# Hilbert-Type Space-Filling Curves

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In Memory of Hans Sagan (1928–2000)

## 1 Introduction

We adopt the definition of a curve given by C. Jordan in 1887: a (plane) curve is the set of points $(\phi(t), \psi(t))$ where $\phi$ and $\psi$ are continuous functions on some closed interval which we take to be $[0, 1]$. It is useful to think of $t$ as time and the curve as the path of a particle starting at $(\phi(0), \psi(0))$ at $t = 0$ and ending at $(\phi(1), \psi(1))$ at $t = 1$. Since only one real parameter is needed to define a curve, it seems reasonable that this would force the curve to be 'one-dimensional'. However, In 1890, G. Peano surprised the mathematical world by discovering a space-filling curve, that is, a curve which passes through each point of a square at least once. Previously, in 1878, G. Cantor demonstrated that there was a one-to-one correspondence between the unit interval and the unit square. In 1879, E. Netto showed that any such mapping could not be continuous. Thus the Peano curve must have multiple points, that is, points which are the images of two or more distinct values of $t$.

In 1891 D. Hilbert discovered another space-filling curve. Whereas Peano's curve was defined purely analytically, Hilbert's approach was geometric. In this paper we shall discuss Hilbert-type space-filling curves in a manner suggested by E. H. Moore in his 1900 paper [**1**] and present some new variants of the Hilbert curve. Many other space-filling curves have been discovered since 1900. These and related ideas, together with a detailed bibliography, may be found in the book by H. Sagan [**2**].

## 2 Hilbert-Type Space-Filling Curves

Let $I = \{t \,|\, 0 \le t \le 1\}$ denote the unit interval and $Q = \{\,(x, y) \,|\, 0 \le x \le 1, 0 \le y \le 1\,\}$ the unit square. For each positive integer $n$ we partition the interval $I$ into $4^n$ subintervals of length $4^{-n}$ and the square $Q$ into $4^n$ subsquares of side $2^{-n}$. We construct a one-to-one correspondence between the subintervals of $I$ and the subsquares of $Q$ subject to two conditions:

**Adjacency Condition** Adjacent subintervals correspond to adjacent subsquares (with an edge in common).

**Nesting Condition** If at the $n$-th partition, the subinterval $I_{n_k}$ corresponds to a subsquare $Q_{n_k}$ then at the $(n+1)$-st partition the 4 subintervals of $I_{n_k}$ must correspond to the 4 subsquares of $Q_{n_k}$.

In Section 3 we show that such a correspondence is possible.

**Theorem 1.** Any correspondence between the subintervals and subsquares that satisfies the Adjacency and Nesting conditions determines a unique continuous function $f$ which maps I onto $Q$.

*Proof.* Given such a correspondence we define a function $f : I \to Q$ as follows: (a) If $t \in I$ is not the endpoint of any of the subintervals, then $t$ belongs to a unique sequence of closed nested subintervals $\{J_n\}_1^\infty$, one from each partition, whose lengths approach zero. The corresponding sequence of closed nested squares $\{S_n\}_1^\infty$ whose diameter approaches zero determines a unique point $q = (q_1, q_2) \in Q$. We define $f(t) = q$. The points $t = 0$ and $t = 1$ may be treated in a similar manner. (b) If $t \in I$ is common to two adjacent intervals $J_n$, $J_n'$ for some value of $n$, say $n = \nu$, then it is common to two adjacent intervals $J_n$ and $J_n'$ for all $n \geq \nu$ and therefore belong to two nested sequences $\{J_n\}_\nu^\infty$ and $\{J_n'\}_\nu^\infty$. However the corresponding sequences of squares $\{S_n\}_\nu^\infty$ and $\{S_n'\}_\nu^\infty$ determine the same point $q \in Q$, since the squares are adjacent and their diagonals approach zero. For such a $t$ we define $f(t) = q$.

We have produced a single valued function $f : I \to Q$. We must show that this function is onto. Each $q \in Q$ lies in (at least) one sequence of closed nested squares which shrink to the point $q$. The corresponding sequence of nested closed intervals shrink to a point $t \in I$ for which $f(t) = q$.

It remains to demonstrate that $f$ is continuous. Suppose $|t_1 - t_2| \leq 4^{-n}$ then $t_1$ and $t_2$ must lie in the same interval or in two adjacent intervals of the $n$-th partition. The corresponding images must lie in, at worst, two adjacent squares forming a rectangle having of sides $2^{-n}$ and $2 \cdot 2^{-n}$. Thus we must have

$$\|f(t_1) - f(t_2)\| \leq \sqrt{5} \cdot 2^{-n} \tag{1}$$

which implies that f is continuous. ∎

Netto's theorem guarantees that the function defined above cannot be one-one. We provide a direct proof of this.

**Theorem 2.** The function defined in Theorem 1 has infinitely many multiple points.

*Proof.* Suppose in the $n$-th partition that a square $S_n$ corresponds to an interval $J_n$. Let $P$ be the center of $S_n$. In the $(n+1)$-th partition shown in Figure 1 the 4 subsquares of $S_n$ must correspond to the four subintervals of $J_n$. Suppose the corresponding squares are as shown in Figure 1. There is a nested sequences of squares lying in square 1 which shrink to $P$. The corresponding sequence of intervals shrinks to a point $t$ in the first subinterval of $J_n$, There is also a nested sequences of squares lying in square 4 which shrink to $P$. The corresponding sequence of intervals shrinks to a point $t' \neq t$ in the fourth subinterval of $J_n$, Therefore therefore at least two distinct points in $I$ that map into $P$. Since the above argument holds in every square in every partition, the curve $f$ has infinitely many multiple points. ■

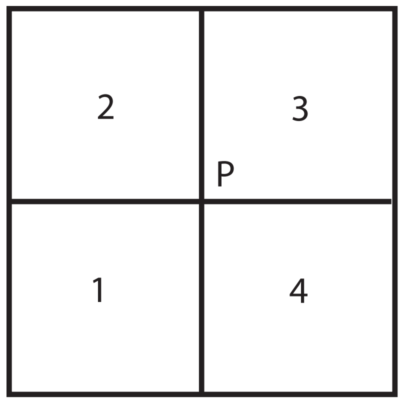


Figure 1: The center of a square in the $n$-th partition

Let $\phi, \psi$ be the coordinate functions of the function $f$ defined in Theorem 1. We have:

**Theorem 3.** The coordinate functions $\phi, \psi$ of $f$ are nowhere differentiable.

*Proof.* The idea for this proof comes from Sagan [**1**, p. 12]. Let $n \geq 2$. For any $t \in I$, pick a $t_n \in I$ so that $|t - t_n| \leq 16 \cdot 4^{-n}$ and so that the $x$-coordinate of $f(t) = (\phi(t), \psi(t))$ and $x$-coordinate of $f(t_n) = (\phi(t_n), \psi(t_n))$ are separated by at least one square of length $2^{-n}$. This is always possible. Since $n \geq 2$, $f(t)$ lies in somewhere in a square consisting of 16 subsquares of side $2^{-n}$ which correspond to points in 16 consecutive time intervals of length $4^{-n}$. From Figure 2 we see that no matter where $f(t)$ is located, we may find an $f(t_n)$ so that the above condition is satisfied. Therefore

$$|\phi(t) - \phi(t_n)|/|t - t_n| \geqslant 2^{-n}/(16 \cdot 4^{-n}) = 2^n/16 \longrightarrow \infty \text{ as } n \longrightarrow \infty.$$

Thus $\phi$ is nowhere differentiable. A similar argument holds for $\psi$. ■

## 3 Hilbert's Space-Filling Curve

For each $n$ we label the $4^n$ subintervals in their natural order from left to right. The correspondence between the intervals and the squares amounts to numbering the squares so that the adjacency and nesting conditions are satisfied. Hilbert's enumeration of the squares is shown in Figure 3 for $n = 1, 2, 3$. Note that the first square is always in the lower left corner and the last square is always in the lower right corner. This means that

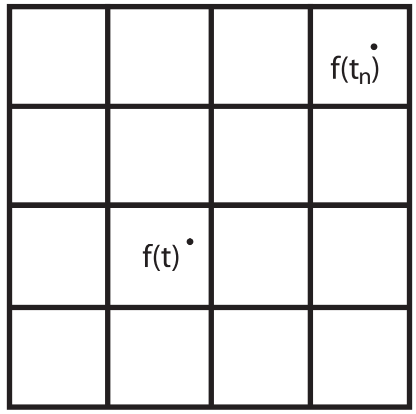


Figure 2: Figure for the non-differentiability argument

the the Hilbert space-filling curve starts at $(0,0)$ at $t=0$ and ends at $(1,0)$ at $t=1$. With the first and last squares of each partition determined there is only one enumeration of the squares that satisfies the adjacency and nesting conditions.

To see more clearly how the $(n+1)$-st partition is obtained from the $n$-th partition look at Figure 4. To obtain the lower left hand quadrant of the $(n+1)$-st partition we shrink the $n$-th partition by a factor of 2 and reflect it in the line $y=x$. The upper left hand quadrant is obtained by shrinking the $n$-th partition by a factor of 2 and adding $4^n$ to each square number. The upper right quadrant is obtained by shrinking by a factor of 2 and adding $2\cdot 4^n$ to the square number. Finally the lower right quadrant is obtained by shrinking by a factor of 2, reflection in the line $x+y=1$ and adding $3\cdot 4^n$ to the number of each square. Clearly if the adjancency and nesting conditions are satisfied in the $n$-th partition, they will also be satisfied in the $(n+1)$-st partition. This particular enumeration of the squares determines a unique space-filling curve mapping $I$ onto $Q$, which we denote by $f_h$ and call the Hilbert space-filling curve. From Figure 4 it is clear that

$$f_h(1/4)=(0,1/2),\ f_h(1/2)=(1/2,1/2),\ f_h(3/4)=(1,1/2). \tag{2}$$

As a visual aid in following the enumeration of the squares Hilbert connected the midpoints of adjacent squares with a straight line segment as shown in Figure 3. The polygonal curve formed in this way may be considered as successive approximations to the Hilbert curve (see Section 5). The polygonal curve exhibits a symmetry about the line $x=1/2$. This suggests that the Hilbert curve $f_h(t)=(\phi_h(t),\psi_h(t))$ has the same symmetry, that is

$$\phi_h(1-t)=1-\phi_h(t) \quad \text{and} \quad \psi_h(1-t)=\psi_h(t) \tag{3}$$

To see why this is true, consider the $n$-th partition. Suppose that $t$ belongs to the $k$-th interval, then $1-t$ belongs to the complementary interval numbered $(4^n+1-k)$. It follows

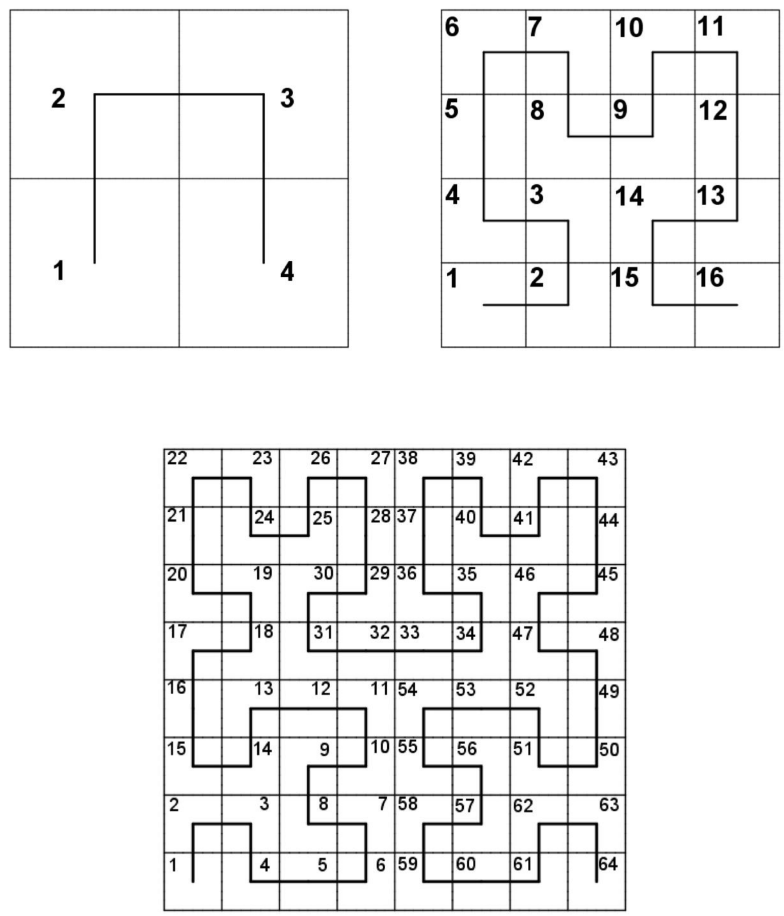


Figure 3: First three stages in the generation of Hilbert's space-filling curve

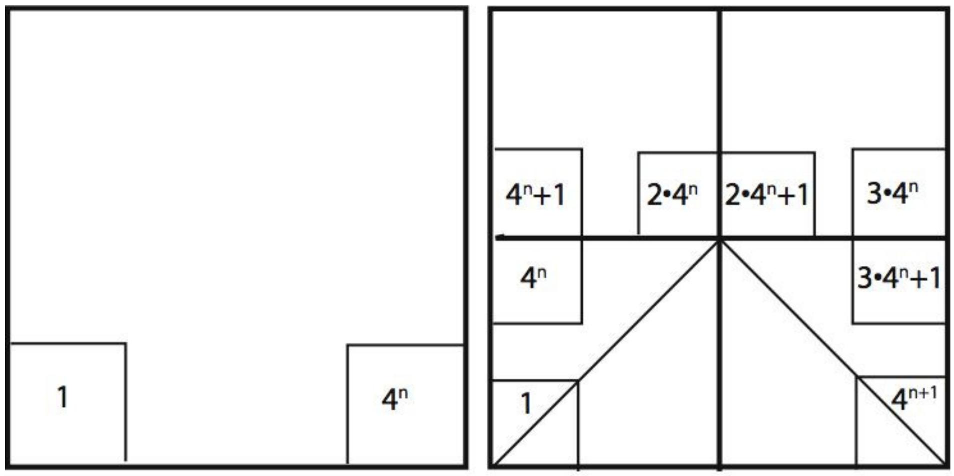


Figure 4: The $n$-th and the $(n+1)$-th partitions

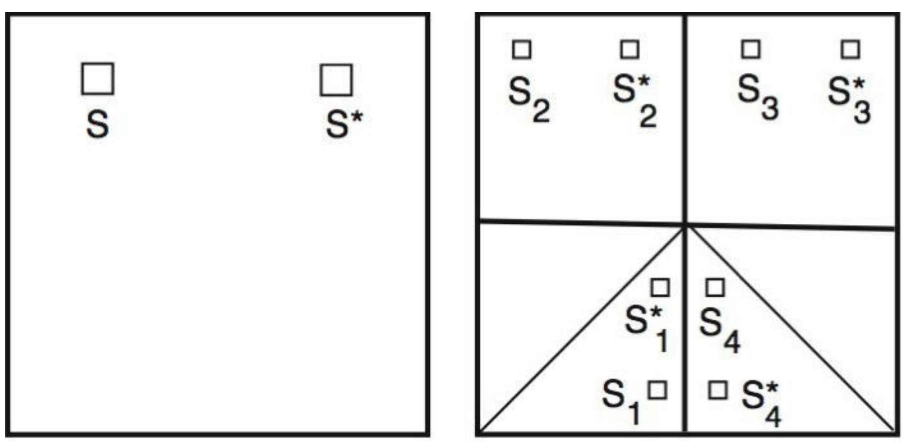


Figure 5: Symmetry at the $n$-th and $(n+1)$-st partitions

that $f_h(t)$ belongs the the $k$-th square $S$ and $f_h(1-t)$ belongs to the complementary square $S^*$ numbered $(4^n+1-k)$. Note that the square numbers of $S$ and $S^*$ add up to $4^n+1$. All we need show is that the squares $S$ and $S^*$ are symmetrical with respect to the line $x=1/2$. This is clear for $n=1,2,3$ from Figure 3. Suppose, in the $n$-th partition that $S$ and $S^*$ are symmetrical with respect to $x=1/2$ (see Figure 5) and that their square numbers add up to $4^n+1$. In passing to the $(n+1)$-th partition the squares $S$ and $S^*$ are mapped into squares $S_i$, $S_i^*$, $(i=1,2,3,4)$. Using the construction shown in Figure 4 it is easy to verify that the square numbers of the symmetrical pairs of squares $(S_1,S_4^*)$, $(S_4,S_1^*)$, $(S_2,S_3^*)$, $(S_3,S_2^*)$ all add up to $4^{n+1}+1$. For example, if the number of the square $S$ and therefore $S_1^*$ is $k$, the number of the square $S_4$ is $4^n+1-k+3\cdot 4^n$, whose sum is $4^{n+1}+1$. Thus the symmetry continues to hold in the $(n+1)$-th partition.

## 4 Analytical Representation of the Hilbert Space-Filling Curve

Hilbert's space-filling curve, $f_h$, described above, starts at $(0,0)$ and ends at $(1,0)$. It is easy to modify $f_h$ to obtain a space-filling curve that starts at any corner of the square and ends at an adjacent corner. Let the corners of the square be denoted by $A$, $B$, $C$, $D$ as shown in Figure 6. Denote the Hilbert space-filling curve that starts at $A$ and and ends at $B$ by $f_{AB}$ with similar designations for any pair of adjacent corners. All of these curves may be described in terms of the coordinates of $f_h$, which henceforth we write as a column vector, $f_h=\begin{bmatrix}\phi_h\\ \psi_h\end{bmatrix}$, as shown in Equations 4–7.

$$f_{AD}=\begin{bmatrix}\phi_h\\ \psi_h\end{bmatrix}\qquad f_{DA}=\begin{bmatrix}1-\phi_h\\ \psi_h\end{bmatrix}\tag{4}$$

$$f_{AB}=\begin{bmatrix}\psi_h\\ \phi_h\end{bmatrix}\qquad f_{BA}=\begin{bmatrix}\psi_h\\ 1-\phi_h\end{bmatrix}\tag{5}$$

$$f_{BC}=\begin{bmatrix}\phi_h\\ 1-\psi_h\end{bmatrix}\qquad f_{CB}=\begin{bmatrix}1-\phi_h\\ 1-\psi_h\end{bmatrix}\tag{6}$$

$$f_{CD}=\begin{bmatrix}1-\psi_h\\ 1-\phi_h\end{bmatrix}\qquad f_{DC}=\begin{bmatrix}1-\psi_h\\ \phi_h\end{bmatrix}\tag{7}$$

Looking at Figure 7 we know that the Hilbert curve $f_h$ starts at $(0,0)$ at $t=0$, spends the first quarter unit of time in square 1, ending at the point $(0,1/2)$, the next quarter unit of time in square 2 ending at the point $(1/2,1/2)$, the third quarter unit of time in square 3 ending at the point $(1,1/2)$, and the fourth quarter unit of time in square 4 ending at the point $(1,0)$. Thus the original space filling curve $f_h$ is composed of 4 one-half size copies of itself (in various orientations) pieced together. Another way to phrase this is to say that the four squares in the first partition are all scaled down Hilbert curves in appropriate orientations. This argument may be repeated for the squares in the 2-nd partition, 3-rd

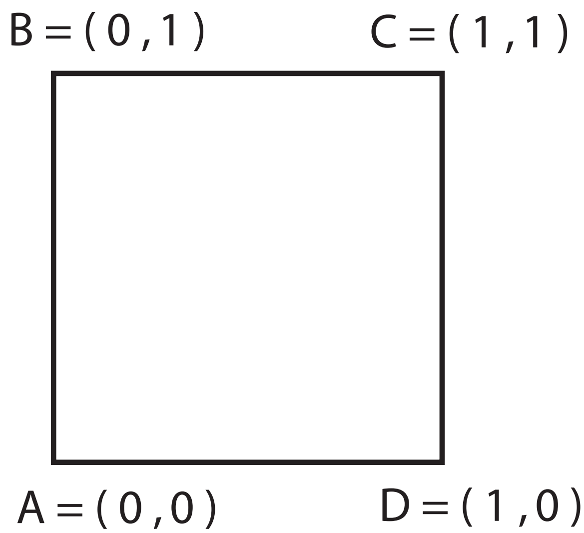


Figure 6: Labeling of the square

partition etc., leading to the conclusion that each square in the n-th partition is a Hilbert curve scaled by the factor of $2^{-n}$ traversed in a time $4^{-n}$.

Looking at the first partition shown in Figure 7, the equations of the four constituent curves are:

$$f_h\left(\frac{t}{4}\right) = \frac{1}{2}f_{AB}(t) \tag{8}$$

$$f_h\left(\frac{1+t}{4}\right) = \frac{1}{2}\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \frac{1}{2}f_{AD}(t) \tag{9}$$

$$f_h\left(\frac{2+t}{4}\right) = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{2}f_{AD}(t) \tag{10}$$

$$f_h\left(\frac{3+t}{4}\right) = \frac{1}{2}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{1}{2}f_{CD}(t) \tag{11}$$

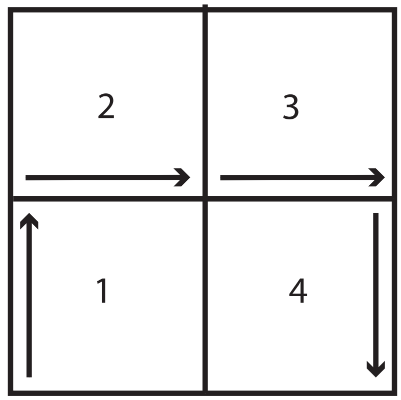


Figure 7: Composition of Hilbert's curve

Using Equations 4–7 we have

$$f_h\left(\frac{t}{4}\right) = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} f_h(t) \equiv \mathfrak{H}_0 f_h(t) \tag{12}$$

$$f_h\left(\frac{1+t}{4}\right) = \frac{1}{2}\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \frac{1}{2} f_h(t) \equiv \mathfrak{H}_1 f_h(t) \tag{13}$$

$$f_h\left(\frac{2+t}{4}\right) = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{2} f_h(t) \equiv \mathfrak{H}_2 f_h(t) \tag{14}$$

$$f_h\left(\frac{3+t}{4}\right) = \frac{1}{2}\begin{bmatrix} 2 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} f_h(t) \equiv \mathfrak{H}_3 f_h(t) \tag{15}$$

With the definitions of $\mathfrak{H}_0$, $\mathfrak{H}_1$, $\mathfrak{H}_2$, $\mathfrak{H}_3$ given in Equations 12–15 we may write equations (8–11) in the compact form:

$$f_h\left(\frac{t_1+t}{4}\right) = \mathfrak{H}_{t_1} f_h(t), \quad t_1 = 0, 1, 2, 3 \tag{16}$$

In particular we have

$$f_h\left(\frac{t_1}{4}\right) = \mathfrak{H}_{t_1} f_h(0), \quad t_1 = 0, 1, 2, 3 \tag{17}$$

$$f_h\left(\frac{t_1}{4} + \frac{t_2}{16}\right) = f_h\left(\frac{t_1 + t_2/4}{4}\right) = \mathfrak{H}_{t_1} f_h\left(\frac{t_2}{4}\right) = \mathfrak{H}_{t_1}\mathfrak{H}_{t_2} f_h(0), \quad t_1, t_2 = 0, 1, 2, 3 \tag{18}$$

Using base 4 notation equations (17, 18) may be written

$$f_h(0_{\dot{4}} t_1) = \mathfrak{H}_{t_1} f_h(0) \tag{19}$$

$$f_h(0_{\dot{4}}\,t_1t_2) = \mathfrak{H}_{t_1}\mathfrak{H}_{t_2}f_h(0) \tag{20}$$

It follows that

$$f_h(0_{\dot{4}}\,t_1t_2\dots t_n) = \mathfrak{H}_{t_1}\mathfrak{H}_{t_2}\dots\mathfrak{H}_{t_n}f_h(0) \tag{21}$$

and since $f$ is continuous and $f_h(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$,

$$f_h(0_{\dot{4}}\,t_1t_2\dots t_n\dots) = \lim_{n\to\infty}\mathfrak{H}_{t_1}\mathfrak{H}_{t_2}\dots\mathfrak{H}_{t_n}\begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{22}$$

Equation 22 may be used to calculate $f_h(t)$ for any $t$ in $0 \le t \le 1$. A simplified version of formula of Equation 22 is given in Sagan [**2**, p. 18], however Equation 22 is sufficient for our purposes.

Let us compute $f_h(1/3)$. Writing 1/3 in base 4 we have $1/3 = 0_{\dot{4}}1111... = 0_{\dot{4}}\overline{1}$. From Equation 22 we find

$$f_h(0_{\dot{4}}\overline{1}) = \lim_{n\to\infty}\mathfrak{H}_1^n\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We find

$$\mathfrak{H}_1\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 \\ 1 \end{bmatrix},\ \mathfrak{H}_1^2\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \frac{3}{4}\begin{bmatrix} 0 \\ 1 \end{bmatrix},\ \dots,\ \mathfrak{H}_1^n\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \frac{2^{n-1}}{2^n}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Thus

$$f_h(1/3) = f_h(0_{\dot{4}}\overline{1}) = \lim_{n\to\infty}\mathfrak{H}_1^n\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

It follows from Equation 3 that

$$f_h(2/3) = f_h(0_{\dot{4}}\overline{2}) = \lim_{n\to\infty}\mathfrak{H}_2^n\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Thus we see that the four corners of the unit square are traversed in times $t = k/3, k = 0, 1, 2, 3$. Each square in the $n$-th partition is traversed in time $1/4^n$, thus the vertices are traversed in times $k/(3\cdot 4^n), k = 0, 1, 2, 3$.

A point on the edge of the unit square cannot be more than a double point. To see that double points occur we calculate

$$f_h(5/12) = f_h(0_{\dot{4}}1\overline{2}) = \lim_{n\to\infty}\mathfrak{H}_1\mathfrak{H}_2^n\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}.$$

From Equation 3 we find that $f_h(7/12)$ has the same image. Therefore the point $(1/2, 1)$ is a double point.

An similar calculation shows

$$f_h(1/6) = f_h(0_{\dot{4}}0\overline{2}) = \lim_{n\to\infty} \mathfrak{H}_0\mathfrak{H}_2^n \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

From Equation 3 we see that $f_h(5/6) = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$. Since we already know that $f_h(1/2) = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$, it follows that the center of the square is a "triple point", i.e. the image of three distinct values of $t$. Moreover, in every partition, each subsquare is a scaled down version of the Hilbert curve, the center of every such subsquare is a triple point.

In like manner it can be shown that the four values

$$t_1 = 0_{\dot{4}}01\overline{2} = 5/48,\ t_2 = 0_{\dot{4}}02\overline{1} = 7/48,\ t_3 = 0_{\dot{4}}31\overline{2} = 41/48,\ t_4 = 0_{\dot{4}}32\overline{1} = 43/48$$

have the same image

$$f_h(t_i) = \begin{bmatrix} 1/2 \\ 1/4 \end{bmatrix},\ i = 1, 2, 3, 4.$$

Therefore the point $(1/2, 1/4)$ is a quadruple point. Again each subsquare in a partition must have a quadruple point.

# 5 Approximating Polygons to the Hilbert Curve

Hilbert constructed approximating polygons to his curve by connecting the midpoints of adjacent squares in the $n$-th partition as shown in Figure 3 for $n = 1, 2, 3$. Figure 8 shows the approximating polygon for $n = 4$. The entry point of the $k$-th square in the $n$-the partition corresponds to the time $t_k = \frac{k}{4^n}$, $(k = 0, 1, 2, \ldots, 4^n - 1)$. Each square in $n$-th partition is traversed in the time $\frac{1}{4^n}$ and reaches the center of the square in time $\frac{1}{2\cdot 4^n}$. Thus the centers of the squares correspond to the times $\tau_k = \frac{(2k+1)}{2\cdot 4^n}$, $k = 0, 1, 2, \ldots, 4^n - 1$. Therefore the Hilbert approximating polygons, $p_n$ are obtained by connecting the images of the times $\tau_k$ with straight line segments. We parametrize the polygons $p_n(t)$ by linear interpolation between the points $p_n(\tau_k)$. Then both $p_n(t)$ and $f_h(t)$ lie in the same square of sidelength $2^{-n}$ for $t_k \le t \le t_{k+1}$. The distance between $p_n(t)$ and $f_h(t)$ cannot exceed the diameter of the square, that is

$$||p_n(t) - f_h(t)|| \le \frac{\sqrt{2}}{2^n}, \quad \text{for all } t \in I.$$

This means that the $p_n$ converges uniformly to $f_h$ and justifies applying the term approximating polygons to the $p_n$. These approximating polygons have no multiple points. However we know that the limiting Hilbert space-filling curve has infinitely many multiple points. Looking closely at Figure 8 one may suspect that multiple points might occur in

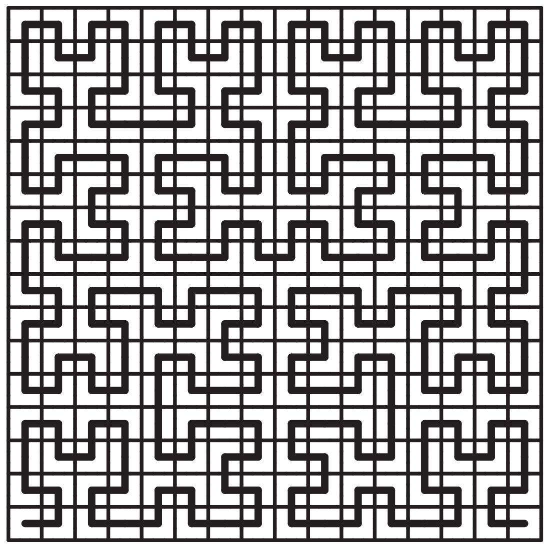

Figure 8: Hilbert approximating polygons for n=4

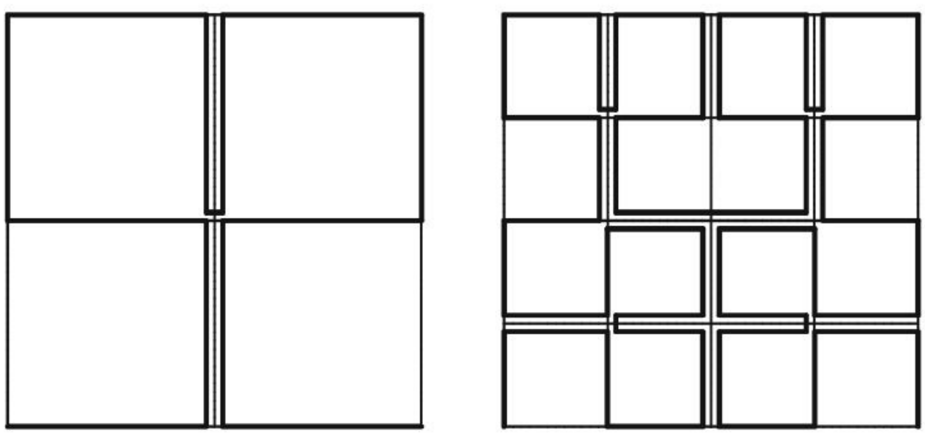

Figure 9: Approximating polygons traversing the vertices of squares with coincident line segments pulled apart

the limiting curve, for example, a double point at $(1/2, 1)$, a triple points at $(1/2, 1/2)$ and a quadruple point at $(1/2, 1/4)$.

Another sequence of approximating polygons can be obtained by connecting the the entry points of each square in the $n$-th partition, that is the points corresponding to the times $t_k = \frac{k}{4^n}$, $(k = 0, 1, 2, \ldots, 4^n - 1)$. For illustrations of these polygons see Sagan [**2**, p. 22].

We will illustrate still another sequence of approximating polygons. In the $n$-th partition we move form one vertex to another along the edges of the squares in the proper order in time. The entry points of each square correspond to the times $t_k = \frac{k}{4^n}$ mentioned in the previous paragraph. We add to these the times it takes to traverse the vertices, namely, the times $\frac{l}{3 \cdot 4^n}$, $l = 0, 1, 2, 3$ to obtain the times $\tau_k = \frac{3k+l}{3 \cdot 4^n}$ or, equivalently, $\tau_k = \frac{k}{3 \cdot 4^n}$, $(k = 0, 1, 2, \ldots, 3 \cdot 4^n)$. Graphs of the third kind of approximating polygons in shown in In Figure 9 for $n = 1, 2$, These polygons have multiple points and double back on themselves. We have "pulled apart" these parts of the curve to more readily follow their progress through the square. Double, triple and quadruple points of the Hilbert curve show up clearly in the right hand graph in Figure 9. For instance $(1/2, 1$ is a double point, $(1/2, 1/2)$ is a triple point and $(1/2, 1/4)$ is a quadruple point. We may argue, as above, that these approximating polygons also approach the Hilbert curve uniformly.

## 6 Variations of Hilbert's Space-Filling Curve

We now examine some variations of the Hilbert space-filling curve. Suppose the curve starts at $(0, 0)$ and we order the squares as shown in Figure 10. The first three squares must be oriented as shown but the last square may be oriented to end up at the center of the square as shown in the left hand diagram in Figure 10. This yields a new space filling curve whose 4-th stage polygonal approximation is shown in the right hand diagram in Figure 10. Using the notation of Section 4 the equations of the four constituent parts of this curve are:

$$\begin{aligned}
f\left(\frac{t}{4}\right) &= \frac{1}{2} f_{AB}(t) && (23)\\
f\left(\frac{1+t}{4}\right) &= \frac{1}{2}\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \frac{1}{2} f_{AD}(t) && (24)\\
f\left(\frac{2+t}{4}\right) &= \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{2} f_{AD}(t) && (25)\\
f\left(\frac{3+t}{4}\right) &= \frac{1}{2}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{1}{2} f_{CB}(t) && (26)
\end{aligned}$$

Each of the right hand sides of (27-30) may be expressed in terms of the Hilbert curve $f_h$ and therefore computed using Equation 22.

Moore [**1**, pp.74–76] came up with a variation of the Hilbert curve which starts at the point $(1/2, 0)$ and ends at the same point. The left-hand diagram of Figure 11 shows that the Moore curve consists of four half-size Hilbert curves placed end-to end with appropriate orientations (apparently Moore did not notice this). The right-hand graph in Figure 11 illustrates the 4-th stage polygonal approximation to the Moore curve. The equations of four parts of the curve $f_m$ are:

$$f_m\left(\frac{t}{4}\right) = \frac{1}{2}f_{DC}(t \tag{27}$$

$$f_m\left(\frac{1+t}{4}\right) = \frac{1}{2}\begin{bmatrix}0\\1\end{bmatrix} + \frac{1}{2}f_{DC}(t) \tag{28}$$

$$f_m\left(\frac{2+t}{4}\right) = \frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix} + \frac{1}{2}f_{BA}(t) \tag{29}$$

$$f_m\left(\frac{3+t}{4}\right) = \frac{1}{2}\begin{bmatrix}1\\0\end{bmatrix} + \frac{1}{2}f_{BA}(t) \tag{30}$$

Once we have a closed curve, clearly we may find a closed space-filling curve that starts and ends at any point of the square.

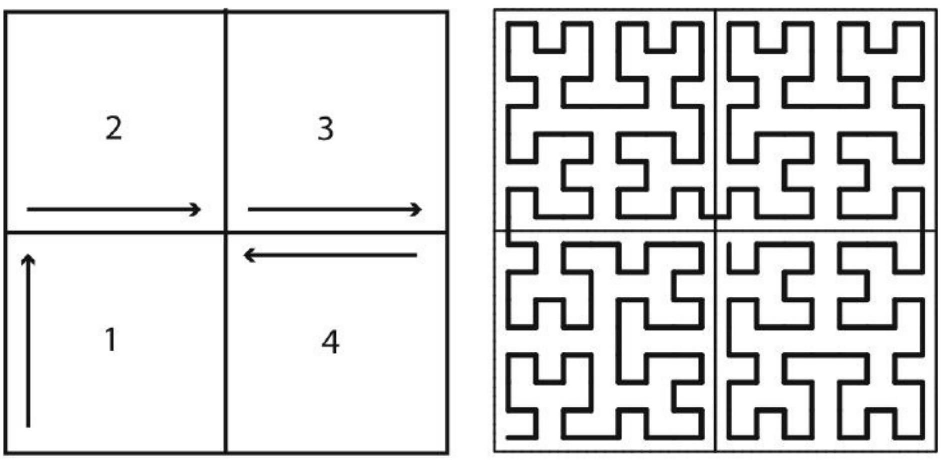


Figure 10: Curve from corner to center

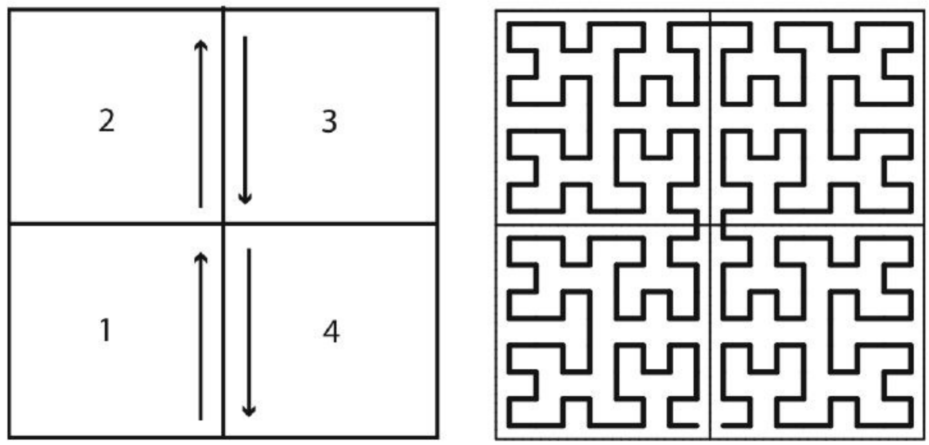


Figure 11: Moore's space-filling curve

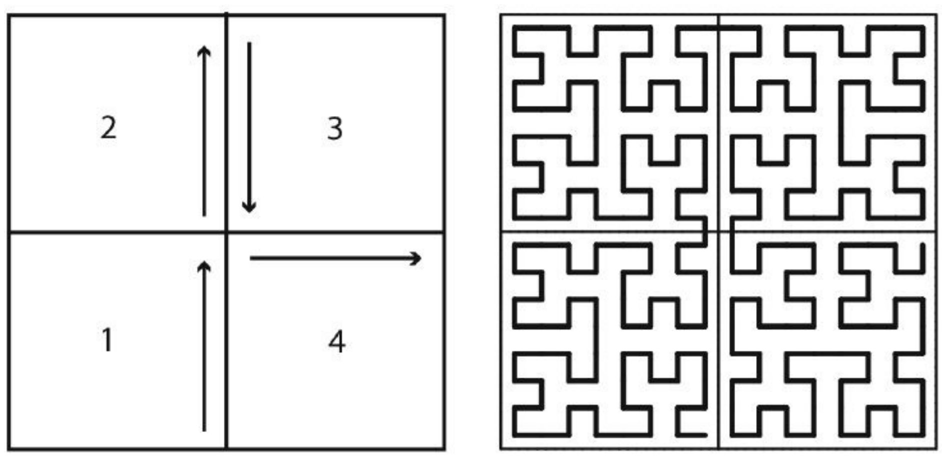


Figure 12: Curve from one side to adjacent side

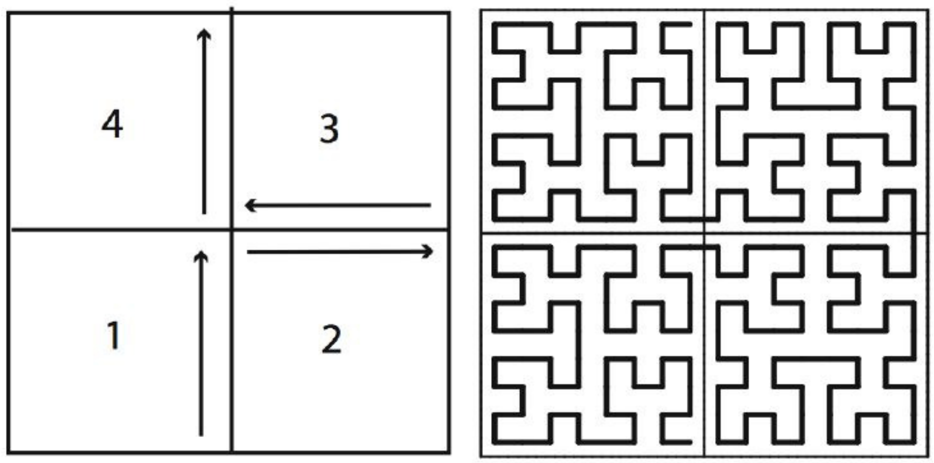


Figure 13: Curve from one side to opposite side

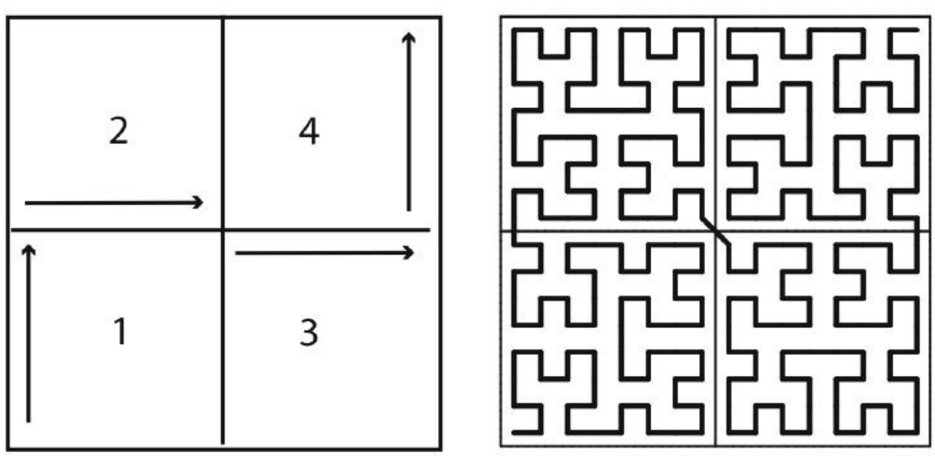


Figure 14: Curve from corner to opposite corner

We may modify the Moore curve to obtain a curve that starts at $(1/2, 0$ and ends at $(1, 1/2)$ as shown in Figure 12.. In Figure 13 we illustrate still another curve that starts at the center of a side of the square and terminates at the center of the opposite side. Equations similar to Equations 27-30 may be written for these curves. Finally we look at the possibilty of a space-filling curve going from one corner of the square to the opposite corner. It is clear that this cannot be done without violating the Adjacency condition. However we may easily do it by continually piecing together four Hilbert curves as we have shown in Figure 14. Equations akin to Equations 27-30 may be written for this curve.

## References


1. Moore, E. H., On Certain Crinkly Curves, *Trans. Amer. Math Soc., 1*, 72-90 (1900).
2. Sagan, H., *Space Filling Curves*, Springer-Verlag, New York (1994).