Newton-Cotes Formula

- The way the trapezoidal rule is derived can be generalized to higher degree polynomial interpolants. Such a quadrature rule is called a *Newton-Cotes formula*.

- Let \( x_0, x_1, \ldots, x_n \) be given nodes in \([a, b]\). Recall that the Lagrange interpolation of a function at these nodes is given by the polynomial
  \[
p(t) = \sum_{j=0}^{n} f(x_j) \ell_j(t)
  \]
  where each \( \ell_j(t) \) is the Lagrange polynomial
  \[
  \ell_j(t) := \prod_{i=0, i \neq j}^{n} \frac{t - x_i}{x_j - x_i}, \quad j = 0, 1, \ldots, n.
  \]

- We therefore have
  \[
  \int_{a}^{b} f(t)dt \approx \int_{a}^{b} p(t)dt = \sum_{j=0}^{n} \omega_j f(x_j)
  \]
  where the weight \( \omega_j \) is determined by
  \[
  \omega_j = \int_{a}^{b} \ell_j(t)dt. \quad (1)
  \]
  - Note that if \( f(t) \) itself is a polynomial of degree \( \leq n \), then
    \[
    f(t) = \sum_{j=0}^{n} f(x_j) \ell_j(t). \quad (Why?)
    \]
    In this case, the Newton-Cotes quadrature rule evaluates \( \int_{a}^{b} f(t)dt \) precisely.
  - The Newton-Cotes quadrature rule has degree of precision at least \( n \).
An Example — Simpson’s Rule

• It is nice to know how the weight \( \omega_j \) should be calculated. However, there are some other concerns:

  ◦ These weights are difficult to evaluate.
  ◦ How high can the degree of precision be pushed?

• Suppose we approximate \( f(t) \) by a quadratic polynomial \( p_2(t) \) that interpolates \( f(a), f\left(\frac{a+b}{2}\right) \) and \( f(b) \).

  ◦ It can be shown that (I derived it in class)

\[
Q_3(f) = \frac{b-a}{6} [f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)].
\] (2)

  ◦ The error part is tricky!

\[\text{Observe that the function (in variable } s) \text{ from Newton’s formula, } g(s) = p_2(s) + f[a, b, \frac{a+b}{2}, t](s - a)(s - b)(s - \frac{a+b}{2}), \text{ interpolates } f \text{ at } a, b, \frac{a+b}{2} \text{ and } t.\]

\[\text{Thinking } t \text{ as arbitrary, we should have}
\]

\[f(t) = p_2(t) + f[a, b, \frac{a+b}{2}, t](t - a)(t - b)(t - \frac{a+b}{2})\]

\[\text{The error } E_3(f) \text{ therefore is given by}
\]

\[
E_3(f) = \int_a^b f[a, b, \frac{a+b}{2}, t] \omega(t) dt \] (3)

with \( \omega(t) := (t - a)(t - b)(t - \frac{a+b}{2}). \)

\[\text{We already know that the degree of precision is } \geq 2. \text{ Can this be better?} \]
• The function $\omega(x)$ changes sign as $x$ crosses $\frac{a+b}{2}$. So we have to analyze $E_3(x)$ by a different approach.

  ◦ Let $\Omega(x) := \int_a^x \omega(t)dt$. Then $\Omega'(x) = \omega(x)$.
  ◦ By integration by parts, we have

    \[ E_3(f) = f[a, b, \frac{a + b}{2}, x] \Omega(x) \bigg|_a^b - \int_a^b f[a, b, \frac{a + b}{2}, x] \Omega(x) dx. \]

  ◦ Observe that $\Omega(a) = \Omega(b) = 0$. Observe also that $\Omega(x) > 0$ for all $x \in (a, b)$.
  ◦ We may apply the mean value theorem to conclude that

    \begin{align*}
    E_3(f) &= -\int_a^b f[a, b, \frac{a + b}{2}, x] \Omega(x) dx \\
    &= -f[a, b, \frac{a + b}{2}, \xi, \xi] \int_a^b \Omega(x) dx \\
    &= -\frac{f^{(4)}(\eta)}{4!} \frac{4}{15} \left( \frac{b - a}{2} \right)^5 \\
    &= -\frac{f^{(4)}(\eta)}{90} \left( \frac{b - a}{2} \right)^5.
    \end{align*}

  ◦ The degree of precision for Simpson’s rule is 3 rather than 2.

• Divide the interval into $2n$ equally space subintervals with $h = \frac{b-a}{2n}$ and $x_i = a + ih$ for $i = 0, 1, \ldots, 2n$. Upon applying Simpson’s rule over two consecutive subintervals $[x_{2j}, x_{2j+2}]$ for $j = 0, 1, \ldots, n-1$ and summing up these integrals, we obtain the composite Simpson’s rule:

\begin{equation}
\int_a^b f(t) dt \approx \frac{h}{3} \left\{ f_0 + 4f_1 + 2f_2 + 4f_3 + \ldots + 2f_{2n-2} + 4f_{2n-1} + f_{2n} \right\}.
\end{equation}

  ◦ The error formula for the composite Simpson’s rule can be obtained in the same way as we derived the error formula for the composite trapezoidal rule:

\begin{equation}
E_{3,cs} = -\frac{(b - a)h^4}{180} f^{(4)}(\eta).
\end{equation}