On an Eigenvalue Problem Involving Legendre Functions

by

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1. INTRODUCTION. The classical eigenvalue problem for the Legendre Polynomials is

$$\text{DE: } \left((1-x^2)y'\right)' + \lambda y = 0, \quad -1 \leq x \leq 1. \tag{1}$$

This is a singular Sturm-Liouville problem. Because $x = \pm 1$ are singular points, no explicit boundary conditions are given at these points; instead it is required that $y(x)$ and $y'(x)$ remain bounded as $x \to \pm 1$. It is well known ([1. p. 186], or [4, p. 325]) that this boundedness condition implies that the eigenvalues must be

$$\lambda = n(n+1), \; n = 0, \; 1, \ldots$$

and the corresponding eigenfunctions are the non-zero multiples of the Legendre polynomials $P_n(x)$ which form an orthogonal set on the interval $-1 \leq x \leq 1$. Another proof of this fact will be given at the end of this paper.

We are mainly interested in the eigenvalue problem

$$\begin{aligned} &\text{DE: } \left((1-x^2)y'\right)' + \lambda y = 0, \; a \leq x \leq 1 \text{ where } -1 < a < 1 \\ &\text{BC: } y(a) = 0, \; y(x) \text{ and } y'(x) \text{ bounded as } x \to 1. \end{aligned} \tag{2}$$

Many books discuss the case when $a = 0$ where it can be shown [1, p. 192] that the eigenvalues are given by $\lambda = (2k+1)(2k+2), \; k = 0, \; 1, \ldots$ and the corresponding eigenfunctions are the odd Legendre polynomials $P_{2k+1}(x)$ which form a orthogonal set on the interval $0 \leq x \leq 1$. In this paper we consider the case when $a \neq 0$. This problem is actually solved in the Hobson treatise [4, p. 444] but seems not to be considered in current textbooks for first courses in boundary value problems or "Advanced Mathematics for Engineers". The reason is clear; the solution requires Legendre functions of arbitrary degree which are not usually considered in such texts.

As will be seen below, this problem provides a nice example of the use of Frobenius series, the Legendre functions of arbitrary degree come up naturally in the solution.. In addition, using a modern comprehensive computing environment such as MAPLE, we can readily find approximations to the eigenvalues, plot the eigenfunctions and test the eigenfunction expansion for specific functions.

The classical problem (1) arises in finding the steady state temperature in a solid homogeneous sphere with given temperature on the surface, assuming the temperature is symmetrical about a diameter [1, p. 193-196]. The problem (2) arises in finding the steady state temperatures in the portion of a sphere cut off by a concentric right-circular cone, where the temperature on the spherical surface is assumed to be symmetrical about the axis of symmetry of the cone and sphere; the temperature on the surface of the sphere is given and the temperature on the surface of the cone is 0 [4, p. 444].

2. SOLUTION OF THE EIGENVALUE PROBLEM. It is shown in [2, p. 294] that the eigenvalues, λ, of problem (2) are nonnegative. It is easy to check that $\lambda = 0$ is not an eigenvalue so that the eigenvalues are all positive. Following the usual custom we write $\lambda = \nu(\nu+1)$, where $\nu > 0$ is a real number.

$$\begin{aligned} &\text{DE: } \left((1-x^2)y'\right)' + \nu(\nu+1)y = 0, \; a \leq x \leq 1, \; (-1 < a < 1), \\ &\text{BC: } y(a) = 0, \; y(x) \text{ and } y'(x) \text{ bounded as } x \to 1. \end{aligned} \tag{3}$$

Since $x = 1$ is a regular singular point we look for a solution as a Frobenius series about this point. To facilitate computations let $x - 1 = t$ and express the differential equation (3) in terms of t. The equation becomes

$$\left((t^2+2t)y'\right)' - \nu(\nu+1)y = 0 \tag{4}$$

where y is considered as a function of t. We look for a Frobenius solution of the form

$$y = \sum_0^\infty a_k t^{k+\alpha} = a_0 t^\alpha + a_1 t^{\alpha+1} + \cdots, \; a_0 \neq 0.$$

Substituting in equation (4) and setting the coefficient of a_0 to zero yields the indicial equation $\alpha^2 = 0$. Since the indicial equation has a double root of zero, we know that there is one power series solution

$$y = \sum_0^\infty a_k t^k. \tag{5}$$

and a second linearly independent solution which has a logarithmic singularity at $t = 0$. We are interested in solutions that are bounded as $t = x - 1 \to 0$, and therefore need only consider the power series solution. Substituting the (5) into the DE (4) we find the recurrence relation

$$a_{k+1} = \frac{(\nu - k)(\nu + k + 1)}{2(k+1)^2} a_k, \; k \geq 0. \qquad (6).$$

From this one finds that

$$a_k = \frac{(\nu + k)(\nu + k - 1)\cdots(\nu + 1)\nu(\nu - 1)\cdots(\nu - (k-1))}{2^k (k!)^2} a_0,$$

and the power series solution about $x = 1$ is is

$$y = a_0 \sum_0^\infty \frac{(\nu + k)(\nu + k - 1)\cdots(\nu + 1)\nu(\nu - 1)\cdots(\nu - (k-1))}{2^k (k!)^2} (x-1)^k.$$

If we set $a_0 = 1$, which means $y(1) = 1$ we obtain the Legendre function of the first kind of order ν:

$$P_\nu(x) = \sum_0^\infty \frac{(\nu + k)(\nu + k - 1)\cdots(\nu + 1)\nu(\nu - 1)\cdots(\nu - (k-1))}{2^k (k!)^2} (x-1)^k. \tag{7}$$

This is the result obtained in ([3], p. 312). From the recurrence relation (6) it is easy to see that that the series (7) converges for $|x - 1| < 2$. If we assume that ν is *not* a positive integer we may multiply numerator and denominator of the coefficients by $\Gamma(\nu - (k-1))$ to obtain

$$P_\nu(x) = \sum_0^\infty \frac{\Gamma(\nu + k + 1)}{2^k (k!)^2 \Gamma(\nu - (k-1))} (x-1)^k.$$

Now we may complete the solution of the boundary value problem. We know that $P_\nu(x)$ satisfies the DE for arbitrary $\nu > 0$ and is bounded at $x = 1$. To satisfy the left hand boundary condition we need only find those values of ν for which

$$P_\nu(a) = 0.$$

We assume that this equation has an infinite number of roots and denote the eigenvalues by $\nu_k = k$th positive root of $P_\nu(a) = 0, \; k = 1, \; 2, \; 3, \ldots$ and the corresponding eigenfunctions by $P_{\nu_k}(x)$. The functions $P_\nu(x)$ are built into MAPLE and are there called LegendreP(ν, x). To get specific numerical results let $a = 1/2$. Figure 1. shows the result of using MAPLE to plot $P_\nu(1/2)$ from $\nu = 0$ to $\nu = 40$.

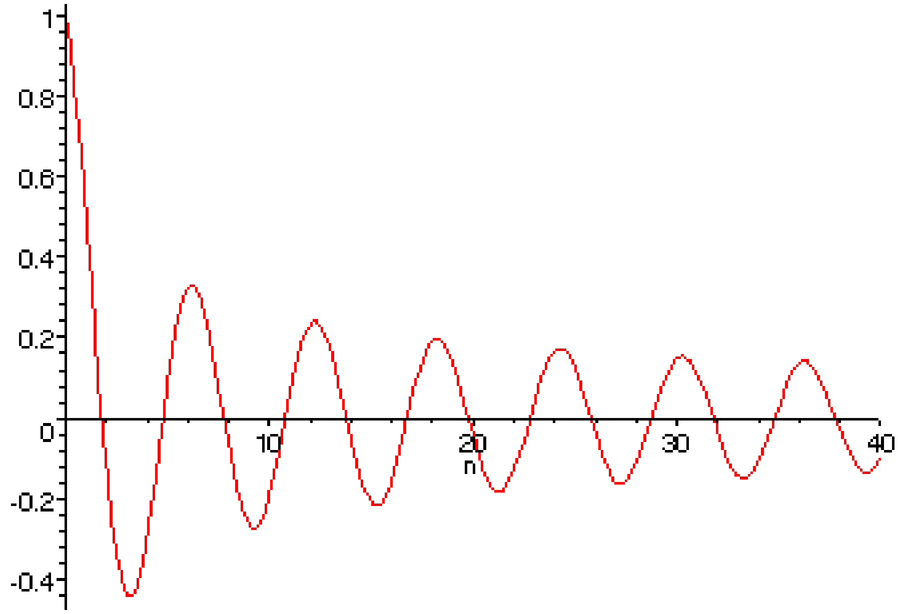

Figure 1.

Using Figure 1. for initial estimates of the eigenvalues, the equation solving property of MAPLE can be used to get more refined estimates. The first four eigenvalues are

$$\nu_1 = 1.777288270, \quad \nu_2 = 4.762779438, \quad \nu_3 = 7.758258853, \quad \nu_4 = 10.75608784.$$

The corresponding eigenfunctions are shown in Figure 2, the number of zeros increases as the suffix increases.

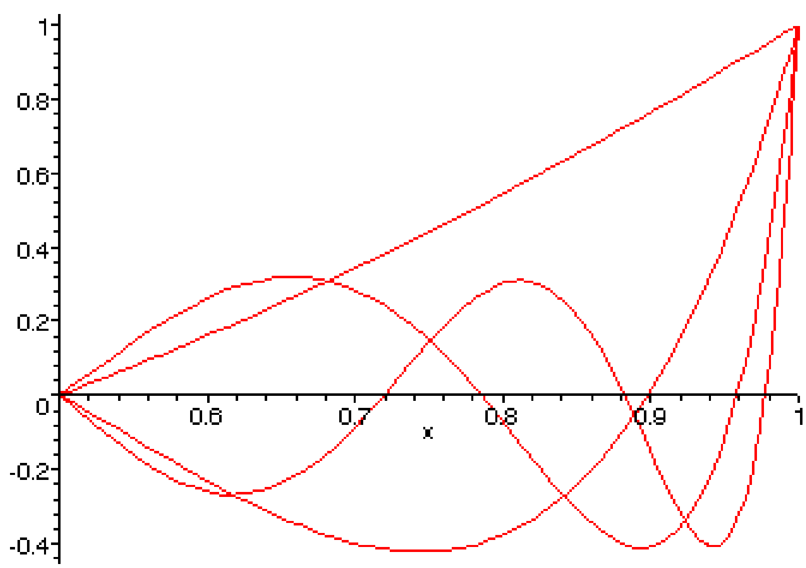

Figure 2.

The eigenfunction expansion of an arbitrary piecewise smooth function is

$$\frac{f(x+0)+f(x-0)}{2} = \sum_{1}^{\infty} A_k P_{\nu_k}(x),\ 1/2 < x < 1, \tag{8}$$

where

$$A_k = \frac{\int_{1/2}^{1} f(x) P_{\nu_k}(x)\,dx}{\int_{1/2}^{1} P_{\nu_k}^2(x)\,dx}.$$

To check out the eigenfunction expansion consider the function

$$f(x) = \begin{cases} x - 1/2, & 1/2 < x > 3/4, \\ x - 1, & 3/4 < x < 1. \end{cases} \tag{9}$$

The result of the first 40 terms of (8) compared with $f(x)$ is shown in Figure 3.

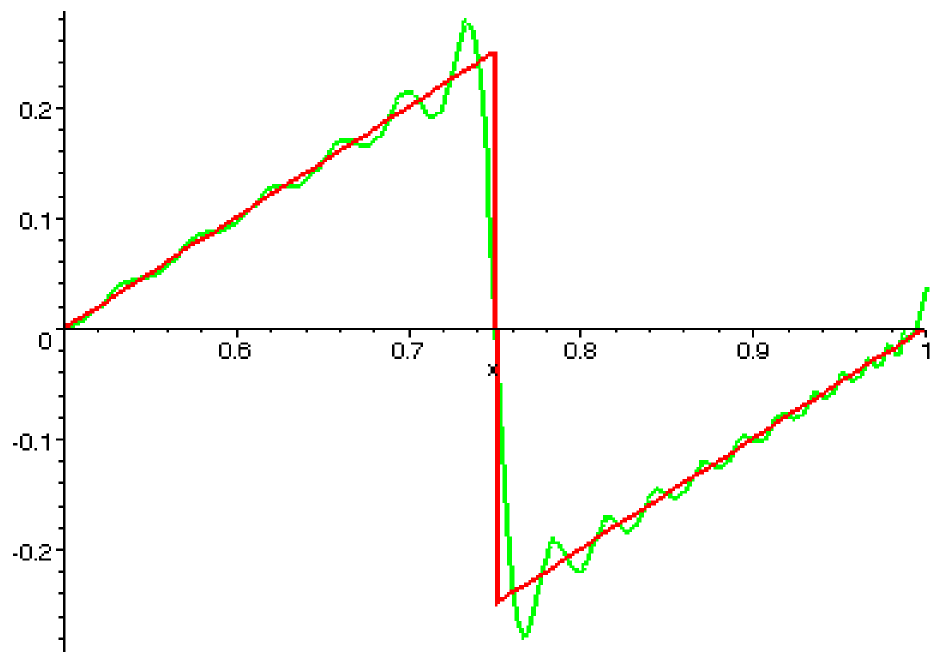

Figure 3.

3. Concluding Remarks. Eigenvalues and eigenfunctions for any value of a in (2) may be found in a similar manner. Also, the boundary condition $y(a) = 0$ may be changed to $y'(a) = 0$ or $y'(a) = hy(a),\ h > 0$ and similar results may be obtained.

In Figure 1. the difference between successive eigenvalues is approximately 3. This is consistent with the asymptotic expansion [4, p. 303]:

$$P_\nu(\cos\theta) = \frac{2}{\sqrt{2\sin\theta}}\frac{(2\nu)!}{2^{2n}(\nu!)^2}\cos\left(\frac{\pi}{4} - \left(\nu + \tfrac{1}{2}\right)\theta\right) + O\left(\frac{1}{\nu}\right). \tag{10}$$

Considered as a function of ν, the first term in (10) shows that the distance between successive zeros of the cosine function is approximately π/θ. In the case of Figure 1, we have $\theta = \pi/3$. so that $\pi/\theta = 3$.

When $\upsilon = n$, a non-negative integer, equation (6) shows that $a_k = 0$, $k > n$ and the solution is a polynomial In this case (7) reduces to a formula for the Legendre Polynomials:

$$P_n(x) = \sum_{0}^{n} \frac{(n+k)!}{2^k (k!)^2 (n-k)!}(x-1)^k. \tag{11}$$

This formula is interesting in that it is not necessary to have different formulae for n even and n odd. On the other hand it is not obvious from (11) that P_n is an even polynomial if n is even and odd if n is odd. Equation (11) may also be written as

$$P_n(x) = \sum_0^n \frac{1}{2^k}\binom{n}{k}\binom{n+k}{n}(x-1)^k. \tag{12}$$

In conclusion we give a proof of the fact stated in the introduction:

The only values of λ *for which (1) has a solution which is continuous for* $-1 \le x \le 1$ *are* $\lambda = n(n+1),\ n = 0,\ 1,\ 2, \cdots$.

Letting $\lambda = \nu(\nu+1)$, where $\nu \ge 0$ is an arbitrary real number we obtain the differential equation DE:

$$\left((1-x^2)y'\right)' + \nu(\nu+1)y = 0.$$

In the previous section it was shown that the only solutions that are bounded at $x = 1$ are multiples of the Legendre function $P_\nu(x)$. From the recurrence formula (6.) see that $P_\nu(x)$ has a polynomial solution if and only if ν is a non-negative integer. Therefore it is sufficient to prove that $P_\nu(-1)$ diverges unless $\upsilon = 0,\ 1,\ .$ Assuming $\upsilon \notin \{0,\ 1,\ 2, \cdots\}$, the Legendre functions are given by a non-terminating infinite series

$$P_\nu(x) = \sum_0^\infty a_k(x-1)^k,$$

where $a_0 = 1$ and the a_k satisify the recurrence relation (6). Therefore

$$P_\nu(-1) = \sum_0^\infty a_k(-2)^k. \tag{13}$$

Let $b_k = a_k(-2)^k$.. The ratio of successive terms is

$$\frac{b_{k+1}}{b_k} = \frac{a_{k+1}(-2)^{k+1}}{a_k(-2)^k} = \frac{(\nu-k)(\nu+k+1)}{2(k+1)^2}(-2) = \frac{(k-\nu)(\nu+k+1)}{(k+1)^2}.$$

If $k > \nu$, the terms b_k have the same sign; assume they are positive. For $k > \nu+1$ we have

$$\frac{b_{k+1}}{b_k} \ge \frac{(\nu+k+1)}{(k+1)^2} \ge \frac{k}{k+1}.$$

It follows that for $j = 2,\ 3, \cdots$ we have

$$b_{\nu+j} \ge \frac{\nu+1}{\nu+j} b_{\nu+1},\ j = 1,\ 2,\ 3\cdots.$$

This implies the series $\sum_{j=1}^\infty b_j$ diverges and therefore $P_\nu(-1)$ diverges also.

References

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