columns are concerned it can be verified that the set \( \{c_1(A), c_3(A), c_4(A)\} \) is an LI set and the other columns depend on these:

\[
c_2(A) = -c_1(A) \\
c_5(A) = 7c_3(A) + 6c_4(A) \\
c_6(A) = -3c_3(A) - c_4(A).
\]

There are systematic methods for finding which sets of rows or columns form LI sets and how to express the remaining rows or columns in terms of the LI sets. However, we will not consider these matters.

We note two other theorems which follow immediately from Theorem 4.

**Theorem 5.** If \( A \) is any matrix then

\[
\text{rank}(A) = \text{rank}(A^T).
\]

**Theorem 6.** If \( A \) is an \( n \times n \) (square) matrix, then

1. If the set of columns is LI so is the set of rows.
2. If the set of columns is LD so is the set of rows.
3. \( A^{-1} \) exists if and only if the set of columns (or rows) is LI.
4. \( Ax = 0 \) has a nontrivial solution if and only if the columns (or rows) are LD.

**Example 10.** Determine if \( Ax = 0 \) has a nontrivial solution where

\[
A = \begin{bmatrix}
1 & 2 & 3 \\
2 & 3 & 4 \\
3 & 4 & 5
\end{bmatrix}.
\]

If one recognizes that \( r_2(A) = (r_1(A) + r_3(A))/2 \), it is clear that the rows are LD. Thus \( \text{rank} A < 2 \) and a nontrivial solution exists.
Rank and Independent Solutions of \( Ax = 0 \)

Let us look at an example of 3 homogeneous equations in 4 unknowns, \( Ax = 0 \), where

\[
A = \begin{bmatrix}
1 & 1 & -1 & 1 \\
1 & 1 & 1 & 1 \\
2 & 2 & 0 & 2
\end{bmatrix}.
\]

To solve this system we find the reduced row echelon form of \( A \)

\[
\text{RREF}(A) = \begin{bmatrix}
1 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]

Since the rank = 2 there are 2 basic variables; these are \( x_1 \) and \( x_3 \). There must be \( n - r = 4 - 2 = 2 \) free variables, which are \( x_2 \) and \( x_4 \). The general solution is

\[
x = \begin{bmatrix}
-x_2 - x_4 \\
x_2 \\
0 \\
x_4
\end{bmatrix} = x_2 \begin{bmatrix}
-1 \\
1 \\
0 \\
0
\end{bmatrix} + x_4 \begin{bmatrix}
-1 \\
0 \\
0 \\
1
\end{bmatrix}.
\]

consider the set of two vectors on the right above, namely

\[
\left\{ \begin{bmatrix}
-1 \\
1 \\
0 \\
0
\end{bmatrix}, \begin{bmatrix}
-1 \\
0 \\
0 \\
1
\end{bmatrix} \right\}.
\]

This set is a LI set since, according to equation (*), \( x = 0 \) if and only if \( x_2 = x_4 = 0 \). This is typical of the general situation.

**Theorem 7.** Let \( A \) be an \( m \times n \) matrix of rank \( r \). If the general solution of \( Ax = 0 \) (obtained from the echelon form of \( A \)) is written in the form

\[
x = t_1v^1 + \ldots + t_{n-r}v^{n-r},
\]

where the \( t_i \) are the free variables, then the set of \( n - r \) solutions

\[
\{v^1, \ldots, v^{n-r}\}
\]

is a LI set.

We omit a formal proof of this theorem.

**Exercises 3.11**

1. Determine by inspection whether the following sets are LI or LD.
   a. \( \{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \end{bmatrix} \} \)
   b. \( \{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \\ 10 \end{bmatrix} \} \)
   c. \( \{[1,0,0,], [0,1,0]\} \)
   d. \( \{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \} \)
   e. \( \{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} 17 \\ 7 \\ 3 \\ 10 \end{bmatrix} \} \)
2. Determine whether or not the following sets are LI or LD.
   a. \[ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \]  
   b. \[ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \]  
   c. \[ \begin{bmatrix} 1 \\ 2 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \\ 1 \end{bmatrix} \]  
   d. \[ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ -1 \\ 3 \end{bmatrix} \]

3. Given that the set of \( n \)-dimensional vectors \( \{ u, v, w \} \) is LI determine whether or not the following sets are LI or LD:
   a. \( \{ u - v, u + w, v - w \} \)
   b. \( \{ u + 2v + 3w, 2u + 3v + 4w, 3u + 4v + 5w \} \)
   c. \( \{ 2u + 3v - w, 3u - 2v - 2w, u + v + w \} \)

4. Let \( x = [2, -3, 5]^T \). Find the coordinates of \( x \) relative to the following bases.
   a. \( \{ e^1, e^2, e^3 \} \) where \( e^1 = [1, 0, 0]^T, e^2 = [0, 1, 0]^T, e^3 = [0, 0, 1]^T \)
   b. \( \{ [1, 2, 3]^T, [2, 1, 3]^T, [3, 0, 0]^T \} \)

5. Complete the following statements where \( A \) is an \( n \times n \) matrix
   a. \( A \) is singular if and only if the rows are______
   b. \( A \) is singular if and only if the columns are______
   c. If the columns of \( A \) are LD then \( Ax = 0 \) has______
   d. If the rows of \( A \) are LD then rank \( A \) is______
   e. If the rows of \( A \) are LD then \( Ax = b \) may have a solution for a particular \( b \) but the solution is______
   f. If columns of \( A \) are LD then \( \text{RREF}(A) \)______

6. Find a set of \( n - r \) LI solutions for \( Ax = 0 \) in each of the following cases
   a. \( A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \end{bmatrix} \)
   b. \( A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \)
   c. \( A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \)
   d. \( A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 2 & 3 & 4 \end{bmatrix} \)

3.12–Determinants

Consider the system

\[
\begin{align*}
a_{11}x_1 + a_{12}x_2 &= b_1 \\
a_{21}x_1 + a_{22}x_2 &= b_2
\end{align*}
\]

Solving by elimination or otherwise, we find that

\[
\begin{align*}
x_1 &= \frac{b_1a_{22} - b_2a_{12}}{a_{11}a_{22} - a_{12}a_{21}}, & \text{if } a_{11}a_{22} - a_{12}a_{21} \neq 0 \\
x_2 &= \frac{b_2a_{11} - b_1a_{21}}{a_{11}a_{22} - a_{12}a_{21}}, & \text{if } a_{11}a_{22} - a_{12}a_{21} \neq 0
\end{align*}
\]
Both numerators and denominators have the same form, namely, the difference of two products. We can write (2) in a compact form if we define the determinant of any $2 \times 2$ matrix $A$, denoted by $\det A$ or $|A|$ to be a number given by

$$|A| = \det \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$  \hspace{1cm} (3)$$

This allows us to write (2) as

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{21} \\ b_2 & a_{22} \end{vmatrix}}{|A|}, \quad x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{22} & b_2 \end{vmatrix}}{|A|}$$  \hspace{1cm} (4)$$

Let $B_i$ be the matrix formed by replacing the $i^{th}$ column of $A$ by $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, then (3) can be written

$$x_1 = \frac{|B_1|}{|A|}, \quad x_2 = \frac{|B_2|}{|A|}, \quad \text{if } |A| \neq 0$$  \hspace{1cm} (5)$$

This is commonly called Cramer’s rule.

Consider three equations in 3 unknowns, $Ax = b$, or

$$\sum_{j=1}^{3} a_{ij}x_j = b_i, \quad i = 1, 2, 3$$  \hspace{1cm} (6)$$

If we solve for $x_1$, $x_2$, $x_3$, we find that the denominator in each case is

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - (a_{13}a_{22}a_{31} + a_{12}a_{21}a_{33} + a_{11}a_{33}a_{22})$$  \hspace{1cm} (7)$$

One way to remember this result is to write the first two columns to the right of the determinant, and then take the product along the diagonals sloping to the right as positive and the products along the diagonals sloping to the left as negative. This scheme is illustrated in the following figure.

Using this definition of a three by three determinant, one may write the solutions of Equation (6), if $|A| \neq 0$ as

$$x_i = \frac{|B_i|}{|A|} \quad i = 1, 2, 3$$  \hspace{1cm} (8)$$
where $B_i$ is the matrix formed by replacing $i^{th}$ column of $A$ by $b$. This is Cramer’s rule for 3 equations. If $|A| \neq 0$, a unique solution is given by above formula.

We shall define the determinant of an $n \times n$ matrix $A$ (or an $n^{th}$ order determinant) inductively, that is, we shall define a determinant of an $n \times n$ matrix in terms of determinants of matrices or order $(n - 1)$ by $(n - 1)$.

**Definition 1.** Let $A$ be an $n \times n$ matrix then

1. the minor $M_{ij}$, of the $(i, j)^{th}$ element is the determinant of the $(n - 1) \times (n - 1)$ matrix formed by striking out the $i^{th}$ row and $j^{th}$ column of $A$.

2. The cofactor, $A_{ij}$, of the $(i, j)^{th}$ element is

$$A_{ij} = (-1)^{i+j}M_{ij}$$

3. The determinant of $A$, $|A|$, is the number defined by

$$|A| = a_{11}A_{11} + a_{12}A_{12} + \ldots + a_{1n}A_{1n}. \quad \text{(9)}$$

The definitions of determinants of orders 2 and 3 given earlier are consistent with this definition.

**Example 1.**

$$\begin{vmatrix} 1 & -2 & -3 \\ 2 & 3 & -1 \\ 3 & -1 & 4 \end{vmatrix} = 1(-1)^2 \begin{vmatrix} 3 & -1 \\ -1 & 4 \end{vmatrix} + (-2)(-1)^3 \begin{vmatrix} 2 & -1 \\ 3 & 4 \end{vmatrix}$$

$$+ (-3)(1)^4 \begin{vmatrix} 2 & 3 \\ 3 & -1 \end{vmatrix}$$

$$= 1(12 - (+1)) + 2(8 - (-3)) - 3(-2 - 9) = 66.$$ 

**Example 2.**

$$\begin{vmatrix} 1 & 2 & -1 & 2 \\ 2 & 3 & 1 & 4 \\ 5 & 6 & 7 & 9 \\ -1 & 0 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 3 & 1 & 4 \\ 6 & 7 & 9 \\ 0 & 0 & 2 \end{vmatrix} - 2 \begin{vmatrix} 2 & -1 & 2 \\ 6 & 7 & 9 \\ 0 & 0 & 2 \end{vmatrix}$$

$$+ 5 \begin{vmatrix} 2 & -1 & 2 \\ 3 & 1 & 4 \\ 0 & 0 & 2 \end{vmatrix} - (-1) \begin{vmatrix} 2 & -1 & 2 \\ 3 & 1 & 4 \\ 6 & 7 & 9 \end{vmatrix}$$

Each of the determinants of order 3 may be evaluated as in Example 1 or by Formula (7).

In Definition 1, a determinant is defined as the sum of products of elements of the first row by their corresponding cofactors. For determinants of order 2 or 3, one may verify that one can use elements of any row or any column instead. This is true in general, as indicated in the following theorem. The proof is rather tedious and will be omitted.

**Theorem 1.** If $A = [a_{ij}]$ is an $n \times n$ matrix then

(a) $|A|$ can be expanded in terms of the elements of the $i^{th}$ row,

$$|A| = \sum_{j=1}^{n} a_{ij}A_{ij} \quad i = 1, \ldots, n$$

(b) $|A|$ can be expanded in terms of the elements of the $j^{th}$ column,

$$|A| = \sum_{i=1}^{n} a_{ij}A_{ij} \quad j = 1, 2, \ldots, n$$
Example 3. We evaluate the determinant of example 1 by expanding in terms of its $2^{nd}$ column
\[
\begin{vmatrix}
1 & -2 & -3 \\
2 & 3 & -1 \\
3 & -1 & 4 \\
\end{vmatrix}
= -(2) \begin{vmatrix}
2 & -1 \\
3 & 4 \\
\end{vmatrix}
+ 3 \begin{vmatrix}
1 & -3 \\
3 & 4 \\
\end{vmatrix}
- (1) \begin{vmatrix}
1 & -3 \\
2 & -1 \\
\end{vmatrix}
= 66
\]

We now establish some basic properties of determinants.

Property 1. If $A$ is an $n \times n$ matrix then $|A| = |A^T|$.

Proof. This follows immediately from Theorem 1.

Property 2. If any row or any column is the zero vector then the value of the determinant is zero.

Proof. Expand in terms of the zero row or column.

Property 3. If a determinant has two equal rows (or columns) its value is zero.

Proof. This is established inductively. For $n = 2$ we have (using equal rows).
\[
\begin{vmatrix}
a & b \\
a & b \\
\end{vmatrix}
= ab - ab = 0.
\]

Assume the result holds for a determinant or order $n-1$. Now expand the determinant (having two equal rows) in terms of one of its non identical rows; each cofactor will be zero by the inductive hypothesis. Thus the determinant equals zero.

We now establish the effect of elementary row (or column) operations on the value of the determinant.

Property 4. If any row (or column) is multiplied by $c$, the value of the determinant is multiplied by $c$.

Proof. Expand in terms of the relevant row (or column).

Corollary $|cA| = c^n |A|$.

Property 5. If a multiple of one row (or column) is added to another row (or column) the value of the determinant is unchanged.

Proof. Let $A = [c^1, \ldots, c^n]$ where $c^i$ is $i^{th}$ column of $A$.

Assume that we perform the operation $kc^1 + c^j \rightarrow c^j$, then by expansion in terms of the $j^{th}$ column we find
\[
\det [c^1, \ldots, c^j, \ldots, kc^1 + c^j, \ldots, c^n] = \det [c^1, \ldots, c^j, \ldots, kc^i, \ldots, c^n] \\
+ \det [c^1, \ldots, c^j, \ldots, c^j, \ldots, c^n].
\]

However,
\[
\det [c^1, \ldots, c^j, \ldots, kc^i, \ldots, c^n] = k \det [c^1, \ldots, c^j, \ldots, c^i, \ldots, c^n] = 0,
\]
since two columns are equal. Thus
\[
\det [c^1, \ldots, c^j, \ldots, kc^1 + c^j, \ldots, c^n] = \det [c^1, \ldots, c^j, \ldots, c^j, \ldots, c^n] = \det A.
\]

Property 6. If two rows (or columns) of a determinant are interchanged, the value of the determinant is multiplied by $(-1)$. 

Section 3.12–Determinants

Proof For simplicity assume we interchange the first two columns. Then we have
\[
\det [c^2, c^1, \ldots] = \det [c^2 - c^1, c^1, \ldots] = \det [c^2 - c^1, c^2, \ldots] = \det [-c^1, c^2, \ldots] = -\det [c^1, c^2, \ldots]
\]
where we have used Property 5 three times and Property 4 at the last step.

Properties 4, 5, 6 are helpful in shortening the work of evaluating a determinant.

Example 4. Evaluate
\[
|A| = \begin{vmatrix} 2 & -3 & 2 & 5 \\ 1 & -1 & 1 & 2 \\ 3 & 2 & 2 & 1 \\ 1 & 1 & -3 & 1 \end{vmatrix}
\]
perform the elementary row operations: \(-2r_2 + r_1 \rightarrow r_1, -3r_2 + r_3 \rightarrow r_3\) and \(-r_2 + r_4 \rightarrow r_4\) to get
\[
|A| = \begin{vmatrix} 0 & -1 & 0 & 1 \\ 1 & -1 & 1 & 2 \\ 0 & 5 & -1 & -5 \\ 0 & 2 & -4 & -3 \end{vmatrix}.
\]
Now expand by the first column to get
\[
|A| = -1 \cdot \begin{vmatrix} -1 & 0 & 1 \\ 5 & -1 & -5 \\ 2 & -4 & -3 \end{vmatrix}.
\]
Perform \(c_3 + c_1 \rightarrow c_1\) to get
\[
|A| = -1 \cdot \begin{vmatrix} 0 & 0 & 1 \\ 0 & -1 & -5 \\ -1 & -4 & -3 \end{vmatrix} = -1 \cdot \begin{vmatrix} 0 & -1 \\ -1 & -4 \end{vmatrix} = 1.
\]
where we expanded the 3rd order determinant by its first row.

Theorem 2. If \(A\) is an \(n \times n\) matrix then \(\det A = 0\) if and only if the rows (or columns) of \(A\) are LD.

Proof Assume the rows are LD and say \(r_1 = \sum_{i=2}^{n} k_i r_i\). Perform the row operation \(-\sum k_i r_i + r_1 \rightarrow r_1\); this will produce a zero row.

Conversely, assume \(\det A = 0\). One can show that in reduction of \(A\) to reduced row–echelon form \(\text{RREF}(A)\), we must have
\[
\det A = k \det \text{RREF}(A) \text{ where } k \neq 0.
\]
Thus \(\det A = 0\) implies \(\det \text{RREF}(A) = 0\). This means \(\text{RREF}(A)\) has a zero row. Thus rank \(A < n\) and the rows are LD.

We are now able to state a condition for the existence of nontrivial solutions to \(Ax = 0\) that will be useful in the remainder of these notes.

Theorem 3. If \(A\) is an \(n \times n\) matrix then \(Ax = 0\) has a nontrivial solution if and only if \(\det A = 0\).

Proof This follows directly from Theorem 2.

Example 5. \(x_1 - x_2 = 0, x_1 + x_2 = 0\).
Clearly this system has only the trivial solution, and \( \det \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = -2 \neq 0 \).

**Example 6.** \( x_1 - 2x_2 = 0, \ x_2 - 2x_3 = 0 \).

Since the second equation is twice the first, only the first equation need be considered. The solution is \( x_1 = 2x_2 \) with \( x_2 \) free. Here we have \( \det \begin{vmatrix} 1 & -2 \\ -2 & 4 \end{vmatrix} = 0 \).

Theorem 3 may be rephrased to provide another condition for the existence of an inverse.

**Theorem 4.** If \( A \) is an \( n \times n \) matrix then \( A^{-1} \) exists if and only if \( \det A \neq 0 \).

**Proof** If \( \det A = 0 \) the rows (and columns) are LD and \( A^{-1} \) does not exist. If \( \det A \neq 0 \), the rows (and columns) of \( A \) are LI, rank \( A = n \), and \( A^{-1} \) exists.

To complete our discussions of determinants we want to show that Cramer’s Rule holds in general and also develop an explicit formula for the inverse of a matrix. We need one preparatory Theorem.

**Theorem 5.** If \( A = [a_{ij}] \) is an \( n \times n \) matrix and \( A_{ij} \) is the cofactor of the \( (i, j)^{th} \) element then

\[
\begin{align*}
\text{a. } & \sum_{j=1}^{n} a_{ij}A_{kj} = \begin{cases} 
\det A, & \text{if } i = k \\
0, & \text{if } i \neq k 
\end{cases} \\
\text{b. } & \sum_{i=1}^{n} a_{ik}A_{ij} = \begin{cases} 
\det A, & \text{if } k = j \\
0, & \text{if } k \neq j 
\end{cases}
\end{align*}
\]

**Proof** Let us prove (a). Note that when \( i = k \) this is the expansion in terms of the \( i^{th} \) row given in Theorem 1. Now let \( B \) denote the matrix obtained from \( A \) by making the \( k^{th} \) row equal to the \( i^{th} \) row of \( A \), leaving all other rows unchanged. Since \( B \) has two equal rows, \( |B| = 0 \). Moreover, the cofactors of the \( k^{th} \) row of \( B \) are the same as the cofactors of the \( k^{th} \) row of \( A \). Expanding \( |B| \) by its \( k^{th} \) row yields equation (a), where \( i \neq k \).

**Theorem 6.** (Cramer’s Rule). If \( A \) is an \( n \times n \) matrix, the equations \( Ax = b \) have a unique solution for every \( b \) if and only if \( \det A \neq 0 \). If \( \det A \neq 0 \) the solutions are

\[ x_i = \frac{|B_i|}{|A|}, \]

where \( B_i \) is the matrix obtained from \( A \) by replacing the \( i^{th} \) column of \( A \) by \( b \).

**Proof** Writing out \( Ax = b \) in full we have

\[
\begin{align*}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_1 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= b_n
\end{align*}
\]

Multiply the first equation by \( A_{11} \), the 2\(^{nd} \) equation by \( A_{21} \), etc., and add to obtain

\[
(A_{11}a_{11} + A_{21}a_{21} + \ldots + A_{n1}a_{n1})x_1 = \sum_{k=1}^{n} A_{ki}b_k,
\]

where, because of Theorem 5, the coefficients of \( x_2, \ldots, x_n \) are zero. Thus we have

\[
|A| \cdot x_1 = \sum_{k=1}^{n} A_{ki}b_k = |B_1|
\]
and

\[ x_1 = \frac{|B_1|}{|A|}. \]

In a similar manner we can show that

\[ x_i = \frac{\sum_{k=1}^{n} A_{ik}b_k}{|A|} = \frac{|B_i|}{|A|}. \]

**Theorem 7.** Let \( A \) be an \( n \times n \) determinant and define the matrix of cofactors, \( \text{cof} \ A \), to be

\[ (\text{cof} \ A)_{ij} = A_{ij} \]

then, if \( |A| \neq 0 \), we have

\[ A^{-1} = \frac{(\text{cof} \ A)^T}{|A|}. \]

**Proof** In the proof of Theorem 6 we found

\[ x_i = \frac{\sum_{k=1}^{n} A_{ik}b_k}{|A|}, \quad i = 1, \ldots, n \]

Writing this out in matrix form we have

\[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_n \\
\end{bmatrix}
= \frac{1}{|A|}
\begin{bmatrix}
  A_{11} & A_{12} & \cdots & A_{1n} \\
  A_{21} & A_{22} & \cdots & A_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  A_{n1} & A_{n2} & \cdots & A_{nn} \\
\end{bmatrix}
\begin{bmatrix}
  b_1 \\
  b_2 \\
  \vdots \\
  b_n \\
\end{bmatrix}.
\]

Therefore the coefficient of the vector \( b \) on the right must be \( A^{-1} \).

\[ A^{-1} = \frac{(\text{cof} \ A)^T}{|A|}. \]

**Example 7.** Let

\[ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ then } \text{cof} \ A = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}, \quad (\text{cof} \ A)^T = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \]

If \( |A| = ad - bc \neq 0 \) we have

\[ A^{-1} = \frac{(\text{cof} \ A)^T}{|A|} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \]

This is a useful formula for the inverse of any \( 2 \times 2 \) matrix (that has an inverse).

**Exercises 3.12**

1. Solve \( Ax = b \) by Cramer’s Rule if

\[ A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 1 & 2 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}. \]
2. Using elementary row or column operations evaluate
\[
\begin{vmatrix}
1 & 2 & 3 & 4 \\
2 & 3 & 4 & 5 \\
3 & 4 & 5 & 6 \\
4 & 5 & 6 & 7
\end{vmatrix}.
\]

3. Using Theorem 7, find \( A^{-1} \) for the matrix \( A \) in problem 1.

4. For what values of \( \lambda \) does the equation \( Ax = \lambda x \) have a nontrivial solution where
\[
A = \begin{bmatrix} 3 & 4 \\ 1 & 3 \end{bmatrix}.
\]

For each such \( \lambda \) find the nontrivial solutions.

Hint: \( Ax = \lambda x \) is equivalent to \( Ax - \lambda x = 0 \) or \( (A - \lambda I)x = 0 \).

5. Evaluate
\[
\begin{vmatrix}
x & a & a & a \\
a & x & a & a \\
a & a & x & a \\
a & a & a & x
\end{vmatrix}.
\]

### 3.13–Eigenvalues and Eigenvectors, An Introductory Example

Consider the system of two first order linear differential equations.

\[
\begin{align*}
\dot{x}_1 &= 4x_1 - 2x_2 \\
\dot{x}_2 &= 3x_1 - x_2.
\end{align*}
\]

We introduce the vector \( \mathbf{x} \) and the matrix \( A \) defined by
\[
\begin{align*}
\mathbf{x} &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \\
A &= \begin{bmatrix} 4 & -2 \\ 3 & -1 \end{bmatrix}.
\end{align*}
\]

The vector \( \mathbf{x} \) is a function of \( t \). We define the derivative of \( \mathbf{x} \) by
\[
\frac{d\mathbf{x}}{dt} = \dot{\mathbf{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix},
\]
that is, a vector is differentiated by differentiating each component. With this notation, equation (1) may be written as
\[
\dot{\mathbf{x}} = A\mathbf{x}.
\]

One solution of (4) is the trivial solution \( \mathbf{x}(t) \equiv 0 \) (\( x_1(t) \equiv 0, \ x_2(t) \equiv 0 \)). We are looking for non-trivial solutions. Since the equations (1) are linear, homogeneous and have constant coefficients, it should be expected that there exist solutions in the form of exponentials. We look for solutions of the form
\[
\mathbf{x}(t) = \mathbf{v}e^{\lambda t}, \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}
\]
where \( \mathbf{v} \) is a constant vector, \( \mathbf{v} \neq 0 \), and \( \lambda \) is a scalar. Equation (5) is equivalent to
\[
\begin{align*}
    x_1(t) &= v_1 e^{\lambda t} \\
    x_2(t) &= v_2 e^{\lambda t}
\end{align*}
\] (6)

Note we are seeking the simplest kind of exponential solutions where the exponentials in \( x_1 \) and \( x_2 \) have the same exponent. In order to find \( \mathbf{v} \) and \( \lambda \), we substitute (5) into (4). Since
\[
\dot{\mathbf{x}} = \frac{d}{dt} \begin{bmatrix} v_1 e^{\lambda t} \\ v_2 e^{\lambda t} \end{bmatrix} = \lambda e^{\lambda t} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \lambda e^{\lambda t} \mathbf{v},
\]
we must have
\[
\lambda e^{\lambda t} \mathbf{v} = A(\mathbf{v} e^{\lambda t}) = e^{\lambda t} A \mathbf{v},
\]
or
\[
(A \mathbf{v} - \lambda \mathbf{v}) e^{\lambda t} = 0 \quad \text{for all } t.
\]
Since \( e^{\lambda t} \) is never zero, we must have
\[
A \mathbf{v} - \lambda \mathbf{v} = 0 \quad \text{or} \quad A \mathbf{v} = \lambda \mathbf{v}
\] (7)
Thus, \( \lambda \), \( \mathbf{v} \) must satisfy this equation. Note that (7) can be written as
\[
(A - \lambda I) \mathbf{v} = \mathbf{0}
\] (8)
and is a system of homogeneous algebraic equations. If a value \( \lambda \) exists for which a non-trivial solution \( \mathbf{v} \) exists, then \( \lambda \) is called an eigenvalue of \( A \) and the corresponding solution \( \mathbf{v} \neq 0 \) is called an eigenvector of \( A \) corresponding to the eigenvalue \( \lambda \).

A non-trivial solution of (8) exists if and only if
\[
\det(A - \lambda I) = 0
\]
This equation determines the eigenvalues. We have
\[
A - \lambda I = \begin{bmatrix} 4 & -2 \\ 3 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 4 - \lambda & -2 \\ 3 & -1 - \lambda \end{bmatrix}
\]
and
\[
\det[A - \lambda I] = (4 - \lambda)(-1 - \lambda) + 6 = 0.
\]
Thus \( \lambda \) must satisfy the quadratic equation
\[
\lambda^2 - 3\lambda + 2 = 0, \quad \text{or} \quad (\lambda - 2)(\lambda - 1) = 0,
\]
so that \( \lambda_1 = 1 \), \( \lambda_2 = 2 \) are the eigenvalues of \( A \).

For each eigenvalue, we are assured that non-trivial solutions of (8) exist. Putting \( \lambda = \lambda_1 = 1 \) in (8) and letting the solution be \( \mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix} \), we have
\[
(A - \lambda_1 I) \mathbf{v} = \begin{bmatrix} 3 & -2 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\]
Therefore is \( b = 3a/2 \), with \( a \) being arbitrary. In vector form, the solutions are
\[
\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ 3a/2 \end{bmatrix} = a \begin{bmatrix} 1 \\ 3/2 \end{bmatrix} = a \begin{bmatrix} 2 \\ 3 \end{bmatrix}.
\]
Let 
\[ \mathbf{v}^1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}. \]

Then \( \mathbf{v}^1 \) is an eigenvector of \( A \) corresponding to \( \lambda_1 = 1 \); of course, any non-zero multiple of \( \mathbf{v}^1 \) is also an eigenvector.

Similarly, for \( \lambda_2 = 2 \), we find
\[ (A - \lambda_2 \mathbf{I}) \mathbf{v} = \begin{bmatrix} 2 & -2 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \mathbf{0}. \]

The solutions are \( b = a \), with a arbitrary, or
\[ \mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ a \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \mathbf{av}^2 \quad \text{where \( \mathbf{v}^2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \).} \]

Thus \( \mathbf{v}^2 \) is an eigenvector of \( A \) corresponding to \( \lambda_2 = 2 \).

Corresponding to \( \lambda_1 = 1 \), \( \lambda_2 = 2 \), we have found the solutions
\[ \mathbf{x}^1 = \mathbf{v}^1 e^{\lambda_1 t} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} e^t, \quad \mathbf{x}^2 = \mathbf{v}^2 e^{\lambda_2 t} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}. \]

It is easy to verify that
\[ \mathbf{x} = c_1 \mathbf{x}^1 + c_2 \mathbf{x}^2 = c_1 \mathbf{v}^1 e^{\lambda_1 t} + c_2 \mathbf{v}^2 e^{\lambda_2 t} \]
(9)
is a solution of the DE (4) for arbitrary values of the scalars \( c_1 \), \( c_2 \). Writing out (9) we have
\[ \mathbf{x} = c_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}, \]
(10)
or
\[ x_1 = 2c_1 e^t + c_2 e^{2t}, \]
\[ x_2 = 3c_1 e^t + c_2 e^{2t}. \]
(11)

Suppose we are given an initial condition \( \mathbf{x}(0) = \mathbf{x}^0 \) where \( \mathbf{x}^0 \) is a given vector. Setting \( t = 0 \) in (9) we find that \( c_1 \), \( c_2 \) must satisfy
\[ \mathbf{x}^0 = c_1 \mathbf{v}^1 + c_2 \mathbf{v}^2 = c_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \]
(12)

Since the eigenvectors \( \mathbf{v}^1 \), \( \mathbf{v}^2 \) are LI, they form a basis for \( \mathbb{C}^2 \), and thus unique constants, \( c_1 \), \( c_2 \), can always be found for any given \( \mathbf{x}^0 \). Thus, with the solution (10), we can solve any initial value problem. In fact, the solution is the general solution, as we shall show later.

Let us find the constants \( c_1 \) and \( c_2 \) in the case \( \mathbf{x}^0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \). Equation (12) becomes
\[ \mathbf{x}^0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}. \]

We can solve these two equations for \( c_1 \) and \( c_2 \) by elimination. Alternatively we can use inverse matrices
\[ \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{-1} \begin{bmatrix} 1 & -1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \]

Thus \( c_1 = 1 \), \( c_2 = 2 \) and the solution satisfying the initial condition is
\[ x_1 = e^t - e^{2t}, \]
\[ x_2 = 3e^t - e^{2t}. \]

**Exercises 3.13**

1. Solve \( \dot{\mathbf{x}} = A \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \) where \( A = \begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix} \).
2. Find the eigenvalues and eigenvectors for \( A = \begin{bmatrix} -1 & 2 \\ 2 & 2 \end{bmatrix} \).

3.14–Eigenvalues and Eigenvectors

**Definition 1.** If \( A \) is an \( n \times n \) matrix, then \( \lambda \) is called an eigenvalue of \( A \) if \( Av = \lambda v \) has a non-trivial solution; any such non-trivial solution \( v \) is called an eigenvector of \( A \) corresponding to the eigenvalue \( \lambda \).

The equation \( Av = \lambda v \) can be written

\[
(A - \lambda I)v = 0. 
\]

This has a non-trivial solution if and only if

\[
\text{det}(A - \lambda I) = 0. 
\]

Since \( A - \lambda I \) simply subtracts \( \lambda \) from the diagonal elements of \( A \), we have

\[
A - \lambda I = \begin{bmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{bmatrix}. 
\]

Thus \( \text{det}(A - \lambda I) \) is a polynomial in \( \lambda \) degree \( n \). The equation

\[
c(\lambda) = \text{det}(A - \lambda I) = 0 
\]

is called the characteristic equation of the matrix \( A \). This equation always has at least one complex root. Therefore, every matrix has at least one eigenvalue and corresponding eigenvector.

**Example 1.** \( A = \begin{bmatrix} 4 & -2 \\ 3 & -1 \end{bmatrix} \). \( |A - \lambda I| = \begin{vmatrix} 4 - \lambda & -2 \\ 3 & -1 - \lambda \end{vmatrix} = 0 \). The characteristic equation is \( \lambda^2 - 3\lambda + 2 = 0 \), the eigenvalues are 1 and 2. As we have seen in the last section, the eigenvectors are

\[
\text{for } \lambda_1 = 1, \quad v_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad \text{and for } \lambda_2 = 2, \quad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. 
\]

Thus we have 2 eigenvalues and 2 eigenvectors. Since \( \text{det} \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} = 1 \neq 0 \), the eigenvectors are LI.

**Example 2.**

\[
A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} 
\]

\[
|A - \lambda I| = \begin{vmatrix} 3 - \lambda & 1 \\ 0 & 3 - \lambda \end{vmatrix} = (3 - \lambda)^2 - 0 = 0. 
\]

The only eigenvalue is \( \lambda = 3 \), which has multiplicity 2. The eigenvector is a solution of

\[
(A - 3I)v = 0 
\]

or

\[
\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]
or
\[ v_2 = 0, \quad v_1 \text{ arbitrary} . \]
Thus \[
\begin{bmatrix} 1 \\ 0 \end{bmatrix}
\]
is an eigenvector. Here, we have only one eigenvalue and one LI eigenvector.

**Example 3.** \[ A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}, \quad |A - \lambda I| = \begin{bmatrix} 3 - \lambda & 0 \\ 0 & 3 - \lambda \end{bmatrix} = (3 - \lambda)^2 = 0 \]

Here \( \lambda = 3 \) is eigenvalue of multiplicity 2. To find the eigenvectors, we solve \((A - 3I)v = 0\) or
\[
\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

Here every vector (except \( \mathbf{0} \)) is an eigenvector. In particular, \( v^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) and \( v^2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \) are eigenvectors. In this case, we have one eigenvalue of multiplicity 2, and 2 LI eigenvectors corresponding to this single eigenvalue.

One can prove that if \( \lambda_1 \) is an eigenvalue of multiplicity \( s \), there may exist anywhere from 1 to \( s \) LI eigenvectors.

**Example 4.** For \( n = 3 \), consider the following three matrices

\[
\begin{align*}
(a) & \quad A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \\
(b) & \quad B = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \\
(c) & \quad C = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}.
\end{align*}
\]

Each matrix has \( \lambda = 2 \) as its only eigenvalue, and the multiplicity of the eigenvalue is 3. To see how many eigenvectors we have, we compute the rank of

\[
\begin{align*}
(a) & \quad A - 2I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\
(b) & \quad B - 2I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\
(c) & \quad C - 2I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.
\end{align*}
\]

Note that \( A - 2I \) has rank 0 and has 3 - 0 = 3 LI eigenvectors, \( B - 2I \) has rank 1 and has 3 - 1 = 2 LI eigenvectors and \( C - 2I \) has rank 2 and has 3 - 2 = 1 LI eigenvectors.

We shall now show that eigenvectors corresponding to distinct eigenvalues are linearly independent.

**Theorem 1.** If \( \lambda_1, \ldots, \lambda_k \) are distinct eigenvalues of \( A \) and \( v^1, \ldots, v^k \), are corresponding eigenvectors then \( \{v^1, \ldots, v^k\} \) is a LI set.

**Proof** If \( \{v^1, \ldots, v^k\} \) is LD, then at least one of the vectors is a linear combination of the others. Suppose
\[
\begin{align*}
v^1 &= \alpha_1 v^2 + \ldots + \alpha_k v^k, \\
\end{align*}
\]
where we can assume \( \{v^2, \ldots, v^k\} \) is LI. Then \( Av^1 = \alpha_1 Av^2 + \ldots + \alpha_k Av^k \), or
\[
\begin{align*}
\lambda_1 v^1 &= \alpha_1 \lambda_2 v^2 + \ldots + \alpha_k \lambda_k v^k.
\end{align*}
\]
Multiply Equation (4) by \( \lambda_1 \) and subtract Equation (5) to get
\[
\alpha_1 (\lambda_1 - \lambda_2) v^2 + \alpha_2 (\lambda_1 - \lambda_3) v^3 + \ldots + \alpha_k (\lambda_1 - \lambda_k) v^k = 0.
\]
Since \( \{v^2, \ldots, v^k\} \) is LI, we must have
\[
\alpha_1 (\lambda_1 - \lambda_2) = 0, \ldots, \alpha_k (\lambda_1 - \lambda_k) = 0.
\]
since \((\lambda_1 - \lambda_i) \neq 0\), it follows that \(\alpha_1 = \alpha_2 \ldots \alpha_k = 0\).

In particular, it follows from Theorem 1 that if an \(n \times n\) matrix has \(n\) distinct eigenvalues, the corresponding \(n\) eigenvectors are linearly independent and form a basis for \(\mathbb{C}^n\).

**Theorem 2.** If \(\lambda_1\) is an eigenvalue of \(A\) then \(\lambda_1^k\) is an eigenvalue of \(A^k\) (\(k\) a positive integer).

**Proof** Since \(Av = \lambda_1 v\), we have

\[
A^2v = \lambda_1 Av = \lambda_1^2v
\]

\[
\vdots
\]

\[
A^kv = \lambda_1^kv
\]

therefore \(\lambda_1^k\) is an eigenvalue of \(A^k\).

**Exercises 3.14**

1. Find the eigenvalues and eigenvectors for
   a. \[
   \begin{bmatrix}
   2 & 5 \\
   3 & 2
   \end{bmatrix}
   \]
   b. \[
   \begin{bmatrix}
   1 & -1 \\
   -1 & 1
   \end{bmatrix}
   \]

2. Determine with as little computation as possible whether or not 3 is an eigenvalue of each of the following matrices. Give reasons.
   a. \[
   \begin{bmatrix}
   1 & 0 & 0 \\
   0 & 2 & 0 \\
   0 & 0 & 0
   \end{bmatrix}
   \]
   b. \[
   \begin{bmatrix}
   1 & 0 & 0 \\
   0 & 1 & 0 \\
   0 & 0 & 3
   \end{bmatrix}
   \]
   c. \[
   \begin{bmatrix}
   4 & -1 & 1 \\
   -1 & 4 & -1 \\
   2 & -2 & 5
   \end{bmatrix}
   \]

3. Given \(A = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 3 \end{bmatrix}\), \(v^1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\), \(v^2 = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}\).
   a. without finding the characteristic equation determine whether or not \(v^1\) is an eigenvector of \(A\), if so what is the eigenvalue.
   b. same as a. for \(v^2\)

4. Find the eigenvalues and a set of three LI eigenvectors for
   a. \[
   \begin{bmatrix}
   3 & -1 & 0 \\
   -1 & 2 & -1 \\
   0 & -1 & 3
   \end{bmatrix}
   \]
   b. \[
   \begin{bmatrix}
   2 & -1 & 1 \\
   1 & 0 & 1 \\
   1 & -1 & 2
   \end{bmatrix}
   \]
   c. \[
   \begin{bmatrix}
   0 & 2 & 3 \\
   1 & 1 & 3 \\
   1 & 2 & 2
   \end{bmatrix}
   \]

5. Find the eigenvalues and eigenvectors for each of the following
   a. \[
   \begin{bmatrix}
   2 & -1 & 0 \\
   -1 & 2 & -1 \\
   0 & -1 & 2
   \end{bmatrix}
   \]
   b. \[
   \begin{bmatrix}
   2 & -1 & 1 \\
   1 & -1 & 1 \\
   1 & 1 & 1
   \end{bmatrix}
   \]
   c. \[
   \begin{bmatrix}
   7 & 1 & 1 \\
   -1 & 7 & 0 \\
   1 & -1 & 6
   \end{bmatrix}
   \]

6. Find the eigenvalues and eigenvectors (they are complex) for

\[
A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}
\]

7. a. If \(Au = 3u\) where \(A\) is an \(n \times n\) matrix and \(u\) is a column vector, find a vector \(x\) such that \(Ax = -2u\).
   b. If 3 is an eigenvalue of a matrix \(A\), what is an eigenvalue of \(A^2 - A + I\).
8. What are the only possible eigenvalues of an $n \times n$ matrix $A$ under the following conditions
   a. $A^2 = 0$ (A need not be the zero matrix).
   b. $A^2 = I$ (A need not be the identity matrix).
   c. $A^2 + 5A + 6I = 0$

9. Prove that $A^{-1}$ exists if and only if 0 is not an eigenvalue of $A$.

10. If $A$ is an $n \times n$ matrix and $u$, $v$ are column vectors such that $Au = 3u$, $Av = -2v$
    a. evaluate $(A^2 - 5A + 6I)u + A^2v$ in terms of $u$ and $v$
    b. find a solution $x$ of $Ax = u + 5v$

11. If $A^{-1}$ exists and $\lambda$ is an eigenvalue of $A$ with eigenvector $v$, find an eigenvalue and eigenvector for $A^{-1}$.

3.15–Solution of Systems of Differential Equations

We consider a system of $n$ first order, linear, homogeneous differential equations with constant coefficients

\[
\begin{align*}
\dot{x}_1 &= a_{11}x_1 + \ldots + a_{1n}x_n \\
\dot{x}_2 &= a_{21}x_1 + \ldots + a_{2n}x_n \\
&\quad \vdots \\
\dot{x}_n &= a_{n1}x_1 + \ldots + a_{nn}x_n
\end{align*}
\]

(1)

Let $x, A$ be defined by

\[
x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \ldots & a_{nn} \end{bmatrix}
\]

(2)

Using this notation the equations (1) can be written in the form

\[\dot{x} = Ax\]

(3)

Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $A$ and $v^1, \ldots, v^n$ be the corresponding eigenvectors. That is $Av^i = \lambda_i v^i$, $i = 1, \ldots, n$. We assume that $A$ has $n$ linearly independent eigenvectors, that is the set \{v^1, \ldots, v^n\} is linearly independent. This will always occur if the $n$ eigenvalues are all distinct, and may occur if some of the eigenvalues are repeated.

We assume a solution of the form

\[x = v e^{\lambda t}\]

(4)

where the scalar $\lambda$ and the constant non-zero vector $v$ must be determined to satisfy the differential equation $\dot{x} = Ax$. Substituting (4) into the differential equation, we find

\[\dot{x} = v \lambda e^{\lambda t} = A(ve^{\lambda t}) = e^{\lambda t}Av.
\]

(5)

Since $e^{\lambda t}$ is never zero, we must have

\[Av = \lambda v, \quad v \neq 0.
\]

(6)

This is just the equation that defines eigenvalues and eigenvectors. We have assumed that there are $n$ linearly independent eigenvectors $v^1, \ldots, v^n$ corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_n$. Thus

\[x = v^i e^{\lambda_i t}, \quad i = 1, \ldots, n\]
are $n$ solutions of the differential equation (3). Since the DE is linear and homogeneous, a linear combination of solutions is again a solution. Thus
\[ x = \alpha_1 v^1 e^{\lambda_1 t} + \alpha_2 v^2 e^{\lambda_2 t} + \ldots + \alpha_n v^n e^{\lambda_n t} \]  
(7)
is a solution for arbitrary values of $\alpha_1, \ldots, \alpha_n$.
In Theorem 1, we will show that Equation (7) represents the general solution of the DE (3). Suppose we are given that the solution must satisfy and initial condition $x(0) = x^0$ where $x^0$ is a given vector. We put $t = 0$ in (2) to obtain
\[ x^0 = \alpha_1 v^1 + \ldots + \alpha_n v^n \]  
(8)
and solve for $\alpha_1, \ldots, \alpha_n$. Since $\{v^1, \ldots, v^n\}$ is a set of $n$ linearly independent vectors in $\mathbb{C}^n$, the set forms a basis for $\mathbb{C}^n$. Thus, there exist unique numbers $\alpha_1, \ldots, \alpha_n$ satisfying (8) and the initial value problem $\dot{x} = Ax$, $x(0) = x^0$ has a solution for every $x^0$.

Example 1. $\dot{x} = Ax$, $x(0) = x^0$
where $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$, $x^0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$
\[ \det(A - \lambda I) = \det \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -6 & -11 & -\lambda \end{bmatrix} = -\lambda \det \begin{bmatrix} -\lambda & 1 \\ -11 & -6 - \lambda \end{bmatrix} - 6 \det \begin{bmatrix} 1 & 0 \\ -\lambda & 1 \end{bmatrix} \]
\[ = -\lambda(\lambda(16 + \lambda) + 11) - 6(1) = -\lambda^3 - 6\lambda^2 - 11\lambda - 6. \]
The eigenvalues are roots of $\lambda^3 + 6\lambda^2 + 11\lambda + 6 = 0$. If there are any integer roots, they must be factors of the constant term. Thus, we try $\lambda = \pm 1, \pm 2, \pm 3$. We find the eigenvalues are $\lambda_1 = -1$, $\lambda_2 = -2$, $\lambda_3 = -3$.
Since the eigenvalues are distinct, the three eigenvectors corresponding to these eigenvalues must be linearly independent. the eigenvectors turn out to be
\[ v^1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad v^2 = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}, \quad v^3 = \begin{bmatrix} 1 \\ -3 \\ 9 \end{bmatrix} \]
The general solution is
\[ x = \alpha_1 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^{-t} + \alpha_2 \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} e^{-2t} + \alpha_3 \begin{bmatrix} 1 \\ -3 \\ 9 \end{bmatrix} e^{-3t} \]
or
\[ x_1 = \alpha_1 e^{-t} + \alpha_2 e^{-2t} + \alpha_3 e^{-3t} \]
\[ x_2 = -\alpha_1 e^{-t} - 2\alpha_2 e^{-2t} - 3\alpha_3 e^{-3t} \]
\[ x_3 = \alpha_1 e^{-t} + 4\alpha_2 e^{-2t} + 9\alpha_3 e^{-3t} \]
To satisfy the initial condition, we must solve the following equations for $\alpha_1$, $\alpha_2$, $\alpha_3$
\[ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ -3 \\ 9 \end{bmatrix} \]
or
\[ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} \]
We find \( \alpha_1 = 3, \alpha_2 = -3, \alpha_3 = 1 \). The unique solution satisfying the initial condition is
\[
\begin{align*}
x_1 &= 3e^{-t} - 3e^{-2t} + e^{-3t} \\
x_2 &= -3e^{-t} + 6e^{-2t} - 3e^{-3t} \\
x_3 &= 3e^{-t} - 12e^{-2t} + 9e^{-3t}
\end{align*}
\]

*Example 2.* Find the general solution of \( \dot{x} = Ax \) where
\[
A = \begin{bmatrix}
2 & 2 & -6 \\
2 & -1 & -3 \\
-2 & -1 & 1
\end{bmatrix}
\]

We find that \( \det(A - \lambda I) = \lambda^3 - 2\lambda^2 - 20\lambda - 24 = 0 \). By trial and error, we find \( \lambda = -2 \) is one root. Therefore \( \lambda + 2 \) is a factor of \( \lambda^3 - 2\lambda^2 - 20\lambda - 24 \). Dividing the cubic by \( \lambda + 2 \), we obtain \( \lambda^2 - 4\lambda - 6 = (\lambda + 2)(\lambda - 6) \). Thus \( \lambda = 6 \) is a simple eigenvalue and \( \lambda = -2 \) is a double eigenvalue. At this point, we cannot be sure that there are two linearly independent eigenvectors corresponding to \( \lambda = -2 \), but this turns out to be the case.

To find the eigenvectors corresponding to \( \lambda = -2 \), we must solve
\[
(A + 2I)v = 0 \quad \text{where} \quad A + 2I = \begin{bmatrix}
4 & 2 & -6 \\
2 & 1 & -3 \\
-2 & -1 & 3
\end{bmatrix}
\]

Reduce this matrix to row echelon form
\[
\begin{pmatrix}
1 & 1/2 & -3/2 \\
2 & 1 & -3 \\
-2 & -1 & 3
\end{pmatrix} \rightarrow \begin{pmatrix}
1 & 1/2 & -3/2 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

We see that \( v_2, v_3 \) are free and \( v_1 = -\frac{1}{2}v_2 + \frac{3}{2}v_3 \) and the general solution is
\[
v = \begin{bmatrix}
v_1 \\
v_2 \\
v_3
\end{bmatrix} = \begin{bmatrix}
-v_2/2 + 3v_3/2 \\
v_2/3 \\
v_3
\end{bmatrix} = v_2 \begin{bmatrix}
-1/2 \\
1 \\
0
\end{bmatrix} + v_3 \begin{bmatrix}
3/2 \\
0 \\
1
\end{bmatrix}
\]

Thus, the two vectors on the right are linearly independent eigenvectors. To avoid fractions, we take as eigenvectors corresponding to \( \lambda = -2 \)
\[
v^1 = \begin{bmatrix}
-1 \\
2 \\
0
\end{bmatrix}, \quad v^2 = \begin{bmatrix}
3 \\
0 \\
2
\end{bmatrix}
\]

Corresponding to \( \lambda = 6 \), we find the eigenvector
\[
v^3 = \begin{bmatrix}
2 \\
1 \\
-1
\end{bmatrix}.
\]

The general solution is therefore
\[
x = \alpha_1 \begin{bmatrix}
-1 \\
2 \\
0
\end{bmatrix} e^{-2t} + \alpha_2 \begin{bmatrix}
3 \\
0 \\
2
\end{bmatrix} e^{-2t} + \alpha_3 \begin{bmatrix}
2 \\
1 \\
-1
\end{bmatrix} e^{6t}.
\]
Example 3. An infinite pipe is divided into 4 sections $S_0$, $S_1$, $S_2$, $S_3$ as shown.

A chemical solution with concentration $y_1(t)$ at times $t$ is in section $S_1$ and the same solution has concentration $y_2(t)$ in section $S_2$. The concentrations in section $S_0$ and $S_3$ are assumed to be zero initially and remain zero for all time. This is reasonable since $S_0$ and $S_3$ have infinite volume. At $t = 0$, the concentrations in $S_1$ and $S_2$ are assumed to be $y_1(0) = a_1$, $y_2(0) = a_2$, where $a_1 > 0$, $a_2 > 0$.

Diffusion starts at $t = 0$ according to the law: at time $t$, the diffusion rate between two adjacent sections equals the difference in concentrations. We are assuming that the concentrations in each section remain uniform. Thus we obtain

$$\frac{dy_1}{dt} = (y_2 - y_1) + (0 - y_1)$$

$$\frac{dy_2}{dt} = (0 - y_2) + (y_1 - y_2)$$

or

$$\dot{y}_1 = -2y_1 + y_2$$

$$\dot{y}_2 = y_1 - 2y_2$$

In matrix form, we have $\dot{y} = Ay$, $y(0) = y^0$, where

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}, \quad y^0 = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

The eigenvalues of $A$ are $\lambda = -1$, $\lambda = -3$ and the corresponding eigenvectors are

$$v^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad v^{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The general solution is

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + \alpha_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t}$$

To satisfy the initial conditions, we must have $\alpha_1 = \frac{a_1 + a_2}{2}$, $\alpha_2 = \frac{a_1 - a_2}{2}$ and the solution is

$$y_1 = \frac{a_1 + a_2}{2} e^{-t} + \frac{a_1 - a_2}{2} e^{-3t}$$

$$y_2 = \frac{a_1 + a_2}{2} e^{-t} - \frac{a_1 - a_2}{2} e^{-3t}$$

We note that $y_1(t) \to 0$ and $y_2(t) \to 0$ as $t \to \infty$ as one would expect. Furthermore, we see that since $a_1 > 0$ and $a_2 > 0$, we must have $y_1(t) > 0$ and $y_2(t) > 0$ for all $t$. Since $e^{-3t}$ approaches zero more rapidly than $e^{-t}$, the concentrations $y_1(t)$ and $y_2(t)$ will be both almost equal to $(a_1 + a_2)e^{-t}/2$ after a short time.

Complex Eigenvalues
The eigenvalues may be complex numbers, this will yield eigenvectors with complex numbers as elements and the solution we get by the above method will be the general complex valued solution. If \( A \) is a real matrix, we can write the final solution in real form. In this regard, the following fact is useful.

If \( A \) is a real matrix and \( \lambda = \alpha + i\beta \) is a complex eigenvalue with complex eigenvector \( \mathbf{w} = \mathbf{u} + i\mathbf{v} \) (\( \mathbf{u}, \mathbf{v} \) are real vectors), then \( \mathbf{w}e^{(\alpha+i\beta)t} \) is a complex valued solution of \( \dot{\mathbf{x}} = A\mathbf{x} \) and the real and imaginary parts of \( \mathbf{w}e^{(\alpha+i\beta)t} \) are real solutions.

We need to find the real and imaginary parts:

\[
\mathbf{w}e^{(\alpha+i\beta)t} = (\mathbf{u} + i\mathbf{v})e^{\alpha t}e^{i\beta t} = e^{\alpha t}(\mathbf{u} + i\mathbf{v})(\cos \beta t + i \sin \beta t) = e^{\alpha t}(\mathbf{u}\cos \beta t - \mathbf{v}\sin \beta t) + ie^{\alpha t}(\mathbf{u}\sin \beta t + \mathbf{v}\cos \beta t)
\]

We see that the real solutions are

\[
e^{\alpha t}(\mathbf{u}\cos \beta t - \mathbf{v}\sin \beta t) \text{ and } e^{\alpha t}(\mathbf{u}\sin \beta t + \mathbf{v}\cos \beta t).
\]

**Example 4.** Consider

\[
\dot{\mathbf{x}} = A\mathbf{x}, \quad A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}
\]

\[
\begin{vmatrix} 1 - \lambda & 1 \\ -1 & 1 - \lambda \end{vmatrix} = 0, \quad \lambda = 1 \pm i.
\]

Consider \((A - \lambda_1 I)\mathbf{w} = 0\) where \(\lambda_1 = 1 + i\), that is

\[
\begin{bmatrix} -i & 1 \\ -1 & i \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

or

\[
-iw_1 + w_2 = 0 \\
-w_1 + iw_2 = 0
\]

Note the 2nd equation is \(i\) times the first equation, so we may use only the first equation. Let \(w_1 = 1\) then \(w_2 = i\) and the eigenvector is

\[
\mathbf{w} = \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\]

Thus \(\mathbf{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\) and \(\mathbf{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\).

The real solutions are \(e^t(\mathbf{u}\cos t - \mathbf{v}\sin t)\) and \(e^t(\mathbf{u}\sin t + \mathbf{v}\cos t)\). Thus, the general real solution is

\[
\mathbf{x} = c_1 e^t(\mathbf{u}\cos t - \mathbf{v}\sin t) + c_2 e^t(\mathbf{u}\sin t + \mathbf{v}\cos t)
\]

or

\[
x_1 = e^t(c_1 \cos t + c_2 \sin t), \\
x_2 = e^t(c_2 \cos t - c_1 \sin t).
\]

Finally, as promised earlier, we shall prove that the solution we have obtained, in the case where \(n\) linearly independent eigenvectors exist, is the general solution.
Theorem 1. If $A$ is an $n \times n$ matrix and has $n$ linearly independent eigenvectors $v^1, \ldots, v^n$ corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_n$, then the general solution of $\dot{x} = Ax$ is
\[ x = \alpha_1 v^1 e^{\lambda_1 t} + \ldots + \alpha_n v^n e^{\lambda_n t}, \]
where $\alpha_1, \ldots, \alpha_n$ are arbitrary numbers. Furthermore, the initial value problem $\dot{x} = Ax$, $x(0) = x^{(0)}$ has a unique solution for every $x^{(0)}$.

Proof. Let $x$ be any solution of $\dot{x} = Ax$. We make a change of variables by letting $x = Ty$, where $T$ is the constant matrix whose columns are the eigenvectors of $A$
\[ T = [v^1, \ldots, v^n]. \]
Since the eigenvectors are linearly independent, $T$ is non-singular. Substituting into the DE we have
\[ \dot{x} = T\dot{y} = ATy \text{ or } \dot{y} = T^{-1}ATy. \]
Now $AT = A[v^1, \ldots, v^n] = [Av^1, \ldots, Av^n]$. Since $Av^i = \lambda_i v^i$, we have
\[ AT = [\lambda_1 v^1, \ldots, \lambda_n v^n] = [v^1, \ldots, v^n] \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}. \]
Letting $D$ be the diagonal matrix on the right
\[ D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}, \]
we have
\[ AT = TD \quad \text{or} \quad T^{-1}AT = D. \]
Thus, $y$ satisfies $\dot{y} = Dy$ which written out is:
\[ \dot{y}_1 = \lambda_1 y_1, \quad \dot{y}_2 = \lambda_2 y_2, \quad \ldots, \quad \dot{y}_n = \lambda_n y_n. \]
We can solve each of these equations individually since each equation has only one unknown (such equations are said to be 'uncoupled'). We have
\[ y_1 = e^{\lambda_1 t} \alpha_1, \quad y_2 = e^{\lambda_2 t} \alpha_2, \ldots, \quad y_n = e^{\lambda_n t} \]
where $\alpha_1, \ldots, \alpha_n$ are arbitrary. Since $x = Ty$ we have
\[ x = Ty = [v^1, \ldots, v^n] \begin{bmatrix} e^{\lambda_1 t} \alpha_1 \\ \vdots \\ e^{\lambda_n t} \alpha_n \end{bmatrix} = \alpha_1 e^{\lambda_1 t} v^1 + \ldots + \alpha_n e^{\lambda_n t} v^n. \]
Since \( \mathbf{x} \) was any solution, we have shown that every solution can be written in the above form.

If we use the initial condition \( \mathbf{x}(0) = \mathbf{x}^0 \) we find as in (9) above that unique constants \( \alpha_1, \ldots, \alpha_n \) are obtained because of the linear independence of the \( n \) eigenvectors. Thus, the initial value problem has a unique solution.

**Exercises 3.15**

1. Find the solution of \( \dot{\mathbf{x}} = \mathbf{A} \mathbf{x} \), \( \mathbf{x}(0) = \mathbf{x}^0 \) in the following cases:
   a. \( \mathbf{A} = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}, \mathbf{x}^0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \)
   b. \( \mathbf{A} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}, \mathbf{x}^0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \)

2. Find the solution of \( \dot{\mathbf{x}} = \mathbf{A} \mathbf{x} \), \( \mathbf{x}(0) = \mathbf{x}^0 \) for the following cases
   a. \( \mathbf{A} = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}, \mathbf{x}^0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \)
   b. \( \mathbf{A} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & 1 \end{bmatrix}, \mathbf{x}^0 = \begin{bmatrix} 2 \\ -1 \\ -4 \end{bmatrix} \)

3. Find the general solution of \( \dot{\mathbf{x}} = \mathbf{A} \mathbf{x} \) in each of the following
   a. \( \mathbf{A} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix} \)
   b. \( \mathbf{A} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix} \)

4. Find the general solution of \( \dot{\mathbf{x}} = \mathbf{A} \mathbf{x} \) in real form if \( \mathbf{A} = \begin{bmatrix} -1 & 2 \\ -5 & -3 \end{bmatrix} \).

5. Tank A contains 100 gals. of brine initially containing 50 lbs. of salt. Tank B initially contains 200 gals. of pure water. Pure water is pumped into Tank A at rate of one gallon per minute. The well-stirred mixture is pumped into Tank B at rate of 2 gals./min. The well-stirred mixture in Tank B is pumped out at 2 gals./min. Half of this is returned to Tank A and the other half is discarded. Set up the DE’s and initial conditions for the amount of salt in each tank at time \( t \).

6. If \( \mathbf{A} \) is a 3 × 3 matrix and \( \mathbf{u}, \mathbf{v}, \mathbf{w} \) are 3–dimensional non-zero vectors such that \( \mathbf{A} \mathbf{u} = 3 \mathbf{u}, \mathbf{A} \mathbf{v} = \mathbf{0}, \mathbf{A} \mathbf{w} = -5 \mathbf{w} \)
   a. What is the general solution of \( \dot{\mathbf{x}} = \mathbf{A} \mathbf{x} \)?
   b. What is the solution of \( \dot{\mathbf{x}} = \mathbf{A} \mathbf{x}, \mathbf{x}(0) = -3 \mathbf{v} + \mathbf{w} \)?

7. Find the general solution of
   \[
   x\ddot{x} + 6\dot{x} + 11\dot{x} + 6x = 0
   \]
   by the following procedure. Let
   \[
   \begin{align*}
   \dot{x} &= y \\
   \dot{y} &= z,
   \end{align*}
   \]
   then
   \[
   \dot{z} = \ddot{y} = \ddot{x} = -6\dot{x} - 11\dot{y} - 6z.
   \]
   Let \( \mathbf{w} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \) and write the above equations in the form \( \dot{\mathbf{w}} = \mathbf{A} \mathbf{w} \) for an appropriate \( \mathbf{A} \). Solve for \( \mathbf{w} \) and then find \( x \).

8. By the method of problem 7 solve
   DE: \( \ddot{x} - 4\dot{x} - \dot{x} + 4x = 0 \)
   IC: \( x(0) = 1, \dot{x}(0) = 0, \ddot{x}(0) = 0 \).
Section 3.16–Systems of Linear Difference Equations

Consider a system of \( n \) first order linear homogeneous difference equations

\[
x_1(k + 1) = a_{11}x_1(k) + \ldots + a_{1n}x_n(k)
\]

\[
\vdots
\]

\[
x_n(k + 1) = a_{n1}x_1(k) + \ldots + a_{nn}x_n(k)
\]

where \( k = 0, 1, 2, \ldots \) and we have written the unknown sequences as \( x_i(k) \) rather than \( (x_i)_k \). Define a matrix \( A \) and a sequence of vectors \( x^k \) by

\[
A = \begin{bmatrix}
a_{11} & \ldots & a_{1n} \\
\vdots & \ddots & \vdots \\
a_{n1} & \ldots & a_{nn}
\end{bmatrix}, \quad x^k = \begin{bmatrix}
x_1(k) \\
x_2(k) \\
\vdots \\
x_n(k)
\end{bmatrix}.
\]

We may now write (1) as

\[
x^{k+1} = Ax^k, \quad k = 0, 1, 2, \ldots
\]

Assuming \( x^0 \) is known, we find by successive substitution

\[
x^1 = Ax^0
\]

\[
x^2 = Ax^1 = A(Ax^0) = A^2x^0
\]

\[
\vdots
\]

\[
x^k = A^kx^0
\]

Thus to solve (1) we must evaluate \( A^kx^0 \) for arbitrary \( k \). We shall do this under the assumption that \( A \) has \( n \) linearly independent eigenvectors. Assume the eigenvalues of \( A \) are \( \lambda_1, \ldots, \lambda_n \) and the set of corresponding eigenvectors \( \{v^1, \ldots, v^n\} \) is LI. The set of eigenvectors is a basis for \( \mathbb{C}^n \). Thus there exist unique scalars \( \alpha_1, \ldots, \alpha_n \) such that

\[
x^0 = \alpha_1v^1 + \ldots + \alpha_nv^n
\]

Thus we have

\[
x^k = A^kx^0 = A^k(\alpha_1v^1 + \ldots + \alpha_nv^n)
\]

\[
= \alpha_1A^k v^1 + \ldots + \alpha_n A^k v^n.
\]

Since the eigenvalues are \( \lambda_i \) and the corresponding eigenvectors are \( v^i \) we have \( A\lambda_i = \lambda_i v^i \), from which it follows that \( A^k v^i = \lambda_i^k v^i \) for any positive integer \( k \). Therefore, from (6) we obtain

\[
x^k = \alpha_1\lambda_1^k v^1 + \ldots + \alpha_n \lambda_n^k v^n, \quad k = 0, 1, 2, \ldots
\]

For arbitrary values of the \( \alpha_i \), equation (7) represents the general solution of (3). If we want to satisfy a given initial condition, we put \( k = 0 \) in (7) and solve for the \( \alpha_i \) (i.e., use equation (5)).

Example 1. Solve \( x^{k+1} = Ax^k, \quad k = 0, 1, 2, \ldots, \) where

\[
A = \begin{bmatrix}
1 & 1 \\
3 & -1
\end{bmatrix}, \quad x^0 = \begin{bmatrix}
1 \\
2
\end{bmatrix}.
\]

The eigenvalues and eigenvectors are

\[
\lambda_1 = 2, \quad v^1 = \begin{bmatrix}
1 \\
1
\end{bmatrix}, \quad \lambda_2 = -2, \quad v^2 = \begin{bmatrix}
-1 \\
3
\end{bmatrix}.
\]
Note that the eigenvectors are LI (they must be, since the eigenvalues are distinct). The general solution is

$$x^k = \alpha_1 2^k \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \alpha_2 (-2)^k \begin{bmatrix} -1 \\ 3 \end{bmatrix}.$$  

Setting $k = 0$ we get

$$x^0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}.$$  

The solutions of this system are $\alpha_1 = 5/4$, $\alpha_2 = 1/4$. The solution of the difference equation satisfying the given initial condition is

$$x^k = \frac{5}{4} 2^k \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{4} (-2)^k \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \quad k = 0, 1, 2, \ldots$$  

If the components of $x^k$ are denoted by $x_1(k)$, $x_2(k)$, these components are

$$x_1(k) = \frac{5}{4} 2^k - \frac{1}{4} (-2)^k$$  

$$x_2(k) = \frac{5}{4} 2^k + \frac{3}{4} (-2)^k.$$

### The Diagonalization Method

There is an alternative method of evaluating $A^k x^0$ which consists of computing the matrix $A^k$. We discuss how this can be done, again assuming that $A$ has $n$ LI eigenvectors.

Let $T$ be the matrix whose columns are the LI eigenvectors.

$$T = [v^1, \ldots, v^n]$$  

Then as shown in the last section (note $T^{-1}$ exists)

$$A = T D T^{-1}$$  

where $D$ is the diagonal matrix whose diagonal entries are the eigenvalues

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}.$$  

Now it is easy to show that

$$A^k = T D^k T^{-1}, \quad k = 0, 1, 2, \ldots$$  

where

$$D^k = \begin{bmatrix} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^k & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^k \end{bmatrix}.$$  

Thus, if the eigenvalues and eigenvectors are known, we have an explicit formula for an arbitrary power of $A$.

#### Example 2

Using the same difference equation as in example 1, we have

$$T = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}, \quad T^{-1} = \frac{1}{4} \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}.$$
Thus

\[ A = T D T^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1/4 & 3/4 \end{bmatrix} \]

This can readily be checked. Equation (11) yields

\[
A^k = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2^k & 0 \\ 0 & (-2)^k \end{bmatrix} \begin{bmatrix} 1/4 & 3/4 \end{bmatrix} \\
= \frac{1}{4} \begin{bmatrix} 3 \cdot 2^k + (-2)^k & 2^k - (-2)^k \\ 3 \cdot 2^k - 3(-2)^k & 2^k + 3(-2)^k \end{bmatrix} 
\]

Now the solution satisfying the initial condition is just

\[
x^k = A^k x^0 \\
= \frac{1}{4} \begin{bmatrix} 3 \cdot 2^k + (-2)^k & 2^k - (-2)^k \\ 3 \cdot 2^k - 3(-2)^k & 2^k + 3(-2)^k \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\
= \frac{1}{4} \begin{bmatrix} 5 \cdot 2^k - (-2)^k \\ 5 \cdot 2^k + 3(-2)^k \end{bmatrix}
\]

which is the same result obtained in example 1.

**Example 3.** Each year 1/10 of the people outside California move in and 2/10 of the people inside California move out. Suppose initially there are \( x_0 \) people outside and \( y_0 \) people inside. What is the number of people inside and outside after \( k \) years. What happens as \( k \to \infty \).

At the end of the \( k+1 \)st year, we have

\[
x_{k+1} = \frac{9}{10} x_k + \frac{2}{10} y_k \\
y_{k+1} = \frac{1}{10} x_k + \frac{8}{10} y_k
\]

This system can be written in the form \( z^{k+1} = A z^k \) where

\[
z^k = \begin{bmatrix} x_k \\ y_k \end{bmatrix}, \quad A = \begin{bmatrix} 9/10 & 2/10 \\ 1/10 & 8/10 \end{bmatrix}
\]

The eigenvalues and eigenvectors are

\[
\lambda_1 = 1, \quad \mathbf{v}^1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \lambda_2 = \frac{7}{10}, \quad \mathbf{v}^2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}
\]

Solving by either of the methods discussed above we find that

\[
z^k = \frac{x_0 + y_0}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{x_0 + 2y_0}{3} \left( \frac{7}{10} \right)^k \begin{bmatrix} 1 \\ -1 \end{bmatrix}
\]

As \( k \to \infty \), we know that \( (7/10)^k \to 0 \), thus

\[
\lim_{k \to \infty} z^k = \frac{x_0 + y_0}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{2x_0}{3} \\ \frac{2y_0}{3} \end{bmatrix} (x_0 + y_0).
\]

Thus, in the long run, there will be \( 2/3 \) of the total population outside and \( 1/3 \) inside. This is true no matter what initial distribution of people inside and outside of California may have been. The limiting vector is just an eigenvector of the matrix corresponding to the eigenvalue 1, that is

\[
\begin{bmatrix} 9/10 & 2/10 \\ 1/10 & 8/10 \end{bmatrix} \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix} = 1 \cdot \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}
\]
If initially there were 2/3 of the people outside and 1/3 inside, the population distribution would never change.

**Example 4.** Solve the 3rd order difference equation

\[ \Delta E: x_{n+3} + 6x_{n+2} + 11x_{n+1} + 6x_n = 0, \quad n = 0, 1, 2, \ldots \]

IC: \( x_0 = 1, \ x_1 = 0, \ x_2 = 0. \)

We reduce this to a system of three first order equations. Let \( x_{n+1} = y_n, \ y_{n+1} = z_n \) then we have

\[
\begin{align*}
x_{n+1} &= y_n \\
y_{n+1} &= z_n \\
x_{n+1} &= y_{n+2} = x_{n+3} = -6x_{n+2} - 11x_{n+1} - 6x_n \\
&= 6x_n - 11y_n - 6z_n.
\end{align*}
\]

or

\[ w^{n+1} = Aw^n \]

where

\[
\begin{bmatrix}
x_n \\
y_n \\
z_n
\end{bmatrix} =
\begin{bmatrix}
x_n^0 \\
y_n^0 \\
z_n^0
\end{bmatrix}, \quad A = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
-6 & -11 & -6
\end{bmatrix}, \quad w^0 = \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}
\]

The solution is

\[ w^n = A^n w^0, \quad n = 0, 1, 2, \ldots \]

If \( A \) has eigenvalues \( \lambda_1, \ \lambda_2, \ \lambda_3 \) and corresponding linearly independent eigenvectors \( \mathbf{v}^1, \ \mathbf{v}^2, \ \mathbf{v}^3 \), we have

\[ w^0 = \alpha_1 \mathbf{v}^1 + \alpha_2 \mathbf{v}^2 + \alpha_3 \mathbf{v}^3, \tag{*} \]

\[ w^n = A^n w^0 = A^n(\alpha_1 \mathbf{v}^1 + \alpha_2 \mathbf{v}^2 + \alpha_3 \mathbf{v}^3) = \alpha_1 \lambda_1^n \mathbf{v}^1 + \alpha_2 \lambda_2^n \mathbf{v}^2 + \alpha_3 \lambda_3^n \mathbf{v}^3. \]

In the last section, we found that the eigenvalues where \( \lambda_1 = -1, \ \lambda_2 = -2, \ \lambda_3 = -3 \) and the corresponding eigenvectors where

\[
\begin{align*}
\mathbf{v}^1 &= \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{v}^2 = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}, \quad \mathbf{v}^3 = \begin{bmatrix} 1 \\ -3 \\ 9 \end{bmatrix}
\end{align*}
\]

Equation \((*)\) becomes

\[
\begin{bmatrix}
1 \\
0
\end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ -3 \\ 9 \end{bmatrix}
\]

Solving, we find \( \alpha_1 = 3, \ \alpha_2 = -3, \ \alpha_3 = 1. \) Thus the solution of the vector equation is

\[ w^n = 3(-1)^n \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} - 3(-2)^n \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} + (-3)^n \begin{bmatrix} 1 \\ -3 \\ 9 \end{bmatrix} \]

Since all we want is \( x_n, \) the first component of \( w^n, \) the solution of our problem is

\[ x_n = 3(-1)^n - 3(-2)^n + (-3)^n, \quad n = 0, 1, 2, \ldots \]
Example 5. Solve $z^{k+1} = Az^k$ where $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$.

The solution is $z^k = A^k z^0$. If the eigenvalues are $\lambda_1$, $\lambda_2$ and the corresponding eigenvectors are $v^1$, $v^3$ and are linearly independent, the solution is

$$z^k = \alpha_1 \lambda_1^k v^1 + \alpha_2 \lambda_2^k v^2$$

We find that the eigenvalues are

$$\lambda_1 = 1 + i = \sqrt{2} e^{i \frac{\pi}{4}}$$
$$\lambda_2 = 1 - i = \sqrt{2} e^{-i \frac{\pi}{4}}$$

The corresponding eigenvectors are

$$v^i = \begin{bmatrix} 1 \\ i \end{bmatrix}, \quad v^2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

Thus $z^k$ is

$$z^k = \alpha_1 (\sqrt{2})^k e^{i k \frac{\pi}{4}} \begin{bmatrix} 1 \\ i \end{bmatrix} + \alpha_2 (\sqrt{2})^k e^{-i k \frac{\pi}{4}} \begin{bmatrix} 1 \\ -i \end{bmatrix} \quad (\ast)$$

This is the general complex valued solution. We would like to get the solution in real form. Let the components of $z^k$ be $x_k$, $y_k$. From equation $(\ast)$ we get

$$x_k = 2^k \left( \alpha_1 e^{i k \frac{\pi}{4}} + \alpha_2 e^{-i k \frac{\pi}{4}} \right)$$
$$y_k = 2^k \left( i \alpha_1 e^{i k \frac{\pi}{4}} - i \alpha_2 e^{-i k \frac{\pi}{4}} \right)$$

Using Euler’s forms we find

$$x_k = 2^k \left( (\alpha_1 + \alpha_2) \cos \frac{k \pi}{4} + i (\alpha_1 - \alpha_2) \sin \frac{k \pi}{4} \right)$$
$$y_k = 2^k \left( i (\alpha_1 - \alpha_2) \cos \frac{k \pi}{4} - ((\alpha_1 + \alpha_2) \sin \frac{k \pi}{4} \right).$$

Let $c_1 = \alpha_1 + \alpha_2$, and $c_2 = i (\alpha_1 - \alpha_2)$ and we obtain

$$x_k = 2^k \left( c_1 \cos \frac{k \pi}{4} + c_2 \sin \frac{k \pi}{4} \right)$$
$$y_k = 2^k \left( c_2 \cos \frac{k \pi}{4} - c_1 \sin \frac{k \pi}{4} \right).$$

If real initial conditions are given for $x_0$, $y_0$ then $c_1$, $c_2$ will be real. If we allow $c_1$, $c_2$ to be arbitrary real numbers, this is the general real solution.

Exercises 3.16

1. Find the solution of $x^{k+1} = Ax^k$, $k = 0, 1, 2, \ldots$ in each of the following cases (use method of example 1).
   a. $A = \begin{bmatrix} 3 & 3 \\ 2 & 4 \end{bmatrix}$, $x^0 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$; b. $A = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}$, $x^0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

2. Solve problems 1a and 1b using the diagonalization method.
3. Find the general solution of $x^{k+1} = Ax^k$ if
   
a. $A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$  
b. $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

4. If $A$ is $3 \times 3$ and $u$, $v$, $w$ are nonzero column vectors such that $Au = 2u$, $Av = v$, $Aw = -3w$
   
a. Find the general solution of $x^{k+1} = Ax^k$.
   
b. Find the solution of $x^{k+1} = Ax^k$, $x^0 = u - 5w$.

5. Solve $x^{k+1} = Ax^k$ where
   
   $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, $x^0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

6. Solve, using matrix methods $\Delta E$: $x_{k+3} - 4x_{k+2} - x_{k+1} + 4x_k = 0$
   
   IC: $x_0 = 1$, $x_1 = 0$, $x_2 = 1$. 