1997 NC STATE UNIVERSITY MATHEMATICS COMPETITION
(Previously the Frank McKee Excellence in Mathematics Competition)
November 8, 1997
Department of Mathematics
North Carolina State University

sponsored by
Wake County Public School System
College of Physical and Mathematical Sciences at
North Carolina State University

DIRECTIONS:

Please do not turn this page until told to do so.

The test is in two parts:

Part I consists of 16 multiple choice problems. Each problem is worth 2 points. You are asked to circle the correct answer.

Part II consists of 12 problems for which you are asked to show your method of solution clearly. Each of these problems is worth 4 points. Please work each problem from Part II on a separate sheet of white paper provided to you. The yellow paper is for scratchwork and will not be scored.

You may allocate the 90 minutes allotted for this test in any manner you wish, but a reasonable strategy would be to spend 30 minutes on Part I and 60 minutes on Part II.

PART I: MULTIPLE CHOICE (*Circle the correct answer*.)

1. $2^{-(2k+1)} - 2^{-(2k-1)} + 2^{-2k}$ is equal to

   (a) 2^{-2k} (b) $2^{-(2k-1)}$ (c) $-2^{-(2k+1)}$ (d) 0 (e) 2

2. Solve for x if $\log_2(\log_3(\log_4 x)) = 0$.

   (a) 16 (b) 9 (c) 64 (d) 81 (e) 8

3. If $1-x$ is used to approximate the value of $\frac{1}{1+x}$, $|x| < 1$, the ratio of the error made to the correct value is

   (a) x (b) x^2 (c) $\frac{1}{1+x}$ (d) $\frac{x}{1+x}$ (e) $\frac{x^2}{1+x}$

4. If $x^2 - 5x + 6 < 0$ and $P = x^2 + 5x + 6$, then

   (a) P can take any real value (b) $20 < P < 30$ (c) $0 < P < 20$
   (d) $P < 0$ (e) $P > 30$

5. Find the minimum value of $\sqrt{x^2+y^2}$ if $5x + 12y = 60$.

   (a) $\frac{60}{13}$ (b) $\frac{13}{5}$ (c) $\frac{13}{12}$ (d) 1 (e) 0

6. Let $f(x) = \log\left(\frac{1+x}{1-x}\right)$ and $g(x) = \frac{3x+x^3}{1+3x^2}$. Then $f(g(x)) =$

   (a) $-f(x)$ (b) $f(x)$ (c) $3f(x)$ (d) $(f(x))^3 - f(x)$ (e) $(f(x))^3$

7. If m and n are roots of $x^2 + mx + n = 0, m \neq 0, n \neq 0,$ then the sum of the roots is

   (a) 3 (b) -1 (c) -3 (d) 1 (e) -2

8. In the expansion of $(2-x)^9$ the coefficient of x^7 is

   (a) -144 (b) -36 (c) 36 (d) 72 (e) -72

9. If $\cos\theta = \frac{a}{b}$ where $0 < a < b$, then the value of $\cos 2\theta$ is

(a) $\frac{a^2-b^2}{b^2}$ (b) $\frac{2a-b^2}{b^2}$ (c) $\frac{2a}{b}$ (d) $\frac{b^2-a^2}{b^2}$ (e) $\frac{2a^2-b^2}{b^2}$

10. When the base of a triangle is increased by 10% and the altitude is decreased by 10%, the change in the area is

(a) 1% increase (b) $\frac{1}{2}$% increase (c) 0% (d) $\frac{1}{2}$% decrease (e) 1% decrease

11. If 25_b represents a two-digit number in the base b, and if 52_b is twice 25_b, then b is:

(a) 7 (b) 8 (c) 9 (d) 11 (e) 12

12. The area of the ring between two concentric circles is $25\pi/2$ square inches. The length of a chord of the larger circle tangent to the smaller circle, in inches, is

(a) $\frac{5}{\sqrt{2}}$ (b) 5 (c) $5\sqrt{2}$ (d) 10 (e) $10\sqrt{2}$

13. A triangle has angles of 30° and 45°. If the side opposite the 45° angle has length 8, then the length of the side opposite the 30° angle is

(a) 4 (b) $4\sqrt{2}$ (c) $4\sqrt{3}$ (d) $4\sqrt{6}$ (e) 6

14. Let $F = .48181\cdots$, with digits 8 and 1 repeating. Then when F is written as a fraction in lowest terms, the denominator exceeds the numerator by

(a) 13 (b) 14 (c) 29 (d) 57 (e) 105

15. The expression $\sqrt{x\sqrt{x\sqrt{x\sqrt{x}}}}$, written as a power of x, is

(a) $x^{1/16}$ (b) $x^{1/4}$ (c) $x^{1/8}$ (d) $x^{7/8}$ (e) $x^{15/16}$

16. If $x = 1 + 2^p$ and $y = 1 + 2^{-p}$ then y in terms of x is

(a) $\frac{x+1}{x-1}$ (b) $\frac{x-1}{x+1}$ (c) $\frac{x}{x-1}$ (d) $\frac{1}{x}$ (e) $2 - x$

PART II: <u>BEGIN EACH PROBLEM ON A NEW SHEET OF WHITE PAPER</u>

To receive full credit, you must carefully organize your work on your answer sheets.

1. Hugh, Al, and Milt, working together, can complete a job in x hours. Working alone, Hugh could complete the job in $x + 6$ hours. Similarly, Al could do the job alone in $x + 1$ hours, and Milt could do the job alone in $2x$ hours. Solve for x.

2. The number $n = 2^{48} - 1$ is exactly divisible by two numbers between 60 and 70. Determine these numbers.

3. A piece of string is cut in two at a point selected at random. Find the probability that the longer piece is at least x times as long as the shorter piece, where $x \geq 1$. (The probability that a random point is in an interval is proportional to its length.)

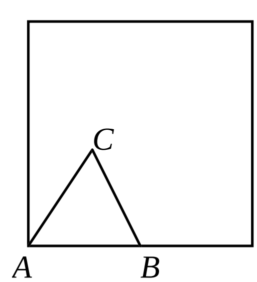
4. Lines are drawn through the point $(3, 4)$ and the trisection points of the segment joining the points $(-4, 5)$ and $(5, -1)$. Find the equations of these two lines.

5. Let $OABC$ be a unit square in the xy-plane with vertices $O(0, 0), A(1, 0), B(1, 1)$, and $C(0, 1)$. Let $u = x^2 - y^2$ and $v = 2xy$ be a transformation of the xy-plane into the uv-plane. Find the image of the square in the uv-plane and sketch it.

6. If $2x - 3y - z = 0$ and $x + 3y - 14z = 0$, $z \neq 0$, then determine the value of

$$\frac{x^2 + 3xy}{y^2 + z^2}.$$

7. An equilateral triangle ABC with side length 2 inches is placed inside a square with side length 4 inches as shown in the figure. The triangle is rotated clockwise about B, then about C, and so on along the sides of the square until C first comes to the original position of A. Find the length of the path in inches traversed by vertex C.



8. Find all values x such that

$$\tan^{-1}(x+1) + \tan^{-1}(x-1) = \tan^{-1}\left(\frac{8}{31}\right).$$

9. The ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ has focus F at $(3, 0)$. For a variable point P on the ellipse, let M be the midpoint of PF. Show that as P varies, the locus of M is an ellipse, and find its center.

10. Find all real solutions of the system of equations

$$\begin{aligned} z^x &= y^{2x} \\ 2^z &= 2(4^x) \\ x + y + z &= 16. \end{aligned}$$

11. Prove that if $|x| < 1$ and $|y| < 1$, then $|x + y| < |1 + xy|$.

12. To number the pages of a bulky volume, the printer used 2989 digits. How many pages has the volume?