

## Summing Every $k$ th Term of a Power Series

by

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### 1. Introduction.

Consider a power series whose radius of convergence is  $R > 1$  and whose sum is  $f(x)$

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \quad 0 \leq x < R. \quad (1)$$

Form a new series by taking every  $k$ th term of the series for  $f(x)$  starting with the  $p$ th term, where  $0 \leq p < k$ . We denote this series by  $S_{k,p}[f(x)]$ , that is

$$S_{k,p}[f(x)] = \sum_{n=0}^{\infty} a_{nk+p} x^{nk+p} = a_p x^p + a_{k+p} x^{k+p} + a_{2k+p} x^{2k+p} + \dots \quad (2)$$

We seek to find  $S_{k,p}[f(x)]$  in terms of  $f(x)$ . If  $f(x)$  is known in closed form, this will yield closed form expressions for the sums (

### 2. The General Formula

To obtain equation (2) from equation (1) we need to multiply the coefficients  $a_n$  in (1) by a sequence  $z_n$  which picks out every  $k$ th term, starting with the  $p$ th term. In other words we want to pick out terms of the form  $n = mk + p$ , or  $n = p \pmod k$ . To this end let

$$z_n = \frac{1}{k} \left( 1 + \omega^{n-p} + (\omega^2)^{n-p} + \dots + (\omega^{k-1})^{n-p} \right) \quad (3)$$

where  $\omega = e^{i2\pi/k}$  is the principal  $k$ th root of unity. It is easy to verify that

$$z_n = \begin{cases} 1, & \text{if } n = p \pmod k; \\ 0, & \text{otherwise.} \end{cases}$$

Therefore

$$S_{k,p}[f(x)] = \sum_{n=0}^{\infty} a_{nk+p} x^{nk+p} = \sum_{n=0}^{\infty} a_n z_n x^n$$

Using the expression for  $z_n$  given in equation (3) we have

$$\begin{aligned} S_{k,p}[f(x)] &= \sum_{n=0}^{\infty} a_{kn+p} x^{kn+p} \\ &= \frac{1}{k} \sum_{n=0}^{\infty} \left( 1 + \omega^{n-p} + (\omega^2)^{n-p} + \dots + (\omega^{k-1})^{n-p} \right) a_n x^n \\ &= \frac{1}{k} \left( \sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} a_n \frac{(\omega x)^n}{\omega^p} + \dots + \sum_{n=0}^{\infty} a_n \frac{(\omega^{k-1} x)^n}{\omega^{(k-1)p}} \right) \\ &= \frac{1}{k} \left( f(x) + \frac{f(\omega x)}{\omega^p} + \dots + \frac{f(\omega^{k-1} x)}{\omega^{(k-1)p}} \right) \end{aligned} \quad (4)$$

We assume that the coefficients in (1) are real. In this case certain pairs of terms in (4) are complex conjugates of each other. First assume that  $k$  is odd. In this case the terms  $\frac{f(\omega^m x)}{\omega^{mp}}$  and  $\frac{f(\omega^{(k-m)x}}{\omega^{(k-m)p}}$  are conjugates for  $m = 1, 2, \dots, \frac{k-1}{2}$ . Combining these terms produces

$$\sum_{n=0}^{\infty} a_{kn+p} x^{kn+p} = \frac{1}{k} \left( f(x) + 2 \sum_{m=1}^{(k-1)/2} \operatorname{Re} \frac{f(\omega^m x)}{\omega^{mp}} \right), \quad k \text{ odd} \quad (5)$$

If  $k$  is even,  $-1$  is one of the  $k$ th roots of unity and the terms  $\frac{f(\omega^m x)}{\omega^{mp}}$  and  $\frac{f(\omega^{(k-m)x}}{\omega^{(k-m)p}}$  are conjugates for  $m = 1, 2, \dots, \frac{k}{2} - 1$ . Combining these terms yields

$$\sum_{n=0}^{\infty} a_{kn+p} x^{kn+p} = \frac{1}{k} \left( f(x) + \frac{f(-x)}{(-1)^p} + 2 \sum_{m=1}^{\frac{k}{2}-1} \operatorname{Re} \frac{f(\omega^m x)}{\omega^{mp}} \right), \quad k \text{ even} \quad (6)$$

Combining (5) and (6) into one formula yields

$$\sum_{n=0}^{\infty} a_{nk} x^{kn+p} = \frac{1}{k} \left( f(x) + \frac{1 + (-1)^k}{2} \frac{f(-x)}{(-1)^p} + 2 \sum_{m=1}^{\lfloor \frac{k-1}{2} \rfloor} \operatorname{Re} \frac{f(\omega^m x)}{\omega^{mp}} \right), \quad (7)$$

where  $[x] = \text{greatest integer } \leq x$ .

Example 1. Let  $f(x) = e^x$  then equation (7) becomes

$$\sum_{n=0}^{\infty} \frac{x^{kn+p}}{(kn+p)!} = \frac{1}{k} \left( e^x + \frac{1 + (-1)^k}{2} \frac{e^{-x}}{(-1)^p} + 2 \sum_{m=1}^{\lfloor \frac{k-1}{2} \rfloor} e^x \cos \frac{2\pi m}{k} \cos \left( x \sin \left( \frac{2\pi m}{k} \right) - \frac{2\pi mp}{k} \right) \right) \quad (8)$$

If  $k = 2$ ,  $p = 0$ ,  $\omega = e^{i\pi} = -1$  and equation (8) becomes

$$\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = \frac{1}{2} (e^x + e^{-x}) = \cosh x. \quad (9)$$

If  $k = 2$ ,  $p = 1$ , we obtain

$$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = \frac{1}{2} (e^x - e^{-x}) = \sinh x. \quad (10)$$

By adding (9) and (10) we recover the familiar identity

$$e^x = \cosh x + \sinh x$$

since  $S_{2,0}[e^x] + S_{2,1}[e^x] = e^x$ .

For  $k = 3$ , we obtain for  $p = 0, 1, 2$

$$\sum_{n=0}^{\infty} \frac{x^{3n}}{(3n)!} = \frac{1}{3} \cdot \left( e^x + 2e^{-x/2} \cos(x\sqrt{3}/2) \right). \quad (11)$$

$$\sum_{n=0}^{\infty} \frac{x^{3n+1}}{(3n+1)!} = \frac{1}{3} \cdot \left( e^x + 2e^{-x/2} \cos(x\sqrt{3}/2 - 2\pi/3) \right). \quad (12)$$

$$\sum_{n=0}^{\infty} \frac{x^{3n+2}}{(3n+2)!} = \frac{1}{3} \cdot \left( e^x + 2e^{-x/2} \cos(x\sqrt{3}/2 - 4\pi/3) \right). \quad (13)$$

Using equations (11), (12), (13), one may verify that

$$S_{3,0}[e^x] + S_{3,1}[e^x] + S_{3,2}[e^x] = e^x.$$

For  $k = 4$ , we find for  $p = 0, 1, 2, 3$

$$\sum_{n=0}^{\infty} \frac{x^{4n}}{(4n)!} = \frac{1}{2} \cdot (\cosh x + \cos x). \quad (14)$$

$$\sum_{n=0}^{\infty} \frac{x^{4n+1}}{(4n+1)!} = \frac{1}{2} \cdot (\sinh x + \sin x). \quad (15)$$

$$\sum_{n=0}^{\infty} \frac{x^{4n+2}}{(4n+2)!} = \frac{1}{2} \cdot (\cosh x - \cos x). \quad (16)$$

$$\sum_{n=0}^{\infty} \frac{x^{4n+3}}{(4n+3)!} = \frac{1}{2} \cdot (\sinh x - \sin x). \quad (17)$$

Again we see that the sum of the equations (14), (15), (16), (17) yields back  $e^x$

If  $k = 5$  and  $p = 1$  we find

$$\sum_{n=0}^{\infty} \frac{x^{5n+1}}{(5n+1)!} = \frac{1}{5} \cdot \left( e^x + 2e^{x \cos 2\pi/5} \cos(x \sin(2\pi/5) - 2\pi/5) + 2e^{x \cos 4\pi/5} \cos(x \sin(4\pi/5) - 4\pi/5) \right).$$

The various  $k$ -term sums for the function  $f(x) = e^x$  are related. We have

$$S_{k,p}[e^x] = \sum_{n=0}^{\infty} \frac{x^{kn+p}}{(kn+p)!}$$

It is easily verified that

$$\frac{d}{dx} S_{k,p}[e^x] = S_{k,p-1}[e^x], \quad k > p \geq 1 \quad \text{and} \quad \frac{d}{dx} S_{k,0}[e^x] = S_{k,k-1}[e^x]. \quad (18)$$

Therefore once  $S_{k,p}[e^x]$  has been found for a particular  $p$ , the other sums may be found by differentiation. One can verify equations (18) for  $k = 2$  in equations (9) and (10); for  $k = 3$  in equations (11), (12), (13) and for  $k = 4$  in equations (14), (15), (16), (17).

Example 2. Let

$$f(x) = \cos(x) = \sum_{j=0}^{\infty} \frac{(-1)^j x^{2j}}{(2j)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \quad (19)$$

We seek the sum of every third non-zero term of this series, i.e. the series

$$\sum_{j=0}^{\infty} \frac{(-1)^j x^{6j}}{(6j)!} = 1 - \frac{x^6}{6!} + \frac{x^{12}}{12!} - \frac{x^{18}}{18!} + \dots$$

In formula (4) all the terms of the series are counted so we must use  $k = 6$ . We have  $\omega = e^{i2\pi/6} = e^{i\pi/3}$ . Thus according to equation (7) we have

$$\begin{aligned} \sum_{j=0}^{\infty} \frac{(-1)^j x^{6j}}{(6j)!} &= \frac{1}{6} \cdot (\cos(x) + \cos(-x) + 2\operatorname{Re}(\cos(\omega x) + \cos(\omega^2 x))) \\ &= \frac{1}{3} \left( \cos x + 2 \cos(x/2) \cosh(x\sqrt{3}/2) \right) \end{aligned}$$

Let  $f(x) = \cos(x)$  and take  $k = 6$  and  $p = 2$ . Looking at the power series for  $\cos x$  given in equation (19), we see that this is the same as starting with the term,  $-x^2/2!$ , and taking every third non-zero term. We have from equation (7) with  $\omega = e^{i\pi/3}$

$$\begin{aligned} \sum_{j=0}^{\infty} \frac{(-1)^{j+1} x^{6j+2}}{(6j+2)!} &= \frac{1}{6} \cdot \left( \cos(x) + \cos(-x) + 2\operatorname{Re} \left( \frac{\cos(\omega x)}{\omega^2} + \frac{\cos(\omega^2 x)}{\omega^4} \right) \right) \\ &= \frac{1}{3} \left( \cos(x) - \cos(x/2) \cosh(x\sqrt{3}/2) - \sqrt{3} \sin(x/2) \sinh(x\sqrt{3}/2) \right). \end{aligned}$$