Mechanical Properties - Stresses & Strains

Types of Deformation: Elastic Plastic Anelastic

Elastic deformation is defined as instantaneous recoverable deformation

Hooke's law: For tensile loading, $\sigma = E \varepsilon$

where $\sigma$ is stress defined as the load per unit area: $\sigma = P/A_0$, N/m², Pa

and strain is given by the change in length per unit length $\varepsilon = \Delta l/l_0$, %

The proportional constant E is the Young's modulus or modulus of Elasticity:

$E \sim 10 \times 10^6$ psi [68.9 GPa] for metals [varying from $10 \times 10^6$ psi for Al, $30 \times 10^6$ for Fe and $59 \times 10^6$ for W].

Poisson's Ratio [$\nu$]: ratio of lateral contraction to longitudinal elongation

$\nu = -\varepsilon_x/\varepsilon_z = -\varepsilon_y/\varepsilon_z$ [for isotropic materials]; in general, $\nu \sim 0.3$

Thus the total contractile strains is less than the expansion along the tensile axis thereby resulting in a slight increase in the volume of the material under stress - this is known as Elastic Dilation.

Modulus Of Rigidity or Shear Modulus [G]: $G = \tau/\gamma$; $G$ is the shear modulus and is related to $E$ and $\nu$, $G = E / 2(1+\nu)$.

Bulk Modulus [$\kappa$]: the change in volume to the original volume is proportional to the hydrostatic pressure [$\sigma_{\text{hyd}}$]: $\Delta V/V = \beta \sigma_{\text{hyd}}$, where $\beta$ is the compressibility.

The inverse of the compressibility is the bulk modulus [$\kappa$]: $\kappa = 1/\beta$.

$\kappa$ is also related to $E$ and $\nu$: $\kappa = E / 3(1-2\nu) = 2G(1+\nu) / 3(1-2\nu)$.

Thus one can evaluate the various elastic moduli from one or more experimentally evaluated constants. Note that the elastic moduli are related to the interatomic bonding and thus decrease [slightly] with increasing temperature. Any change in the crystal structure, for example following a phase change [polymorphism], one notes a distinct change in the elastic moduli.
### Stress - Strain Curve

#### Definitions

<table>
<thead>
<tr>
<th>Nominal (engineering)</th>
<th>True</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S = \frac{P}{A_0}, \quad e = \frac{\Delta l}{l_0} )</td>
<td>( \sigma = \frac{P}{A}, \quad \varepsilon = \ln \left( \frac{1}{l_0} \right) )</td>
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| Proportional limit (PL) | \( \sigma = S (1+e) \quad \& \quad \varepsilon = \ln (1+e) \) |
| Yield strength \((S_y) 0.2\% \text{ offset}; \quad (S_{LY})\) | \( true \ \text{Yield stress} \quad (\sigma_y) 0.2\% \text{ offset} \) |
| Tensile strength \((TS \text{ or } UTS \text{ or } S_{UTS})\) | \( true \ \text{Tensile strength} \quad (TS \text{ or } UTS \text{ or } \sigma_{UTS}) \) |
| Fracture strength \((S_F)\) | \( true \ \text{Fracture strength} \quad (\sigma_F) \) |
| Uniform elongation \((e_u)\) | \( true \ \text{Uniform strain} \quad (\varepsilon_u) \) |
| Total elongation \((ductility) \quad (e_t \text{ or } e_f \text{ in } 2\” )\) | \( true \ \text{Total elongation} \quad (ductility) \quad (e_t \text{ or } e_f \text{ in } 2\” ) \) |
| Necking strain \((e_n = e_t - e_u)\) | \( true \ \text{Necking strain} \quad (e_n = e_t - e_u) \) |
| Reduction in area \((ductility) \quad (RA)\) |  |
| Volume increases \((Elastic Dilation)\) | \( Volume \ is \ conserved \ A_{0l_0} = Al \) |

### Energy to fracture

- Elastic \( (\sigma = E \varepsilon) \)
- Plastic \( (\sigma = K \varepsilon^n) \)

- Resilience \( (U_{el} = \frac{\sigma^2}{2E}) \) units \( (\text{J/m}^3) \)
- Toughness \( (J = K \frac{\varepsilon^{n+1}}{n+1}) \)
**σ - ε curves**

smooth (SSs & fcc) with yield point (steels & bcc)

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**Rate Effects**

Plastic deformation is rate dependent

(generally at high temperatures) : \( \sigma \equiv f(\dot{\varepsilon}) = A \dot{\varepsilon}^m \), \( m = \text{SRS} = \left( \frac{d \ln \sigma}{d \ln \dot{\varepsilon}} \right) T, \varepsilon \)

\( m \sim 0 \) at low temperatures \( m_{\text{max}} = 1 \)

\[ m \uparrow \quad \dot{\varepsilon} \uparrow \quad \text{vs} \quad n \uparrow \quad \varepsilon_u \uparrow \]

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**Group Work :**

left of the instructor : (1) Derive relation between \( \sigma \) and \( S \) :

right of the instructor : (2) Derive relation between \( \varepsilon \) and \( e \) :

all (3) Show that \( \varepsilon_u = n \)
Concept of Stress

\[ \sigma = \lim_{A \to 0} \frac{F}{A} \]

Force extended on reference section by remaining sections \( \Leftarrow \) body in equilibrium

Normal and Shear Stresses

\[ \sigma_N = \frac{F}{A} \cos \theta \]

\[ \sigma_{\text{shear}} = \tau = \frac{F}{A} \sin \theta \]

\( \tau_x = \) \\
\( \tau_y = \)

recall RSS \((\tau_{\text{RSS}})\):
Stress - Strain Relationships

Elastic Behavior

$\mathbf{F}$, Force is a vector (1st rank tensor)

while $\sigma_{ij}$ should be specified with 2 directions: plane normal and force direction - acts on plane perpendicular to i along j direction

**Sign Convention (Fig. 2.2)**

\[
\sigma = \begin{pmatrix}
\sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\
\sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\
\sigma_{zx} & \sigma_{zy} & \sigma_{zz}
\end{pmatrix}
\]

no net moment $\Rightarrow \sigma_{ij} = \sigma_{ji}$ or only 6 components or

\[
\sigma = \begin{pmatrix}
\sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\
\sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\
\sigma_{zx} & \sigma_{zy} & \sigma_{zz}
\end{pmatrix}
; \text{ book notation } \Rightarrow \begin{pmatrix}
\sigma_x & \tau_{xy} & \tau_{xz} \\
\tau_{yx} & \sigma_y & \tau_{yz} \\
\tau_{zx} & \tau_{zy} & \sigma_z
\end{pmatrix}
\]

$\Rightarrow$ Values of $\sigma_{ij}$ depend on the choice of reference axes (see 2-D example 2.3) $\Leftarrow$

{can determine these components using tensor transformations, Mohr circle, etc.}

First, we look at 3 important examples of Stress States
1. Plane Stress (p.20) 2. Hydrostatic & Deviatoric Stresses (p.46)
   3. Principal Stresses (various sections such as 2.14)
Plane Stress (p. 20)
- stresses are zero in one of the primary directions (or 2-D stress state) -

Examples:
1. Thin sheet with loaded in the plane (stresses are zero along the thickness direction)
2. Pressurized thin cylinder (stresses along r or thickness direction are zero for cylinders when wall-thickness is about 1/10th of diameter):

\[ \sigma_0 = \frac{P_r}{t}, \quad \sigma_z = \frac{P_r}{2t} \text{ with } \sigma_r \approx 0 \]

Principal Stresses (p. 22)

Principal Planes are the planes on which maximum normal stresses act with no shear stresses and these stresses are the Principal Stresses

Designate \( \sigma_1, \sigma_2, \sigma_3 \) \( \Leftrightarrow \) implies no shear stresses or \( \sigma_{ij} = \sigma_{ij} \delta_{ij} \)

Proof is clear from Mohr’s circle representation (see text Fig.2.6) for 2-D

\[
\text{Note: } \tau_{\text{max}} = \frac{\sigma_1 - \sigma_3}{2}
\]

Hydrostatic and Deviatoric Stresses (p. 46)

Total stress tensor can be divided into two components (Fig. 2-18)

Hydrostatic or mean stress tensor \( (\sigma_m) \) involving only pure tension or compression & Deviatoric stress tensor \( (\sigma_{ij}') \) representing pure shear with no normal components

\[
\sigma_m = \frac{\sigma_{kk}}{3} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3}
\]
\[
\sigma_{ij}' = \sigma_{ij} + \frac{1}{3} \delta_{ij} \sigma_{kk}
\]
Example on Hydrostatic and Deviatoric Stresses

Given the stress state: \( \sigma_{ij} = \begin{pmatrix} 80 & 20 & -50 \\ 20 & -40 & 30 \\ -50 & 30 & 50 \end{pmatrix} \),

**a. Find the hydrostatic part of the stresses.**

**b. Find the deviatoric part of the stresses.**

**Ans.** (a) \( \sigma_{ij}^{\text{hyd}} = \sigma_m \delta_{ij} = \frac{1}{3} (\sigma_{11} + \sigma_{22} + \sigma_{33}) \delta_{ij} \) where \( \sigma_m = \frac{1}{3} (80 - 40 + 50) = 30 \) so that

\[
\sigma_{ij}^{\text{hyd}} = \begin{pmatrix} 30 & 0 & 0 \\ 0 & 30 & 0 \\ 0 & 0 & 30 \end{pmatrix}.
\]

(b) By definition, \( \sigma_{ij}^{\text{dev}} = \sigma_{ij} - \sigma_{ij}^{\text{hyd}} \) where

\[
\sigma_{ij}^{\text{dev}} = \begin{pmatrix} 80 & 20 & -50 \\ 20 & -40 & 30 \\ -50 & 30 & 50 \end{pmatrix} - \begin{pmatrix} 30 & 0 & 0 \\ 0 & 30 & 0 \\ 0 & 0 & 30 \end{pmatrix} = \begin{pmatrix} 50 & 20 & -50 \\ 20 & -70 & 30 \\ -50 & 30 & 20 \end{pmatrix}
\]

Note that the mean hydrostatic stress for \( \sigma_{ij}^{\text{dev}} = (\sigma_{11}^{\text{dev}} + \sigma_{22}^{\text{dev}} + \sigma_{33}^{\text{dev}}) = 0 \), as expected.
Principal Stresses (p. 22)

**Principal Planes** are the planes on which maximum normal stresses act with no shear stresses and these stresses are the **Principal Stresses**

Designate $\sigma_1, \sigma_2, \sigma_3$ implies no shear stresses or $\sigma_{ij} = \delta_{ij}$

Proof is clear from Mohr’s circle representation (see text Fig.2.6) for 2-D

For 3-D such an analogy is not useful and these are determined from the roots of $\sigma$ of the determinant (cubic in $\sigma$):

$$
\begin{vmatrix}
\sigma_{11} - \sigma & \sigma_{12} & \sigma_{13} \\
\sigma_{21} & \sigma_{22} - \sigma & \sigma_{23} \\
\sigma_{31} & \sigma_{32} & \sigma_{33} - \sigma \\
\end{vmatrix} = 0 ; \quad \text{Expand the determinant} \quad \Gamma
$$

$$
\sigma^3 - I_1 \sigma^2 - I_2 \sigma - I_3 = 0
$$

where I’s are invariants of the stress tensor (Eqs. on p.28 of Text):

$$
I_1 = (\sigma_{11} + \sigma_{22} + \sigma_{33}) \\
I_2 = -(\sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11} - \sigma_{12}^2 - \sigma_{23}^2 - \sigma_{31}^2) \\
I_3 = \sigma_{11}\sigma_{22}\sigma_{33} + 2\sigma_{12}\sigma_{23}\sigma_{31} - \sigma_{11}\sigma_{23}^2 - \sigma_{22}\sigma_{33}^2 - \sigma_{33}\sigma_{12}^2
$$

\[\text{[text - x, y, z / 1, 2, 3]}\]

(a) Uniaxial stress : $\begin{pmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

(b) Biaxial stress : $\begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

(c) Hydrostatic pressure : $\begin{pmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{pmatrix}$

(d) Pure shear : $\begin{pmatrix} 0 & \sigma & 0 \\ \sigma & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

- Special Stress States -
Normal and Shear Stresses on a Given Plane

[Cut-Surface Method]

Given \( \sigma_{ij} \) in reference system 1 2 3;

- \( \hat{n} \) is the unit vector normal to the plane = \( n_1 \ n_2 \ n_3 \)
- \( \hat{m} \) is the unit vector in the plane = \( m_1 \ m_2 \ m_3 \)

\( \sigma_N \) = normal stress along \( \vec{n} \)

\( \tau \) = shear stress along \( \vec{m} \)

Note: \( \hat{n} \cdot \hat{m} = 0; \ n_1^2 + n_2^2 + n_3^2 = 1 \ and \ m_1^2 + m_2^2 + m_3^2 = 1 \)

Note: if \( \vec{n} = 1, 2, 5, \Rightarrow \hat{n} = \frac{1}{\sqrt{30}}, \frac{2}{\sqrt{30}}, \frac{5}{\sqrt{30}} \ \Leftrightarrow \hat{n} \) is a unit vector

where \( \sqrt{n_1^2 + n_2^2 + n_3^2} = \sqrt{30} \) so that \( n_1^2 + n_2^2 + n_3^2 = 1. \)

1. Find the stress vector (\( \vec{S} \))

\{the stress vector : the vector force per unit area acting on the cut\}:

\[
\begin{align*}
\begin{pmatrix}
S_1 \\
S_2 \\
S_3
\end{pmatrix}
&=
\begin{pmatrix}
\sigma_{11} & \sigma_{12} & \sigma_{13} \\
\sigma_{21} & \sigma_{22} & \sigma_{23} \\
\sigma_{31} & \sigma_{32} & \sigma_{33}
\end{pmatrix}
\begin{pmatrix}
n_1 \\
n_2 \\
n_3
\end{pmatrix}
\Rightarrow
S_i = \sum_{k=1}^{3} \sigma_{ik} n_k; \ i.e. \ S_1=\sigma_{11}n_1+\sigma_{12}n_2+\sigma_{13}n_3; \ etc.
\end{align*}
\]

2. \( \sigma_N \) and \( \tau \) follow as: \( \sigma_N = \vec{S} \cdot \hat{n} = S_1 n_1 + S_2 n_2 + S_3 n_3 \)

\( \tau = \vec{S} \cdot \vec{m} = S_1 m_1 + S_2 m_2 + S_3 m_3 \)

and \( \tau_{\text{max}} \) occurs when \( n, S \) and \( m \) are in the same plane, or from Fig.

\[
|\vec{S}|^2 = \sigma_N^2 + \tau_{\text{max}}^2
\]
Normal and Shear Stresses on a Given Plane

[Cut-Surface Method]

**EXAMPLE**

A stress state in a given reference frame is (MPa): \( \sigma_{ij} = \begin{pmatrix} 8 & 2 & -5 \\ 2 & -4 & 3 \\ -5 & 3 & 6 \end{pmatrix} \).

Assume that the stresses are independent of position (uniform stress state). A plane "cut" is made through the body such that the normal to the cut is \( \vec{n} = \sqrt{2}, 2, -2 \).

**a.** What is the normal stress \( \sigma_N \) on the plane?

**b.** What is the shear stress \( \tau \) along the direction \( \vec{m} = 0, 1, 1 \) in the plane?

**c.** What is the maximum shear stress in the plane (consider all directions in the plane)?

**Answer:**

\[ \hat{n} = \frac{\sqrt{2}}{\sqrt{10}}, \frac{2}{\sqrt{10}}, \frac{-2}{\sqrt{10}} \text{ and } \hat{m} = 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}. \]

\[
S_1 = \sigma_{11}n_1 + \sigma_{12}n_2 + \sigma_{13}n_3 = \frac{1}{\sqrt{10}} \left[ 8 (\sqrt{2}) + 2 (2) + (-5) (-2) \right] = 8 \text{ MPa},
\]

Similarly, \( S_2 = -3.53 \text{ MPa} \) and \( S_3 = -4.13 \text{ MPa} \).

(a) \( \sigma_N = S_1n_1 + S_2n_2 + S_3n_3 = \frac{1}{\sqrt{10}} \left[ 8 (\sqrt{2}) + (-3.53) (2) + (-4.13) (-2) \right] = 3.96 \text{ MPa} \)

(b) \( \tau = \vec{S} \cdot \hat{m} = S_1 m_1 + S_2 m_2 + S_3 m_3 = \frac{1}{\sqrt{2}} \left[ 8 (0) + (-3.53) (1) + (-4.13) (1) \right] = -5.42 \text{ MPa}. \)

Note: "-" sign means the shear stress acts in the \( -\hat{m} \) direction.

(c) To find the maximum shear stress in the plane: Since \( \tau = \vec{S} \cdot \hat{m} \), the maximum projection of \( \vec{S} \) along \( \hat{m} \) will occur when these 2 vectors are coplanar (containing \( \hat{n} \) also) - see Fig. below:

\[
|\vec{S}|^2 = S_1^2 + S_2^2 + S_3^2 = (8)^2 + (-4.13)^2 + (-5.42)^2 = 93.5.
\]

From the figure, note that

\[
|\vec{S}|^2 = \sigma_N^2 + \tau_{\max}^2 \text{ or } \tau_{\max} = \sqrt{|\vec{S}|^2 - \sigma_N^2} = \sqrt{93.5 - (3.96)^2} = 8.82 \text{ MPa}.
\]
Example: Stress Tensor Transformations

Given \( \sigma_{ij} = \begin{pmatrix} 10 & 5 & 0 \\ 5 & 20 & 0 \\ 0 & 0 & 0 \end{pmatrix} \) wrt x,y,z axes.

New axes (x',y',z') are rotated 45° around z-axis.

Need to find \( \sigma'_{ij} \). \( \sigma'_{ij} = a_{ik} a_{jn} \sigma_{kn} \)

a. Calculate \( \tau'_{xy} \).

\[
\begin{align*}
\tau'_{xy} &= a_{xx} a_{yx} \sigma_{xx} + a_{xx} a_{yy} \sigma_{xy} + a_{xx} a_{yz} \sigma_{xz} \\
&\quad + a_{xy} a_{yx} \sigma_{yx} + a_{xy} a_{yy} \sigma_{yy} + a_{xy} a_{yz} \sigma_{yz} \\
&\quad + a_{xz} a_{yx} \sigma_{zx} + a_{xz} a_{yy} \sigma_{zy} + a_{xz} a_{yz} \sigma_{zz} \\
&= \frac{1}{\sqrt{2}} (-\frac{1}{\sqrt{2}})(10) + \frac{1}{\sqrt{2}} \left( \frac{1}{\sqrt{2}} \right)(5) + 0 \\
&\quad + \frac{1}{\sqrt{2}} \left( -\frac{1}{\sqrt{2}} \right)(5) + \frac{1}{\sqrt{2}} \left( \frac{1}{\sqrt{2}} \right)(20) + 0 \\
&\quad + 0 + 0 + 0 + 0 \\
&= 5 \text{ MPa}
\end{align*}
\]

b. Show that \( \sigma'_x = 20 \text{ MPa} \) and \( \sigma'_y = 10 \text{ MPa} \).

Thus \( \sigma'_{ij} = \begin{pmatrix} 20 & 5 & 0 \\ 5 & 10 & 0 \\ 0 & 0 & 0 \end{pmatrix} \). Note that \( \sigma_x + \sigma_y = \sigma'_x + \sigma'_y (= 30 \text{ MPa}) \).
Example 2: Same as above but in 2-D for a general case

\[ x', y' \text{ rotation by } \theta \]

\[
\theta_{ij}: \begin{array}{c|cc}
  & x & y \\
\hline
x' | & \theta & 90-\theta \\
y' | & 90+\theta & \theta \\
\end{array}
\]

\[
a_{ij} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}
\]

\[
\tau_{xy}' (\text{or } \tau_{xy}') = a_{xx} a_{yx} \sigma_{xx} + a_{xx} a_{yy} \tau_{xy} + a_{xy} a_{yx} \tau_{yx} + a_{xy} a_{yy} \sigma_{yy} \\
= -\sin \theta \cos \theta \sigma_{xx} + \cos^2 \theta \tau_{xy} - \sin^2 \theta \tau_{xy} + \sin \theta \cos \theta \sigma_{yy} \\
= \frac{-\sigma_{yy} - \sigma_{xx}}{2} \sin 2\theta + \tau_{xy} \cos 2\theta \Rightarrow \text{Eq. 2.7}
\]

Similarly find \[\sigma_{x}' = \frac{\sigma_{x} + \sigma_{y}}{2} + \frac{\sigma_{x} - \sigma_{y}}{2} \cos 2\theta + \tau_{xy} \sin 2\theta \] (Eq 2.5)
and \[\sigma_{y}' = \frac{\sigma_{x} + \sigma_{y}}{2} - \frac{\sigma_{x} - \sigma_{y}}{2} \cos 2\theta - \tau_{xy} \sin 2\theta \] (Eq 2.6)

& Fig. 2.4 shows variation of these 3 stresses with \(\theta\).

Note: \[\sigma_{x} + \sigma_{y} = \sigma_{x}' + \sigma_{y}'\] as should be since \(I_1\) is invariant ⇔
i.e., sum of normal stresses on mutually perpendicular planes is invariant.
same thing can be done using Mohr's circle representation (easier for 2-D case)

If \(\tau_{xy} = 0\), principal planes and stresses: \[\tan 2\theta = \frac{2\tau_{xy}}{\sigma_{x} - \sigma_{y}} \] (Eq. 2.8)

whereas maximum shear stresses when \[\tan 2\theta = -\frac{\sigma_{x} - \sigma_{y}}{2\tau_{xy}} \] (Eq.2.10) & \(\tau_{\text{max}}\) given by Eq. 2.11.